Experimental Combinatorics via Rule-Algebraic Methods

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Combinatorics and Arithmetic for Physics: special days

IHES
Institut des Hautes Études Scientifiques

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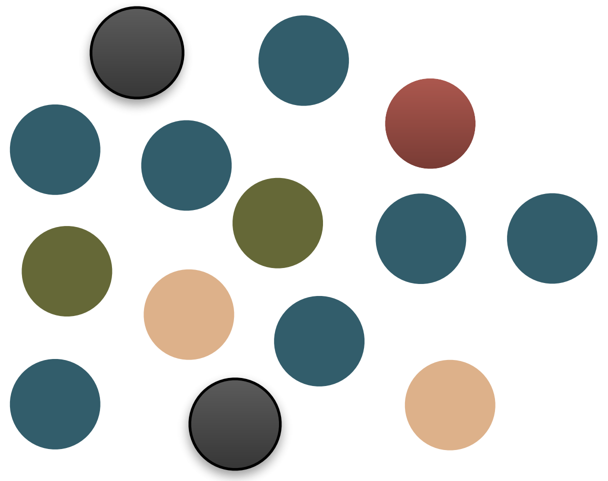
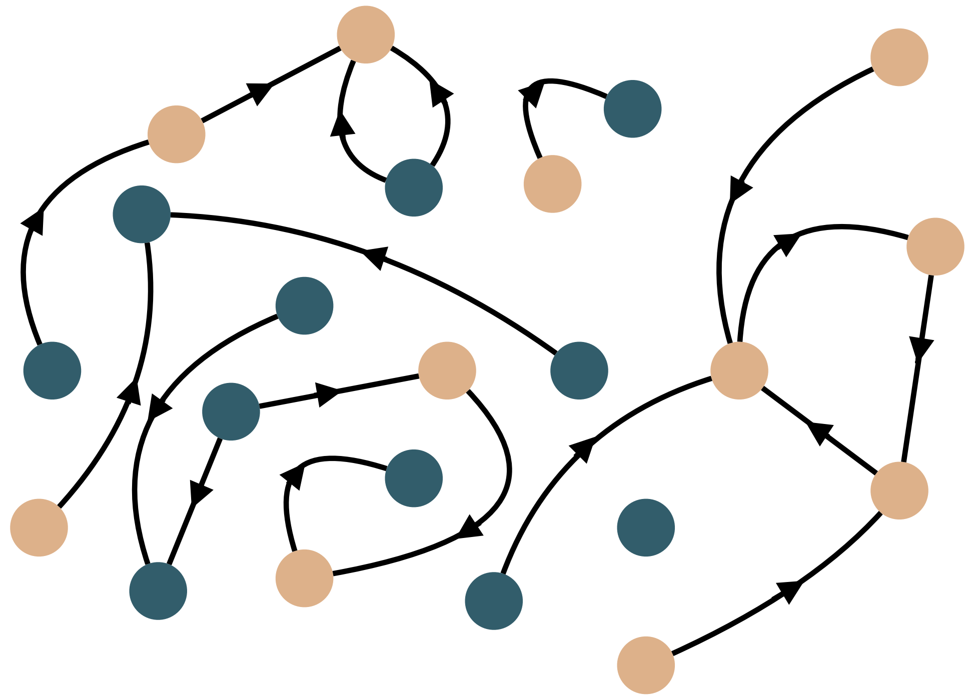
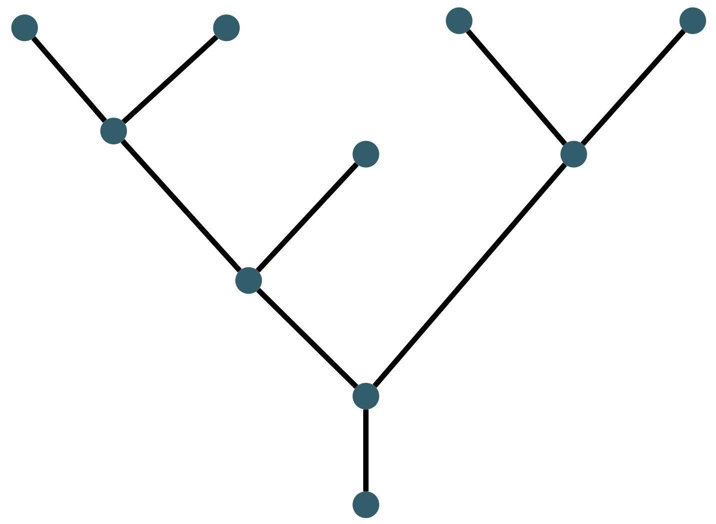
Centre de conférences Marilyn et James Simons
Bures-sur-Yvette, France

structures



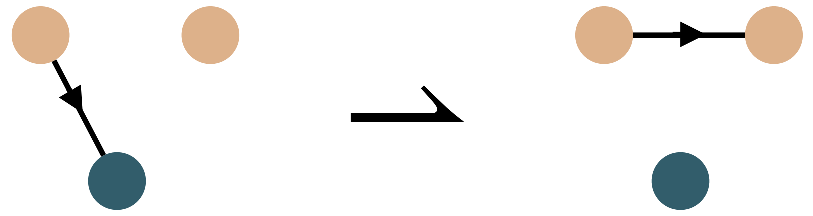


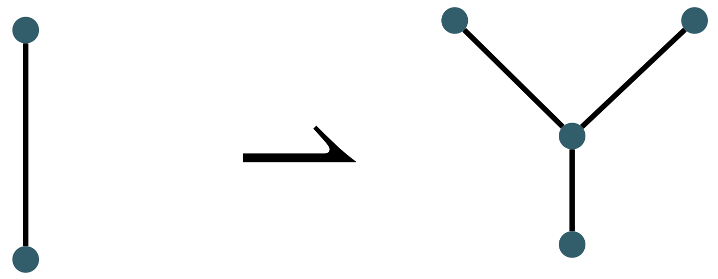


transitions $I \rightharpoonup O$

find occurrence
of **input pattern I**…

… **replace** with occurrence
of **output pattern O**





applications

+ **continuous-time Markov chain (CTMC)** semantics for transitions
  $\Rightarrow$ **CHEMISTRY**

+ **CTMC** or **discrete-time Markov chain (DTMC)** semantics for transitions
  $\Rightarrow$ **DYNAMICAL NETWORKS, AUTOMATA, RANDOM GRAPHS, …**

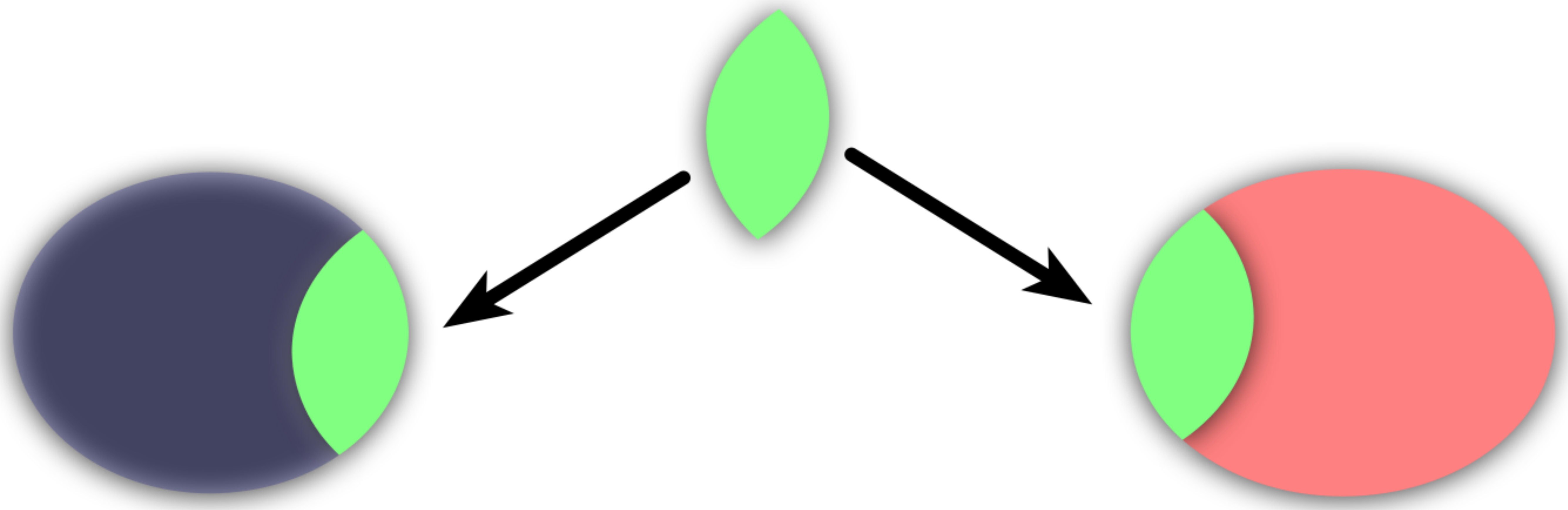
+ **pattern counting**
+ **generating function theory**
  $\Rightarrow$ **COMBINATORICS**

Plan

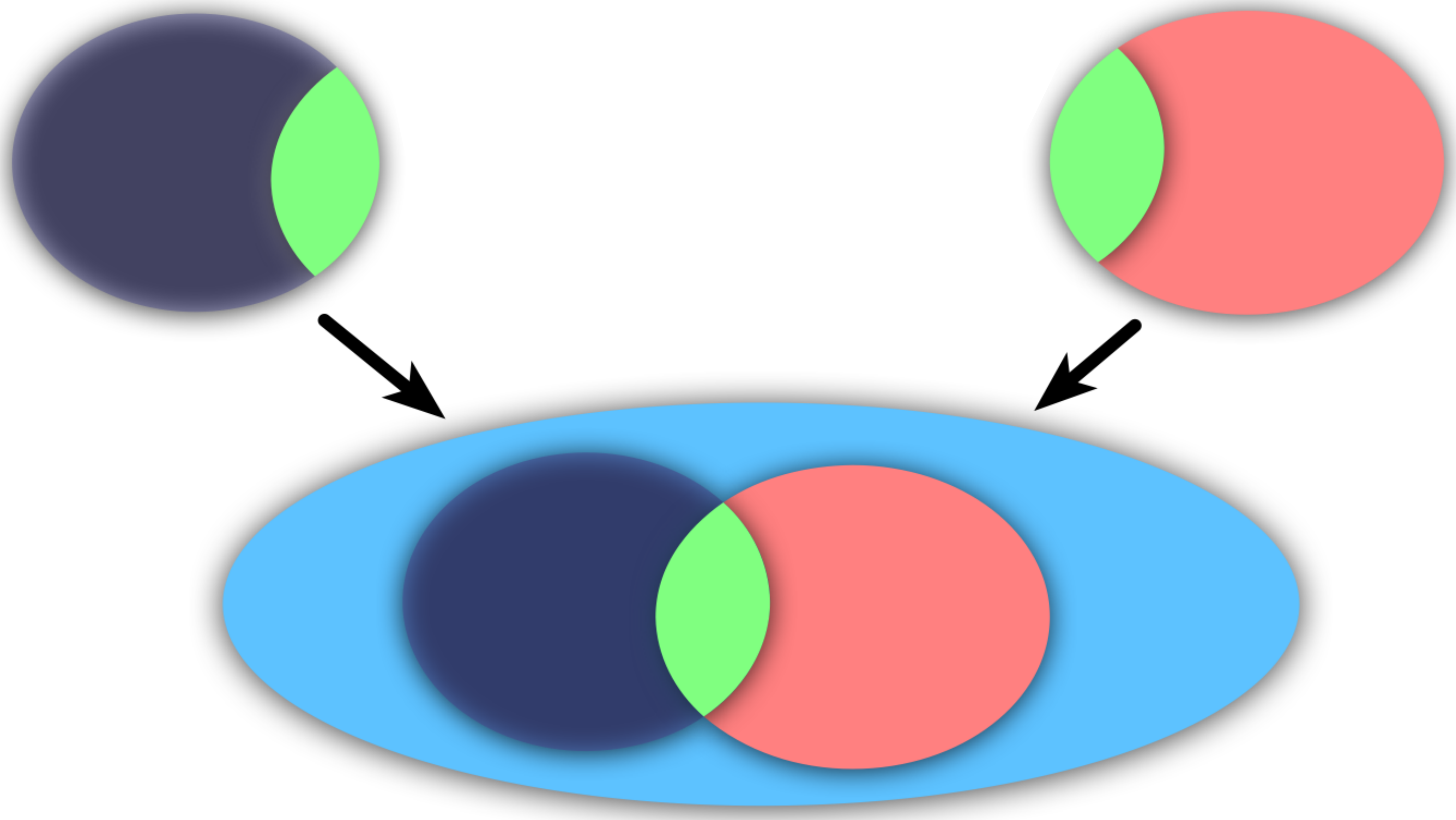
I. Categorical rewriting theory
II. Rule algebra formalism
III. Experimental combinatorics

I. Categorical rewriting theory

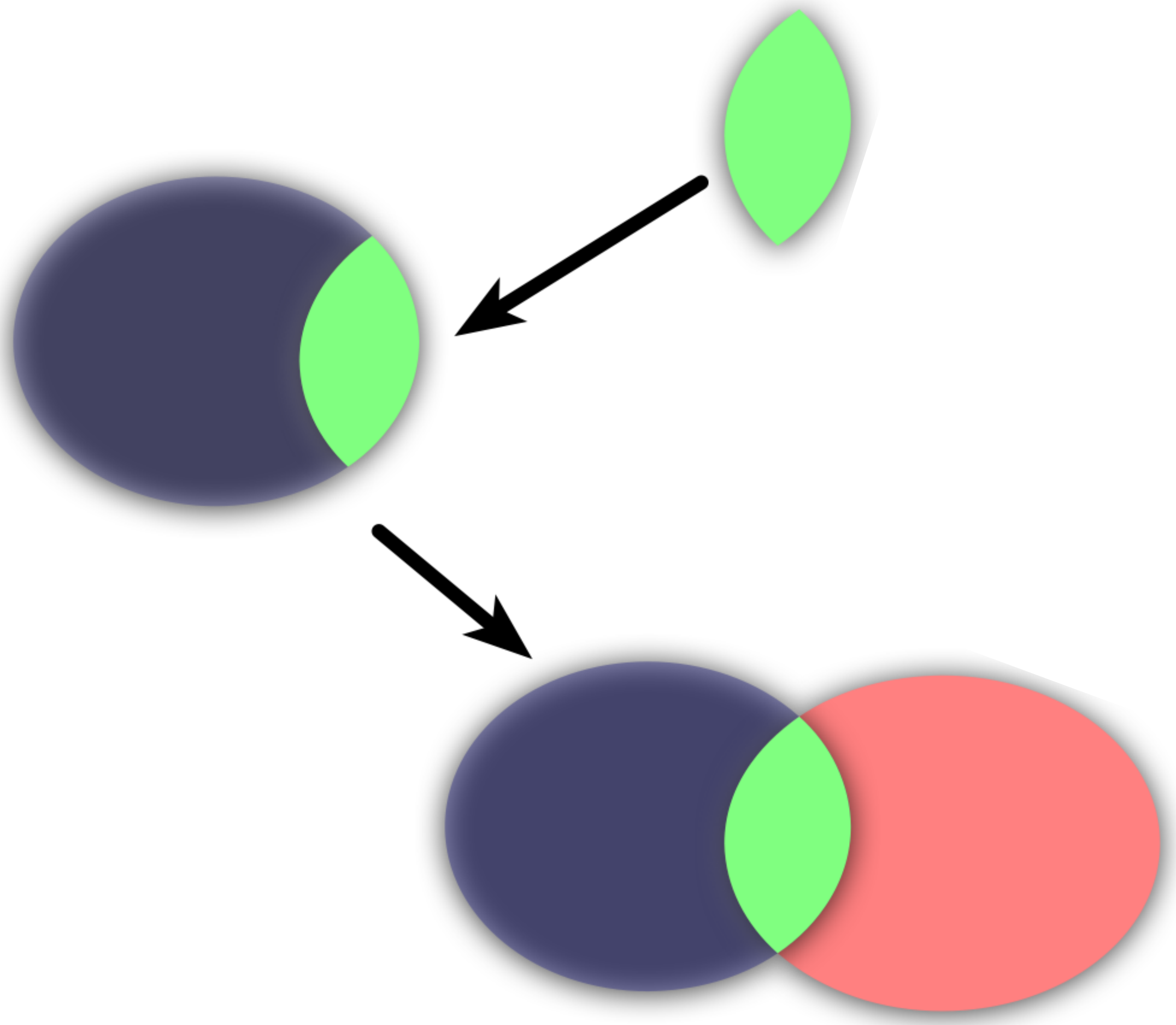
Set union



Set intersection (of two sets within another set)

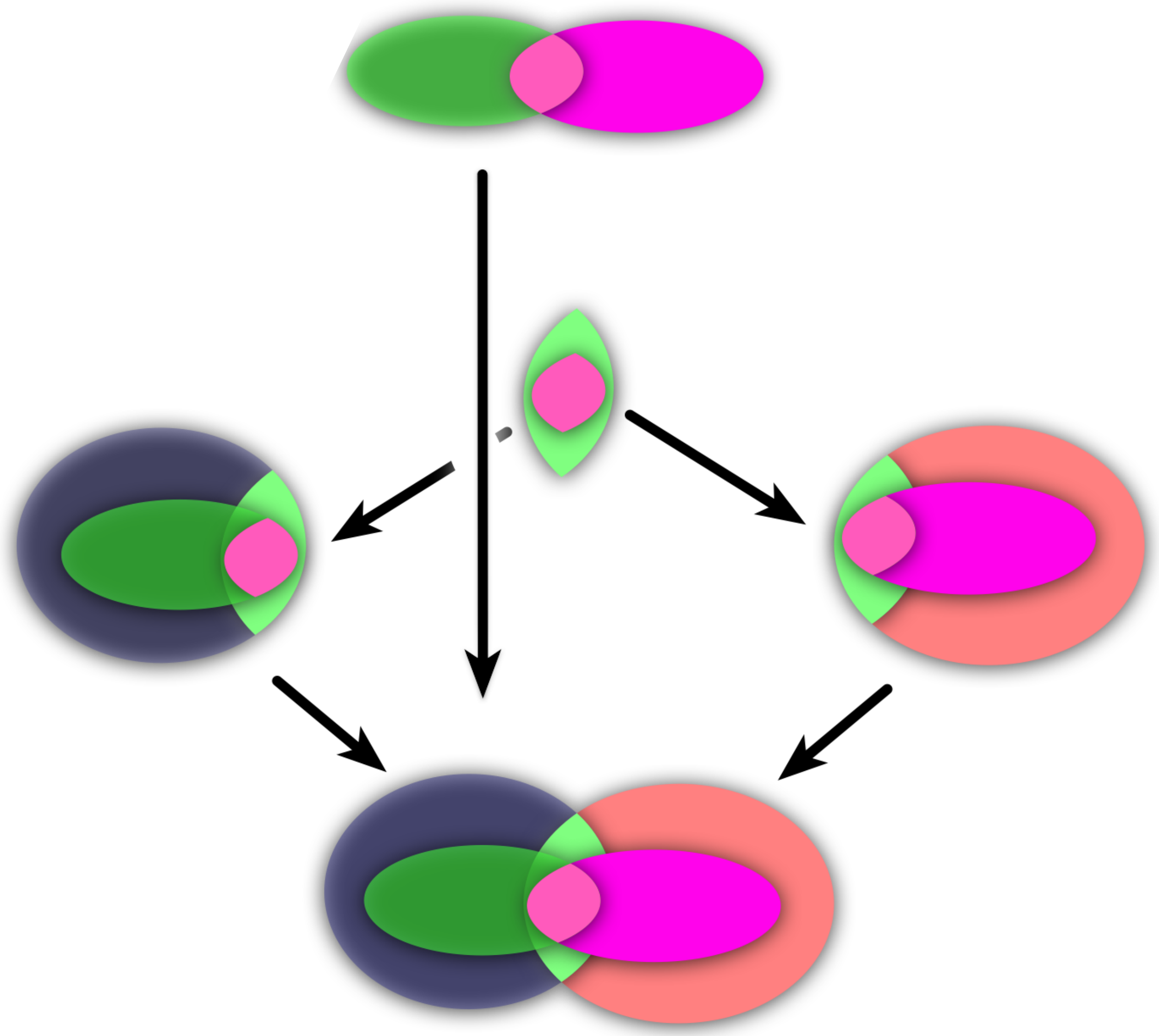


Set complement



Van Kampen property (*Lack & Sobocinski 2003*)

If the bottom square is a **pushout**



The basic prerequisites for category-theoretical rewriting theories

A category **C** is **adhesive** if

1. **C** has **all pullbacks**
2. **C** has **pushouts along monomorphisms**
3. Pushouts along monomorphisms are **van Kampen squares**.

S. Lack & P. Sobociński **(2005)**. *Adhesive and quasiadhesive categories.* RAIRO-Theoretical Informatics and Applications, 39(3), 511-545.

A category **C** is **finitary** if every object $X \in$ ob(**C**) has only **finitely many subobjects** (up to isomorphism).

K. Gabriel, B. Braatz, H. Ehrig & U. Golas **(2014)**. *Finitary M-adhesive categories.* Mathematical Structures in Computer Science, 24(4).

A category **C** possesses a **strict initial object** $\varnothing \in |\mathbf{C}|$ (the "*empty object*") if

1. $\forall X \in \mathrm{ob}(\mathbf{C}) : \exists!(\iota_X : \varnothing \hookrightarrow X) \in \mathrm{mono}(\mathbf{C})$
2. $\forall X \in \mathrm{ob}(\mathbf{C}) : \exists(X \to \varnothing) \;\Rightarrow\; X \cong \varnothing$

Example: presheaves

Definition: For $\mathbb{S}$ a (small) category, the category $\hat{\mathbb{S}}$ of **presheaves over** $\mathbb{S}$ has

- **objects** of $\hat{\mathbb{S}}$ are **functors** $F : \mathbb{S}^{op} \to \mathbb{SET}$
- **morphisms** of $\hat{\mathbb{S}}$ are **natural transformations** $\phi : F \dot{\to} G$

Special case: (directed) **multigraphs** as $\hat{\mathbb{G}}$, where $\mathbb{G} : V \underset{t}{\overset{s}{\rightrightarrows}} E$

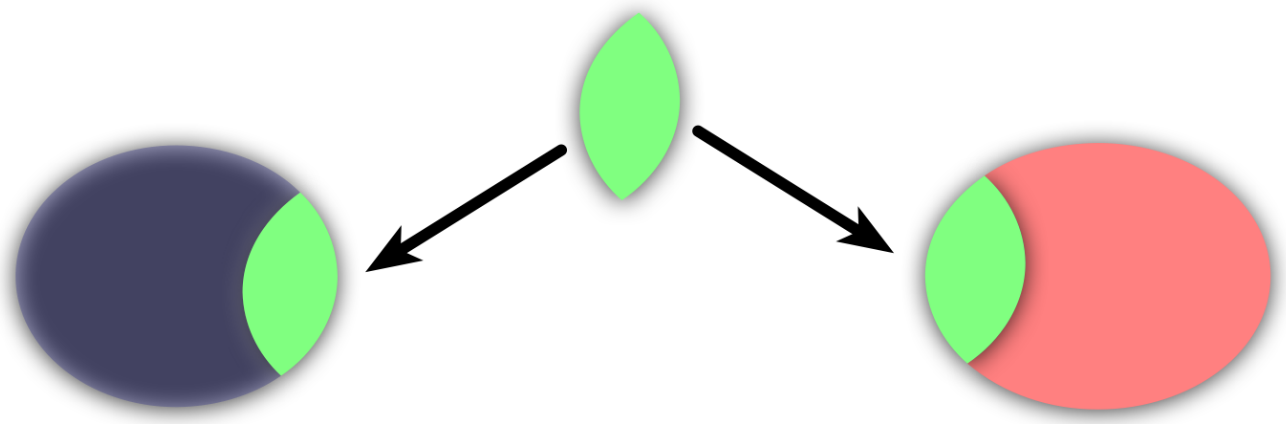
$\Rightarrow$ a **graph** G is given by the data $G(V)$ (set of *vertices*), $G(E)$ (set of *edges*) and two morphisms $G(s), G(t) : G(E) \to G(V)$ (*source/target maps*)

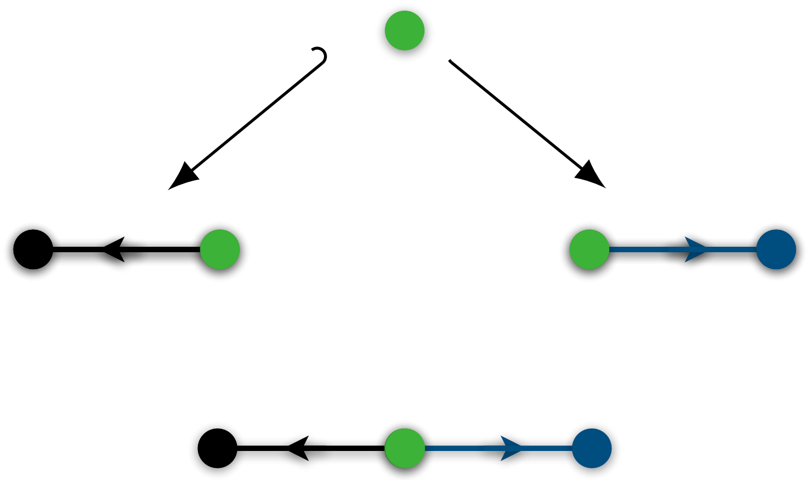
$\Rightarrow$ a **graph homomorphism** $\phi = (\phi_V, \phi_E) : G_1 \to G_2$ is a natural transformation, i.e.

$$
\begin{tikzcd}
G_1(E) \arrow[r, "G_1(s)"] \arrow[d, "\phi_E"'] & G_1(V) \arrow[d, "\phi_V"] \\
G_2(E) \arrow[r, "G_2(s)"'] & G_2(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
G_1(E) \arrow[r, "G_1(t)"] \arrow[d, "\phi_E"'] & G_1(V) \arrow[d, "\phi_V"] \\
G_2(E) \arrow[r, "G_2(t)"'] & G_2(V)
\end{tikzcd}
$$

commute

Set union is a special case of ***pushouts***



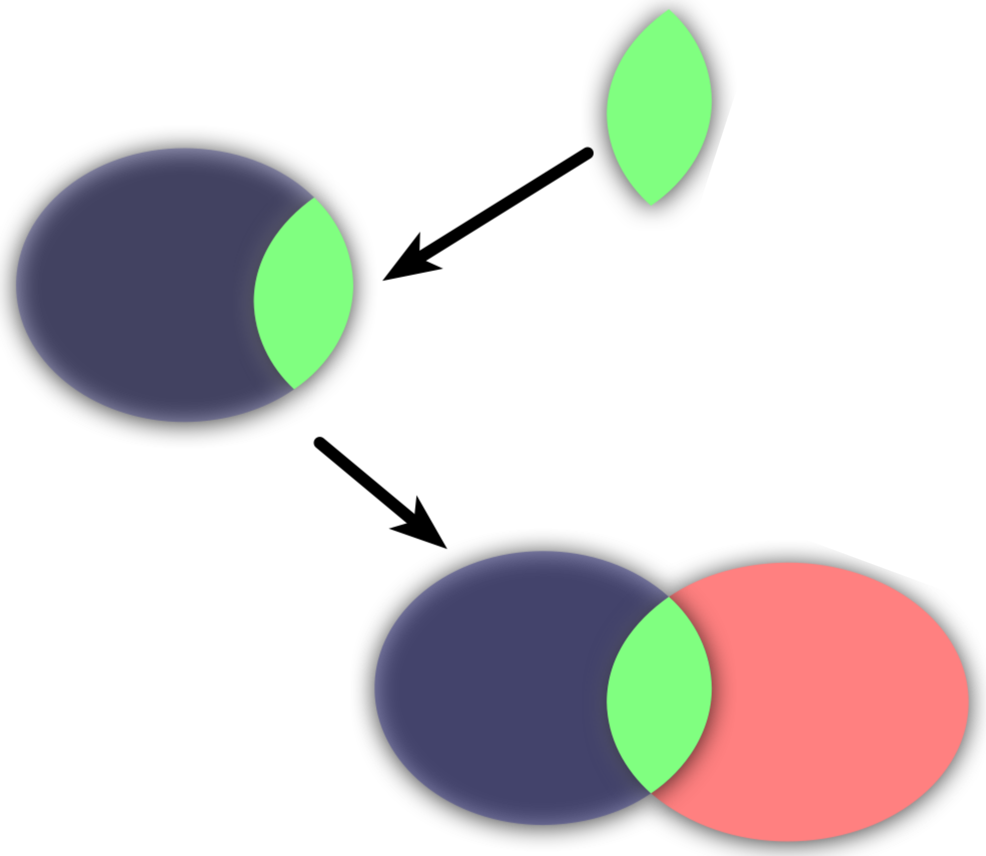


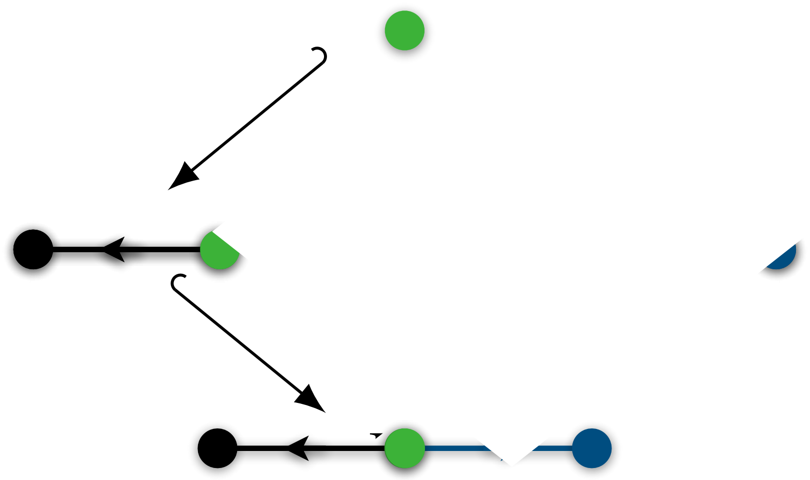
Interpretation:

A — **intersection** of B and C in D

D — **union** of B and C along A

Set complements are a special case of ***pushout complements***

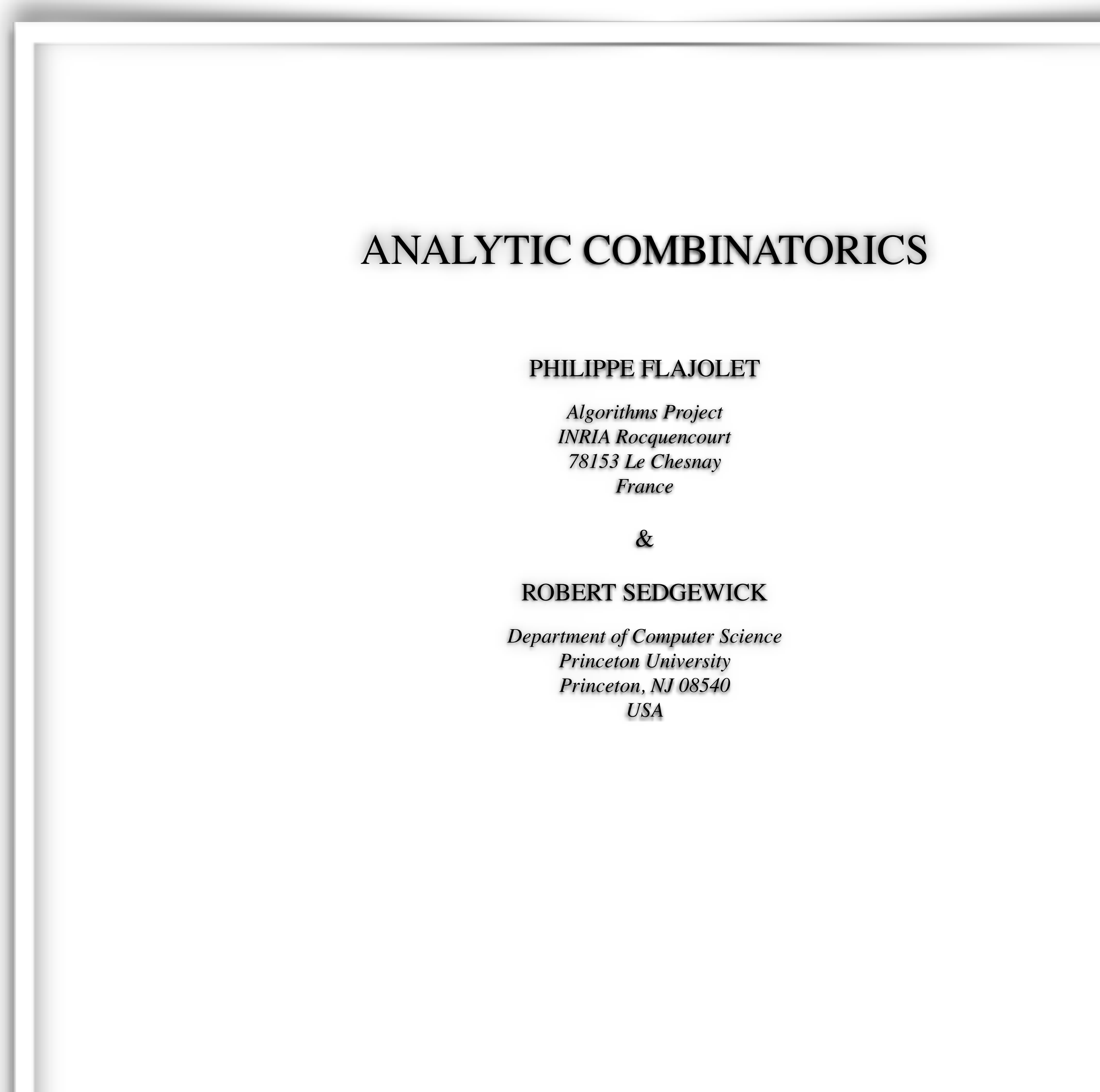




Sources of constructions of adhesive categories

- Every **topos** (elementary or Grothendiek) is an adhesive category.

S. Lack & P. Sobociński (2006). *Toposes are adhesive*. In Proceedings of the Third international conference on Graph Transformations (pp. 184-198). Springer-Verlag.


ANALYTIC COMBINATORICS

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Constraints on adhesive categories

Definition: For an adhesive, extensive and finitary category **C**, **constraints** are recursively defined as follows: let $(m : X \hookrightarrow Y) \in mono(C)$ be a **monomorphism**.

The central “workflow” of categorical rewriting theory

Fix an **adhesive finitary category C**.

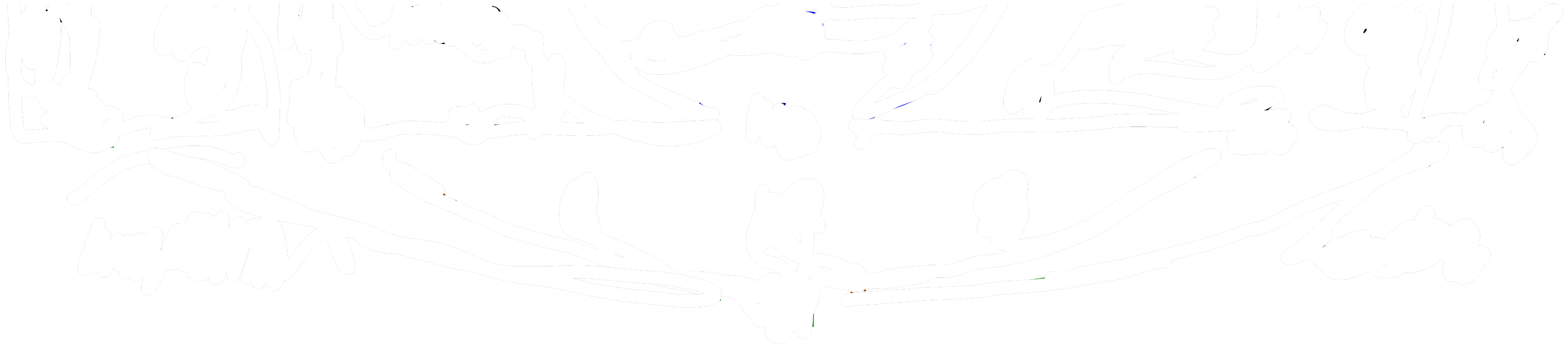
- Isomorphism classes of **objects** will model the **configurations**.
- Isomorphism classes of **spans of monomorphisms** will model the **transitions**, also referred to as **rewriting rules**:

$$r \equiv (O \xleftarrow{r} I) \equiv (O \xhookleftarrow{o} K \xhookrightarrow{i} I)$$

Examples for rewriting rules:

Sequential composition of linear rules along a match

Fix an **adhesive finitary category C**. Let

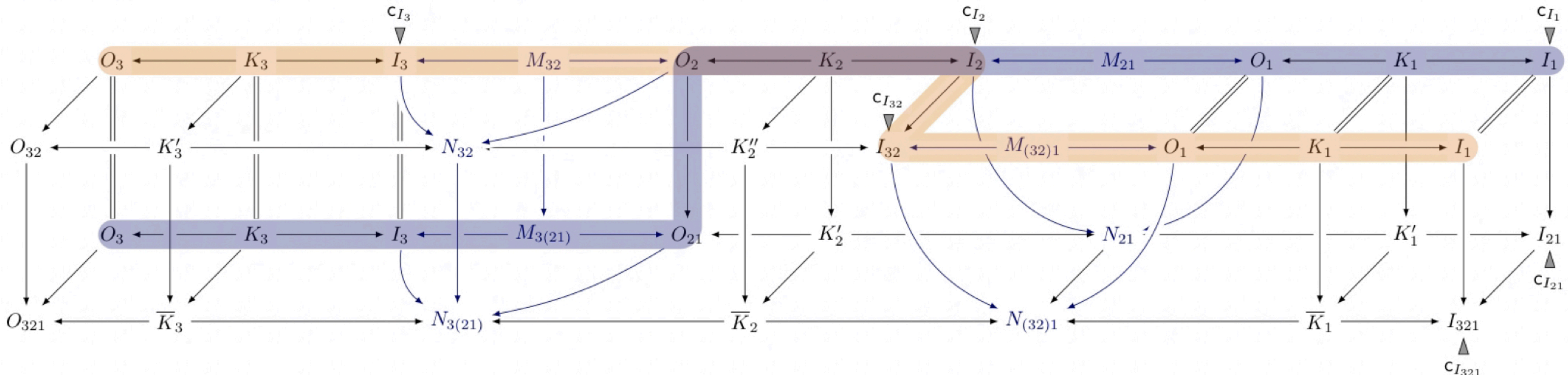


II. Rule algebra formalism

Encoding non-determinism of rule compositions

- Given two **rules** r_1, r_2 and a **match** $\mu \in M_{r_2}(r_1)$, let the **composition** be denoted as $r_2 \overset{\mu}{\blacktriangleleft} r_1$.

Theorem (NB et al.): For every **adhesive, finitary** and **extensive category C** (possibly with **constraints**), the **rule algebra** $(R_C, *_{R_C})$ is an **associative unital algebra**, with **unit element** $r_\varnothing := \delta(\varnothing \hookleftarrow \varnothing \hookrightarrow \varnothing)$.

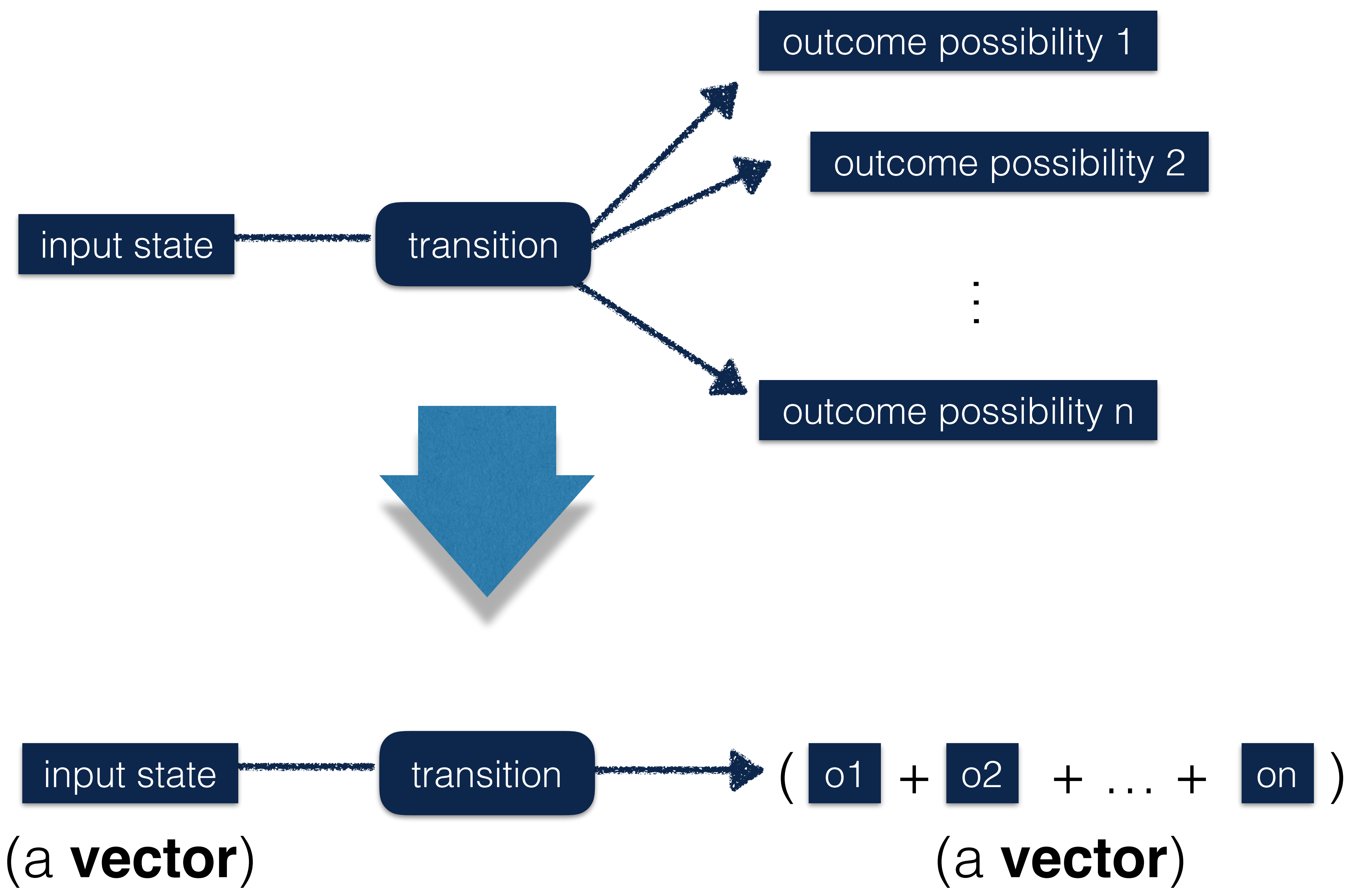


Nicolas Behr and Pawel Sobocinski. "Rule Algebras for Adhesive Categories". In: *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*. Ed. by Dan Ghica and Achim Jung. Vol. 119. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Sept. 2018, 11:1–11:21

Nicolas Behr. "Sesqui-Pushout Rewriting: Concurrency, Associativity and Rule Algebra Framework". In: *arXiv preprint 1904.08357* (2019)

Nicolas Behr and Jean Krivine. "Compositionality of Rewriting Rules with Conditions". In: *arXiv preprint 1904.09322* (2019)

Encoding non-determinism of rule applications



A possibility to encode non-determinism:

map **multiple possibilities** of transitions . . .

. . . into **"sum of possibilities"**

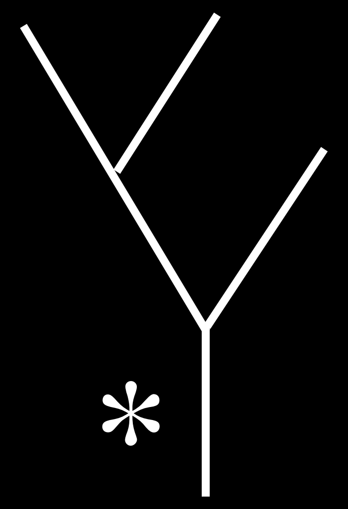
(via employing the notion of a **vector space of states** and of transitions as **linear operators** on this space)

Encoding non-determinism of rule applications

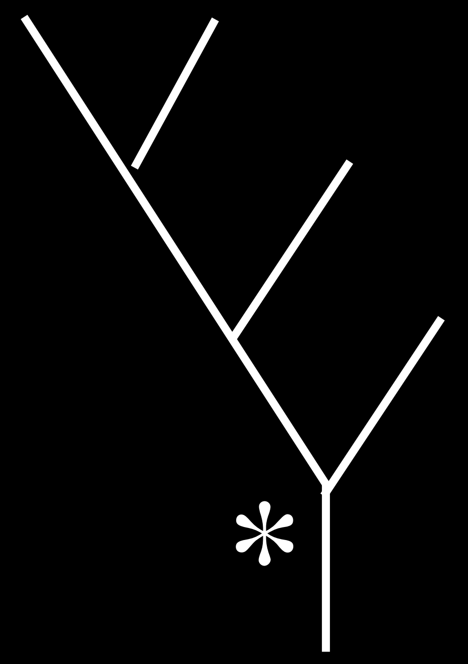
- Given an **object** $X \in \mathbf{C}$, a **rule** r and a **match** $m \in M_r(X)$, let the **application of** r **to** X **along** m be denoted as $r_m(X)$.

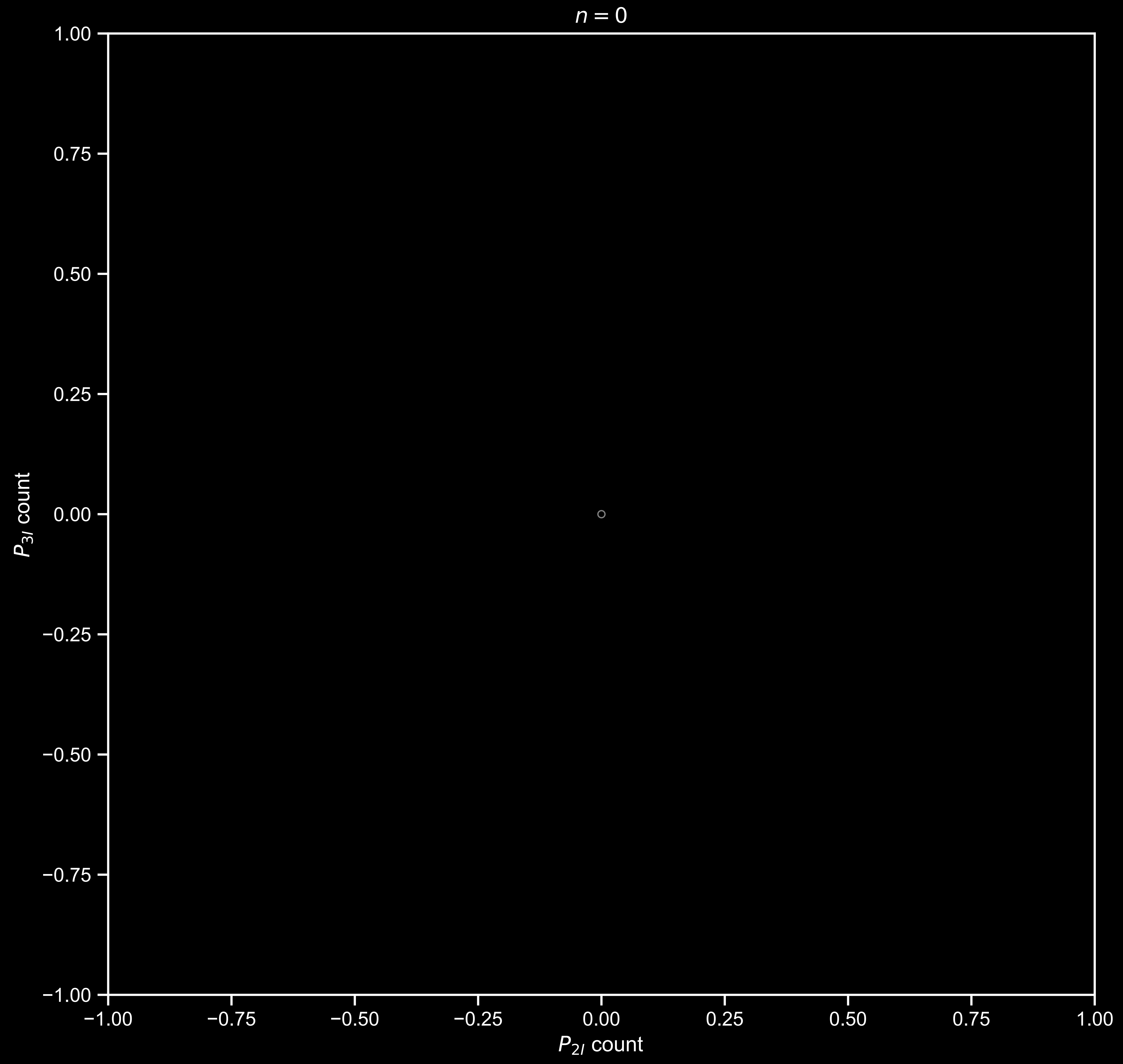
III. Experimental combinatorics

$P_{2I} :=$

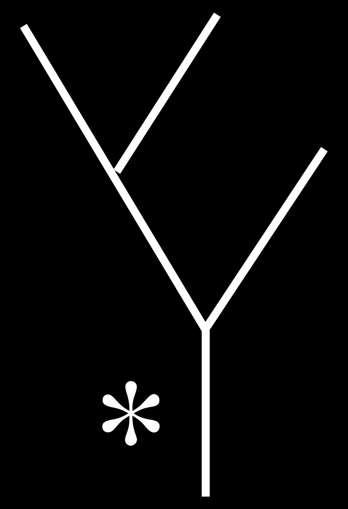


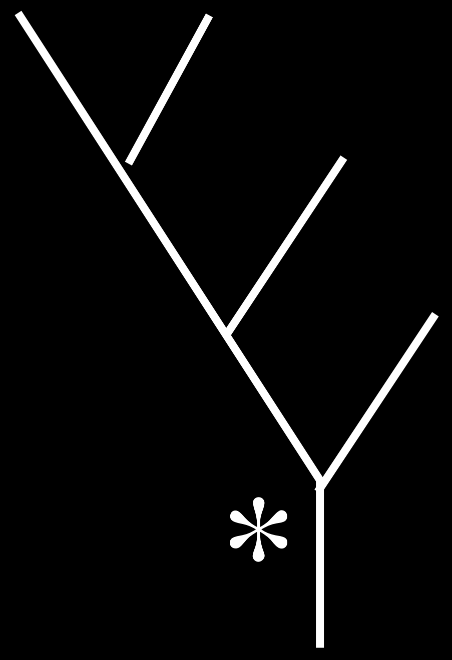
$P_{3I} :=$

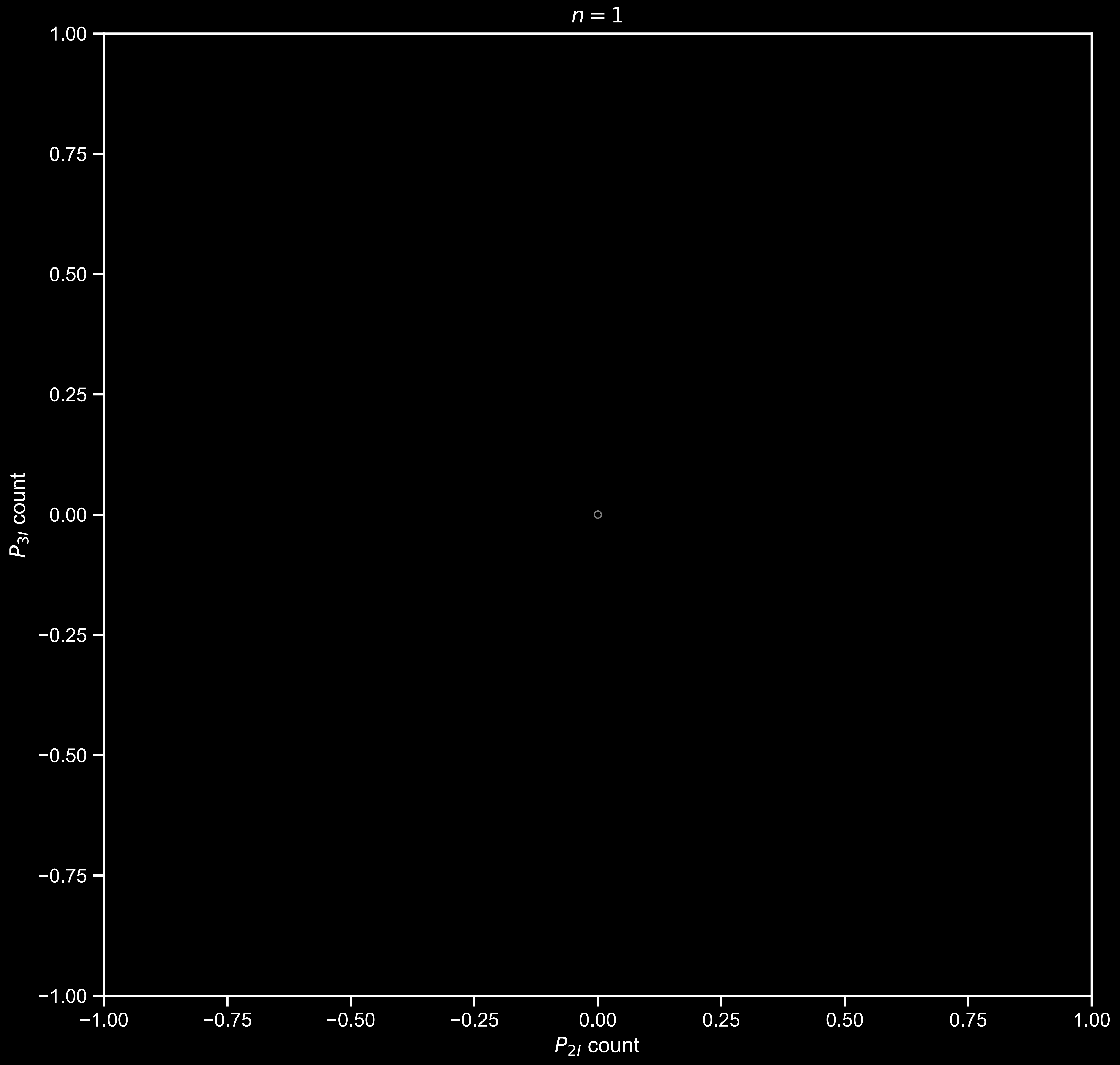


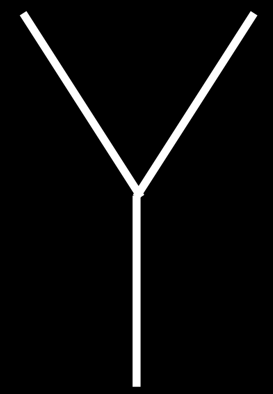


$$\mathscr{T}_0 = \{ \mid \}$$

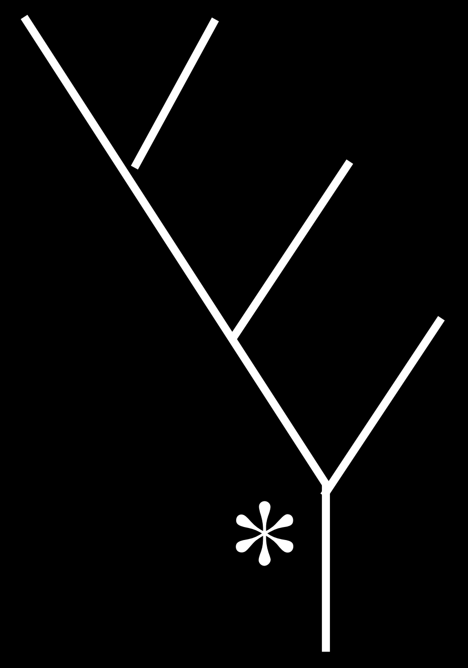
$P_{2I} :=$ 

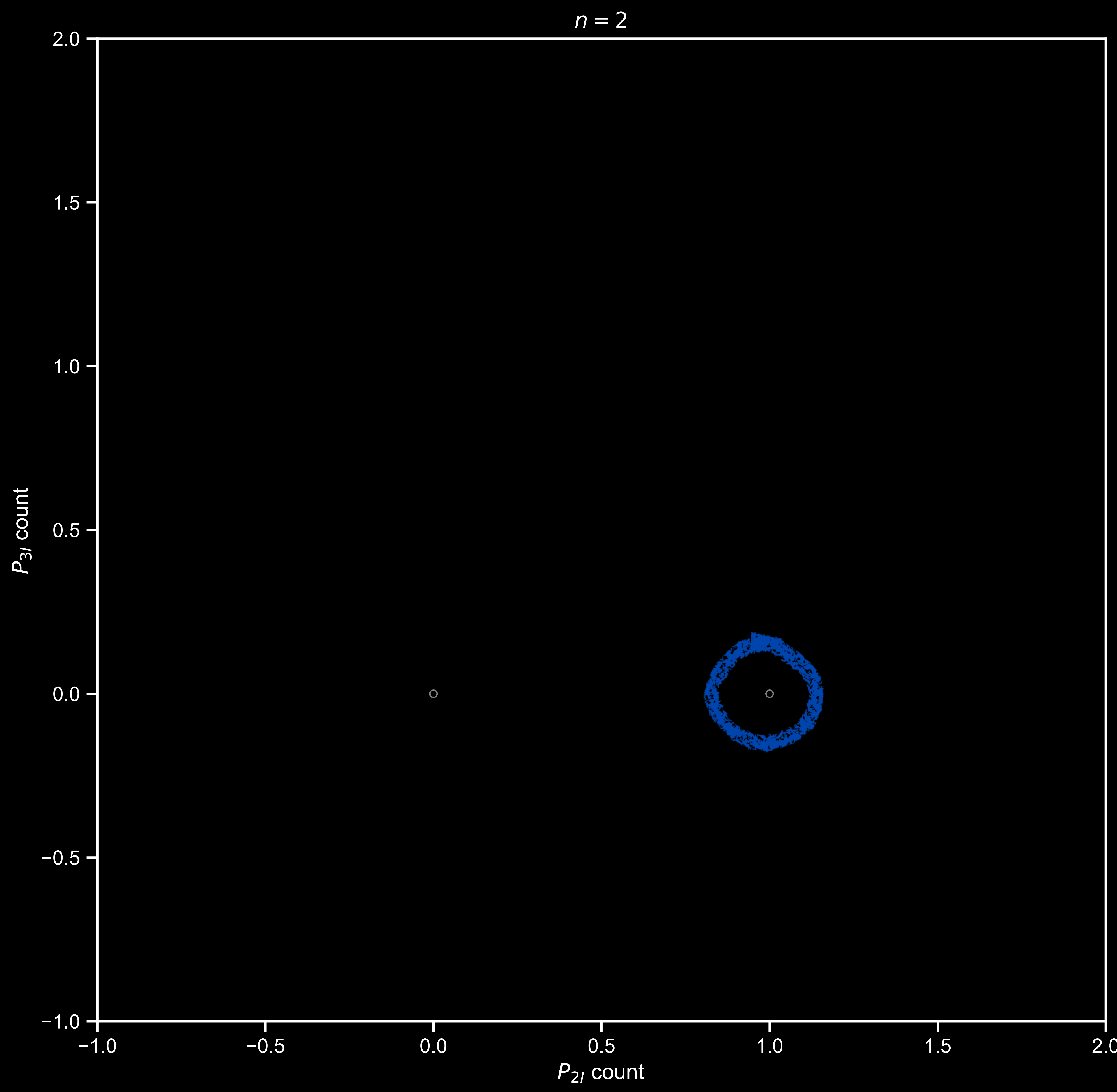
$P_{3I} :=$ 



$\mathscr{T}_1 = \{$  $\}$

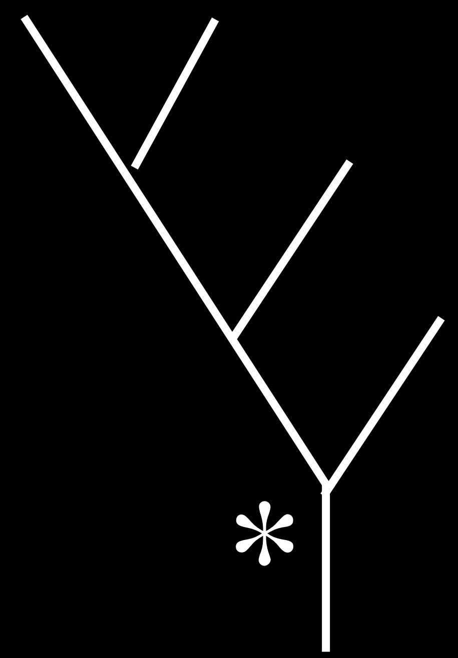
$P_{2I} :=$ 

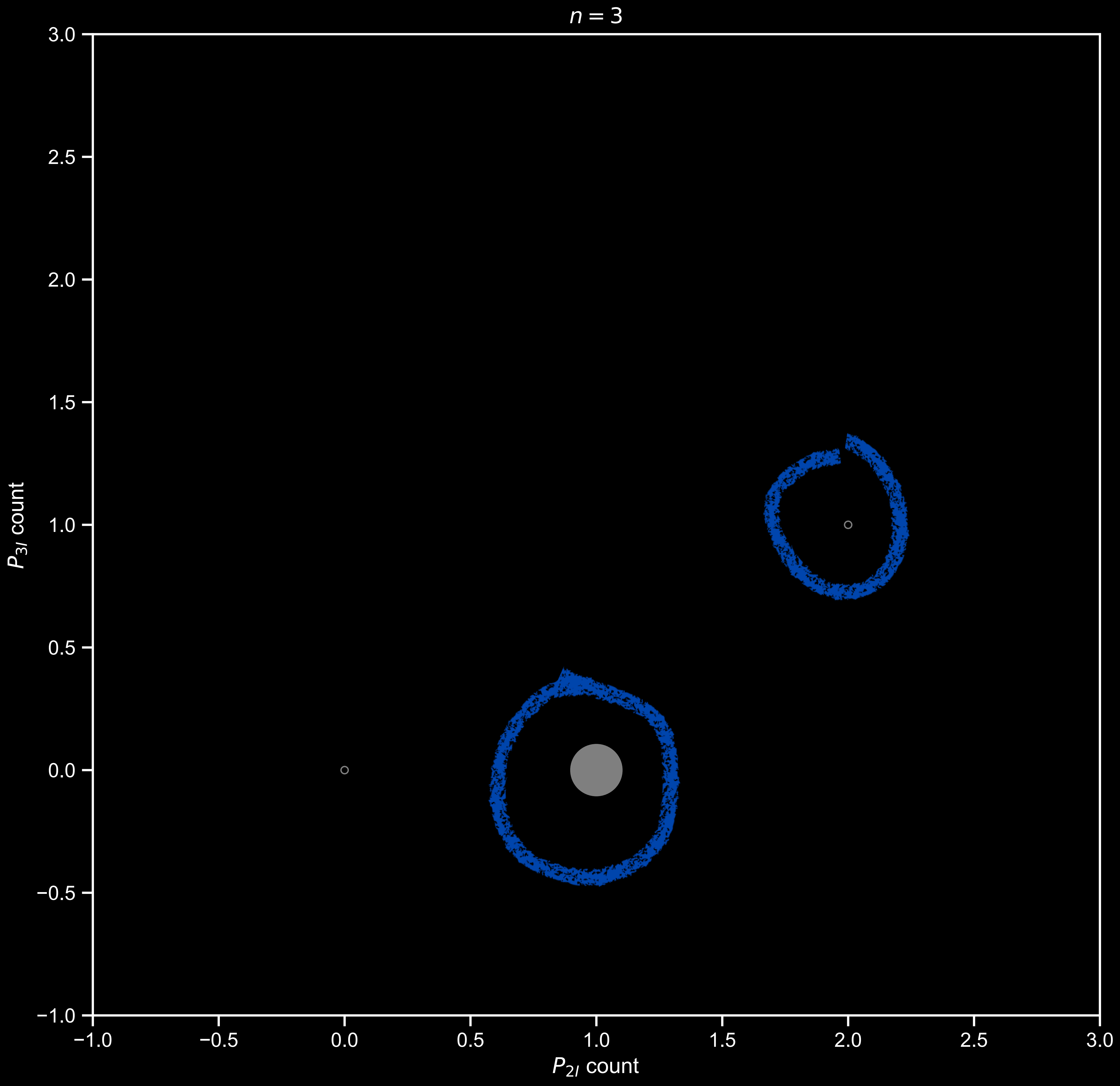
$P_{3I} :=$ 

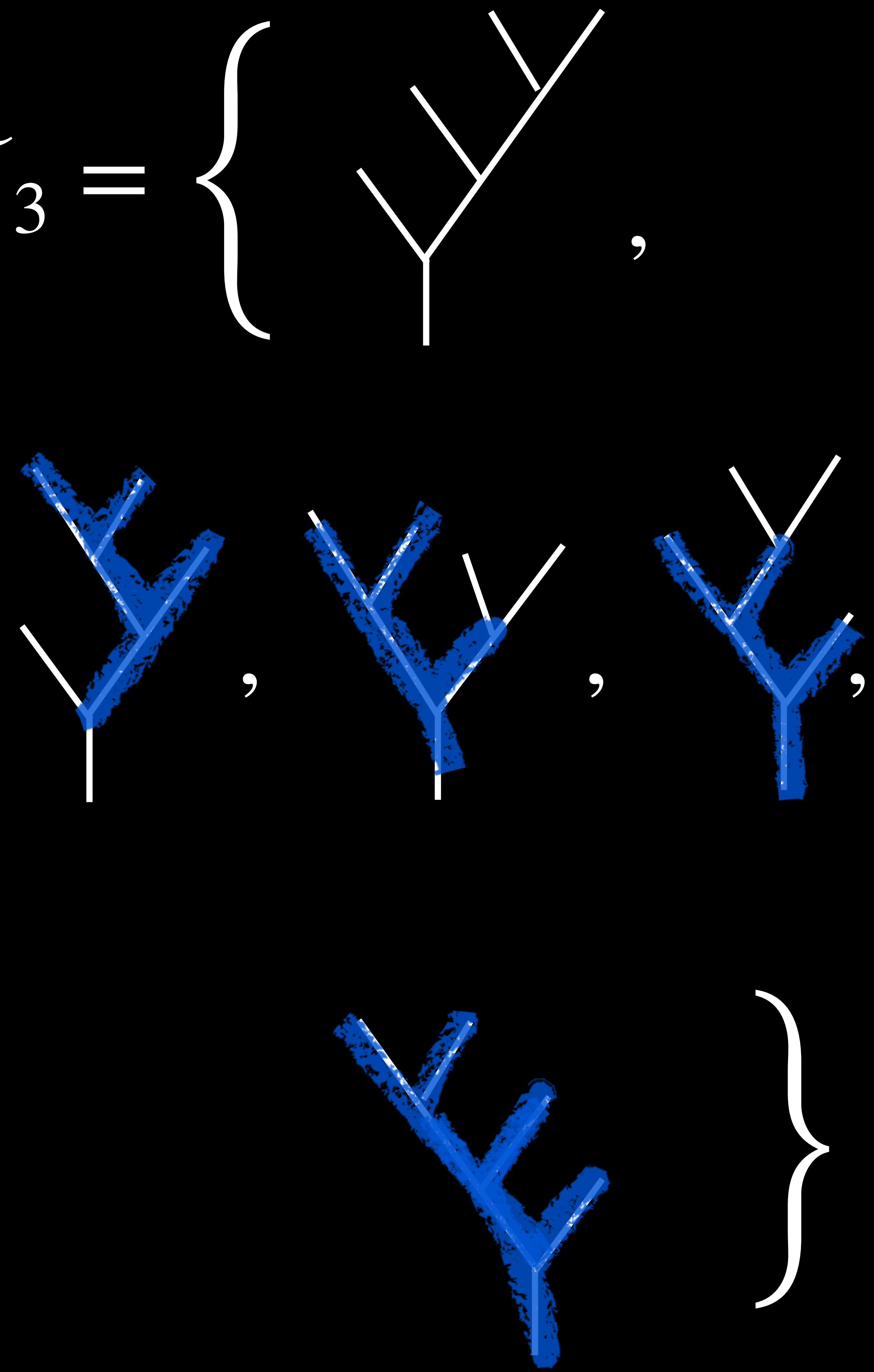


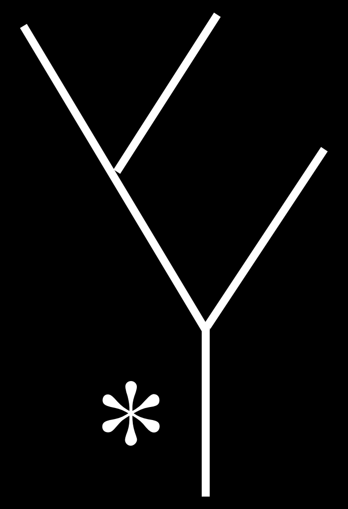
$\mathscr{T}_2 = \{$  ,  $\}$

$P_{2I} :=$ 

$P_{3I} :=$ 



$\mathscr{T}_3 = \{$  $\}$

$P_{2I} :=$ 

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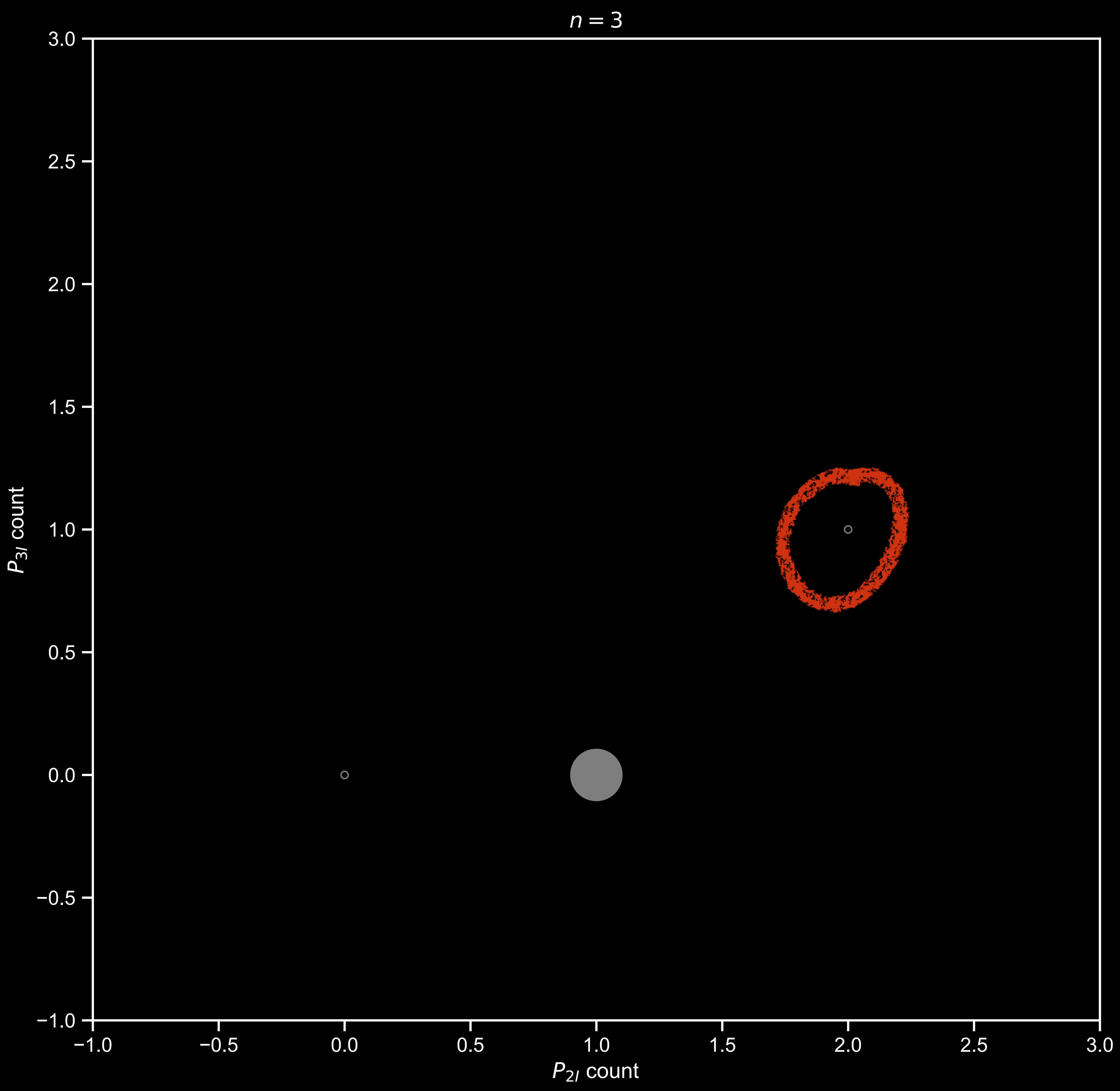
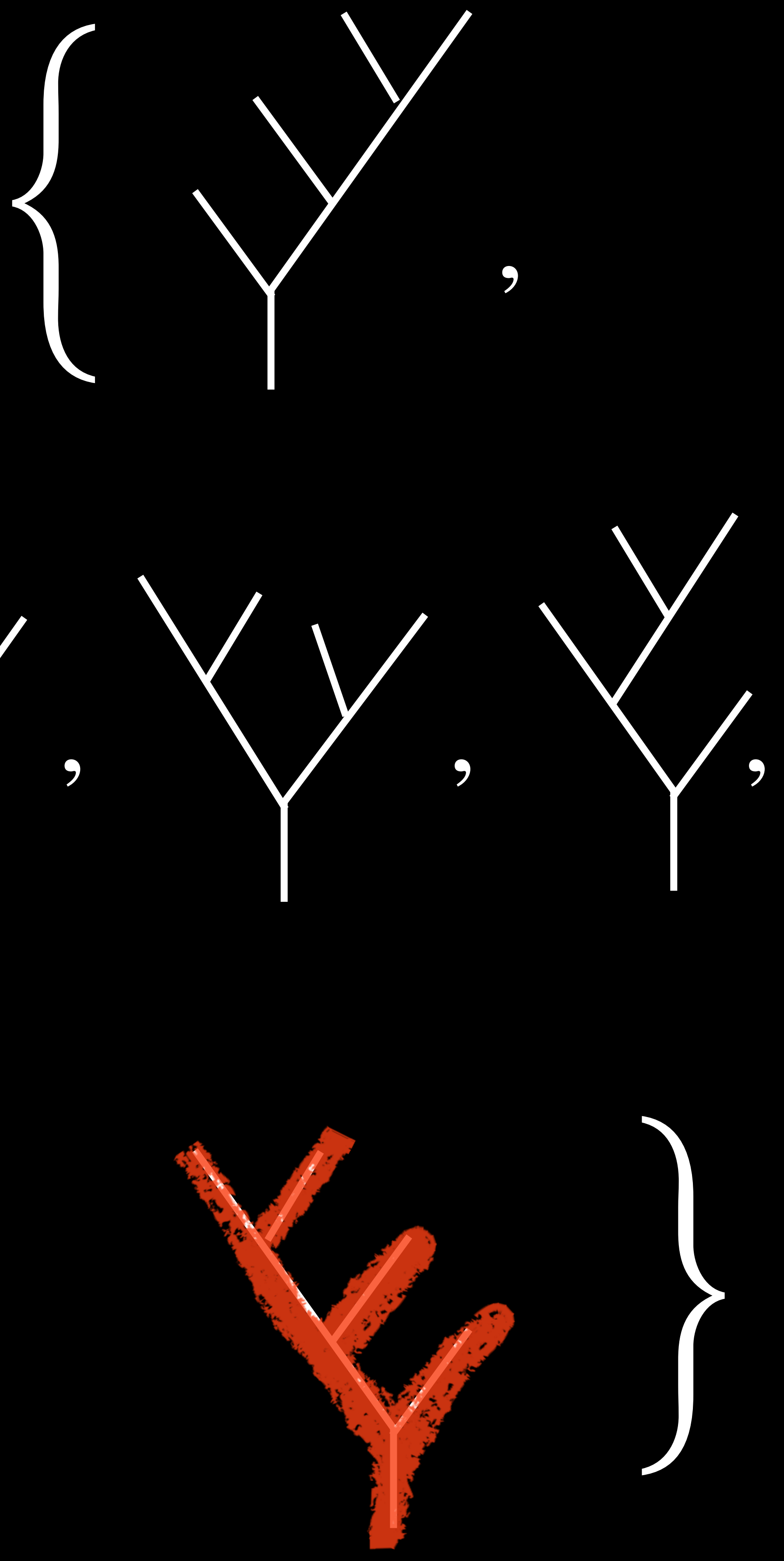
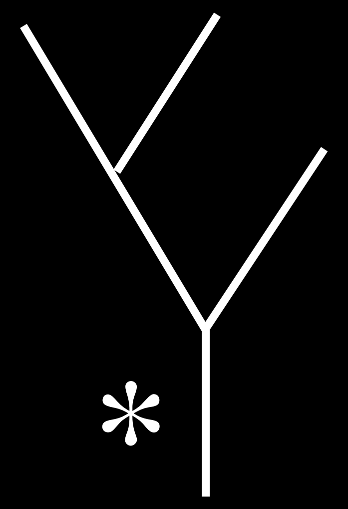
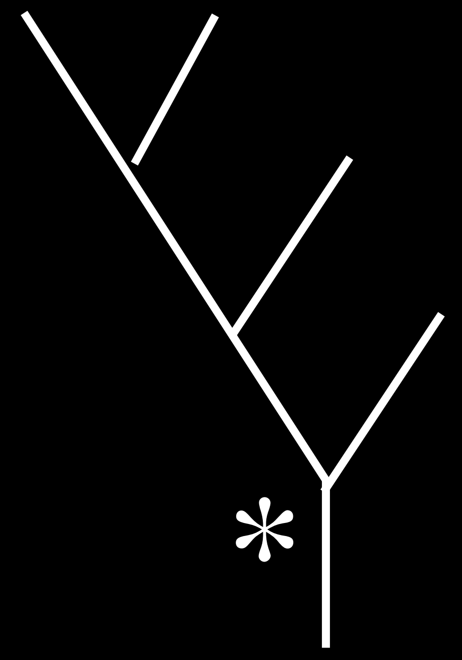
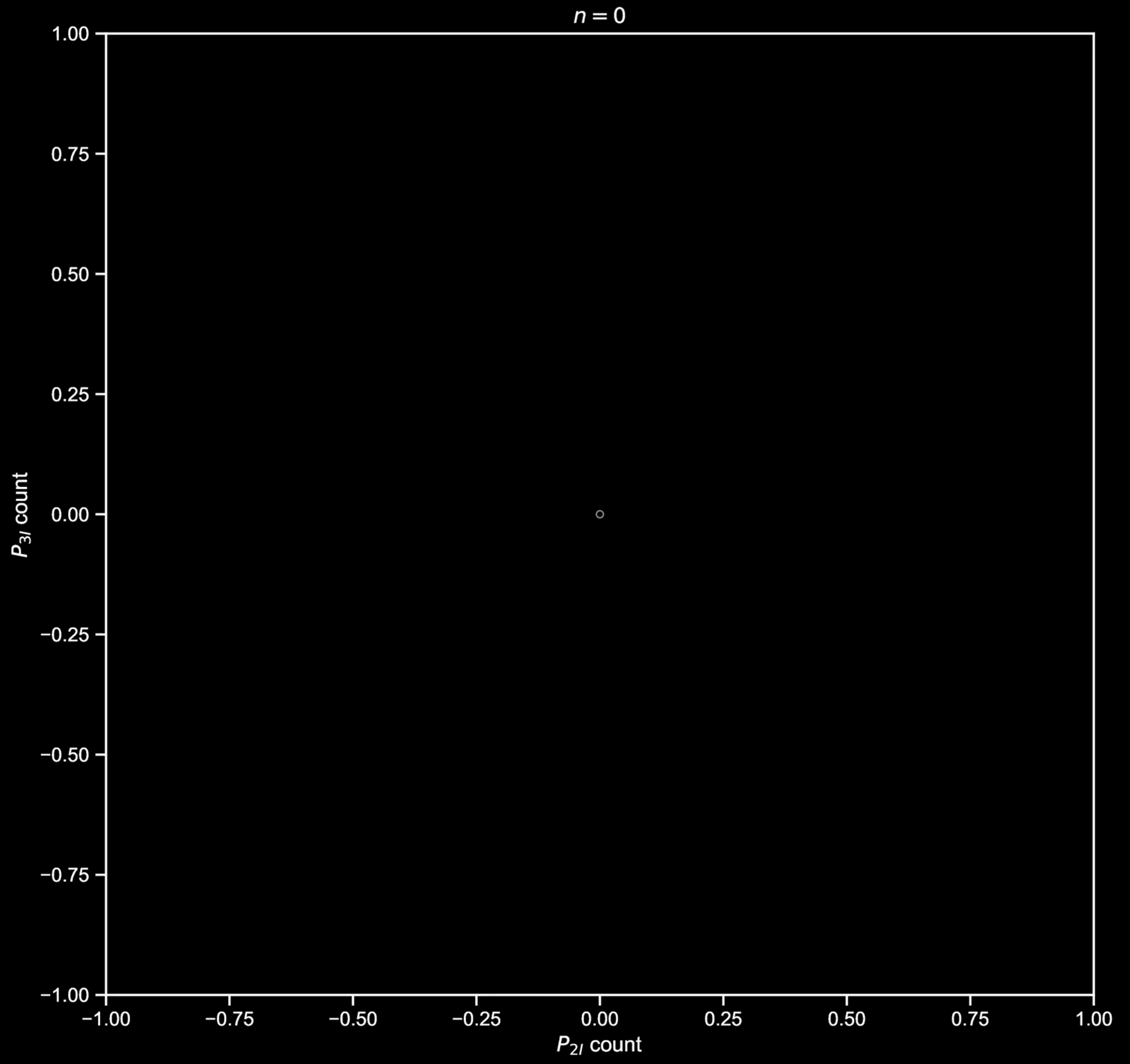


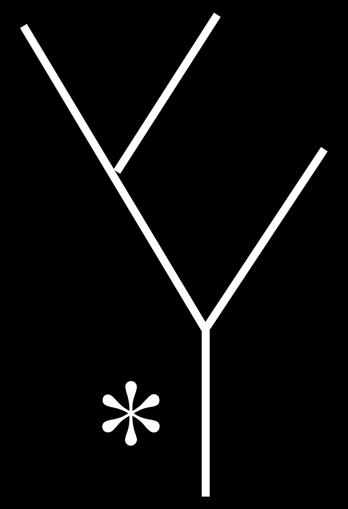
$\mathscr{T}_3 = \{$  $\}$

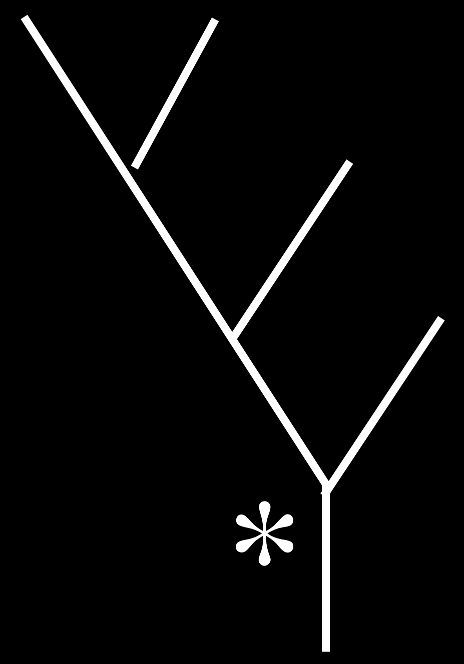
$P_{2I} :=$ 

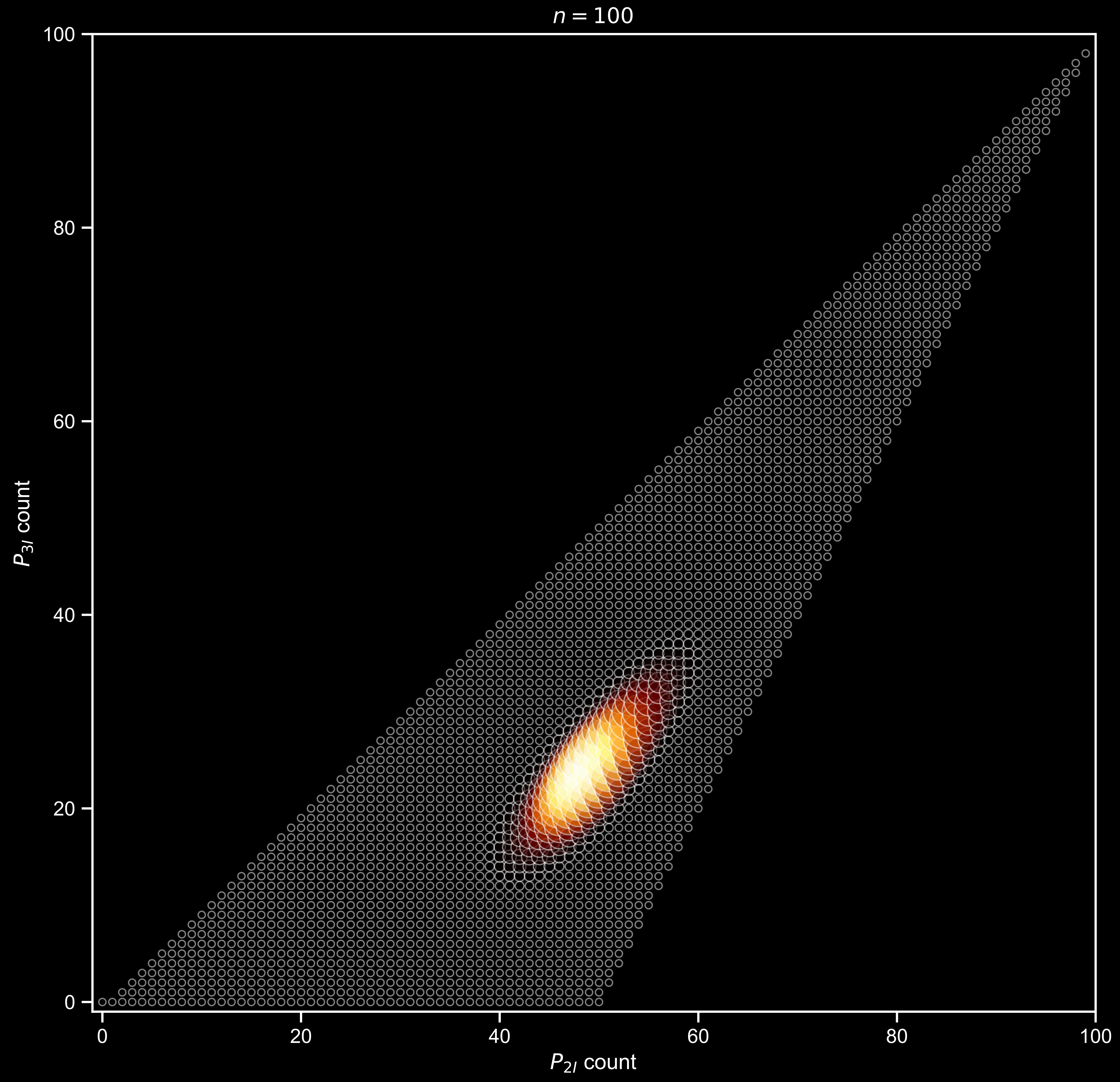
$P_{3I} :=$ 

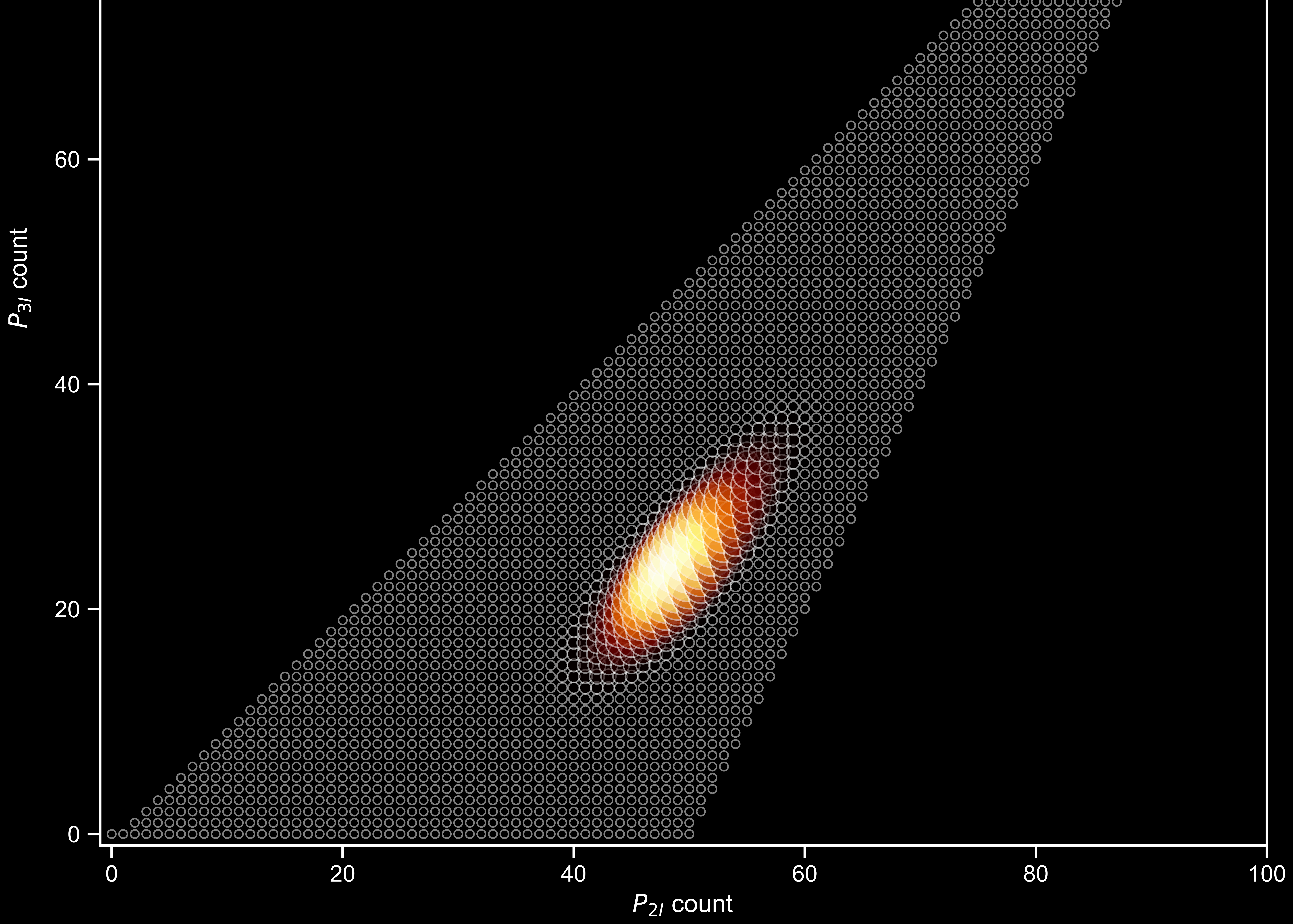


$$|\mathscr{T}_{100}| = \frac{200!}{100!} \approx 10^{217}$$

$P_{2I} :=$ 

$P_{3I} :=$ 





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P_{3l} count
P_{2l} count

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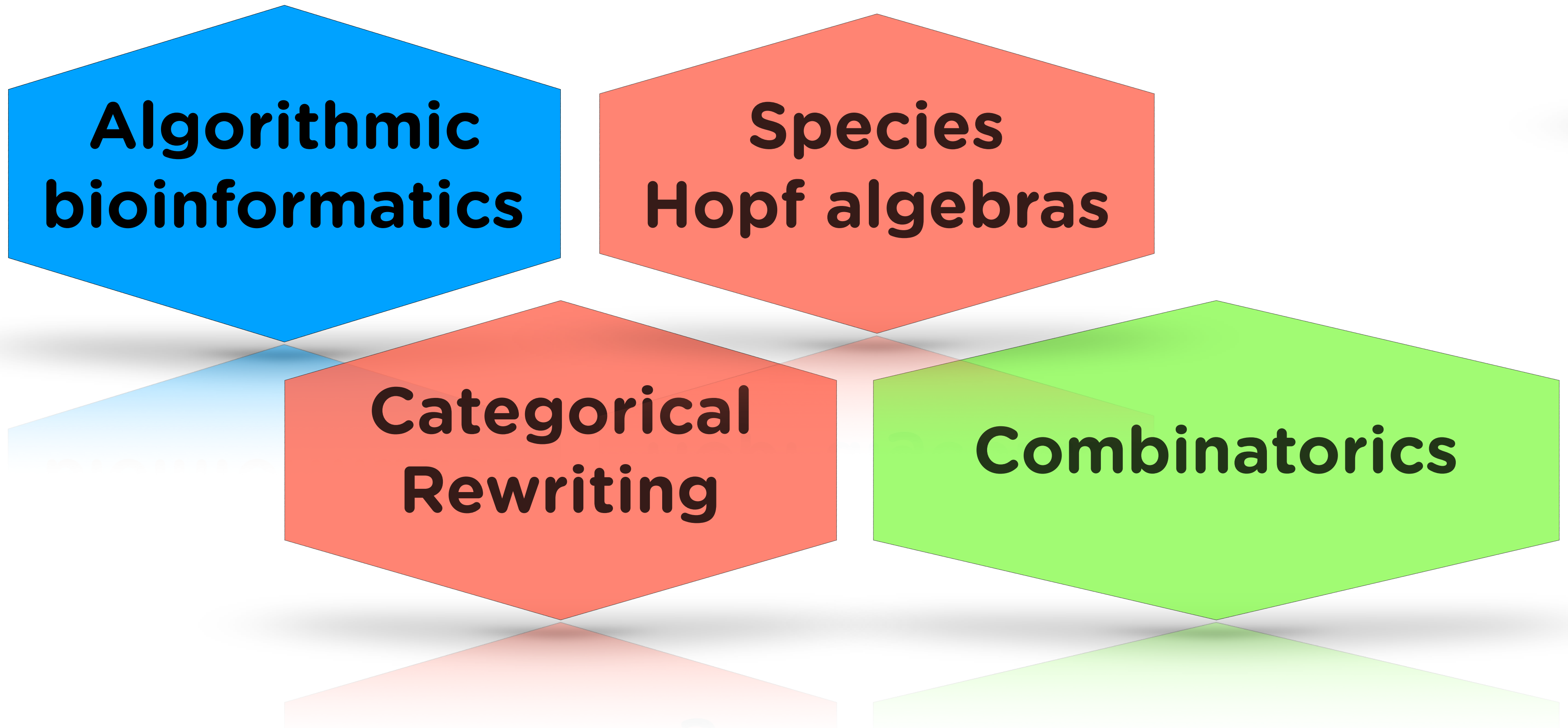
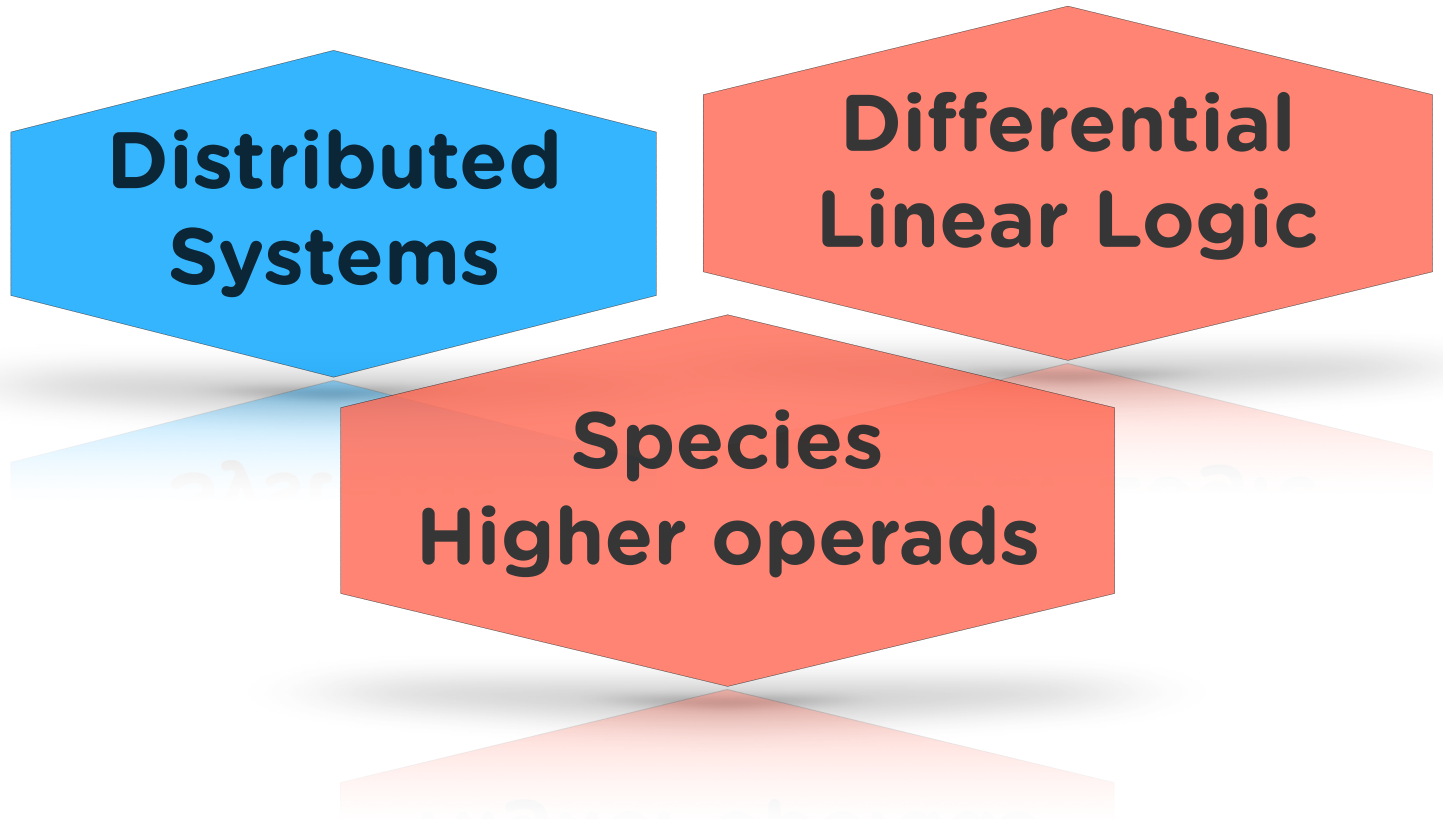
Perspectives

- Jean Krivine (IRIF)
- Vincent Danos (ENS Paris)
- Reiko Heckel (U Leicester)

- Joachim Kock (U Barcelona)





- Christine Tasson (IRIF)

- Pawel Sobocinski (U Tallinn)
- Paul-André Melliès (IRIF)
- Noam Zeilberger (LIX)

- Giuseppe Dattoli (ENEA Frascati)
- Gérard Duchamp (LIPN)
- Karol Penson (UPMC)

Thank you!

Further materials
(references, slides and videos):
nicolasbehr.com