

# On the (non)-freeness of operads considered as pre-Lie algebras

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# Outline

- 1 Pre-Lie algebras on operads
- 2 A bi-Pre-Lie algebra on planar trees

# Pre-Lie algebras on operads

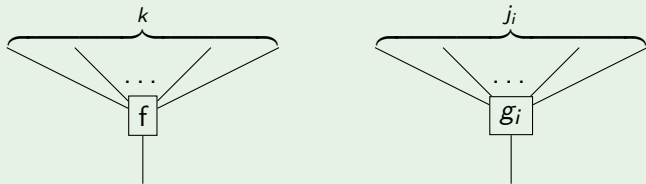
# Outline

- 1 Pre-Lie algebras on operads
  - Operads
  - Pre-Lie algebras
  - Pre-Lie product on operads
- 2 A bi-Pre-Lie algebra on planar trees

# What is an operad ? [May, Boardman-Vogt, 70s]

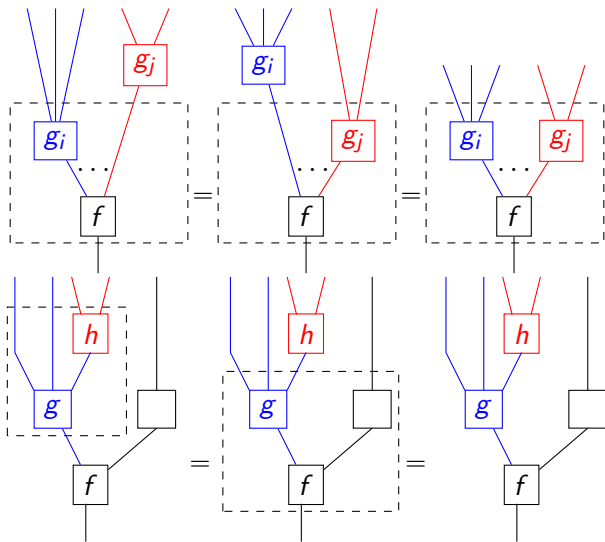
Let  $V$  be a vector space. Consider the space of multilinear endomorphisms:

$$(\text{End})(V)(n) = \text{Hom}(V^{\otimes n}, V)$$



endowed with the composition of endomorphisms.

# What is an operad ?



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## Definition (May, Boardman-Vogt, 70s)

A (symmetric) operad  $\mathcal{P}$  is a pair formed by:

- a family  $\{\mathcal{P}(n)\}_{n \geq 1}$  of finite dimensional  $\mathfrak{S}_n$ -modules (=vector spaces endowed with an action of the symmetric group),

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- associative partial composition maps, compatible with the action of the symmetric group, given by:

$$o_i : \mathcal{P}(k) \otimes \mathcal{P}(l) \rightarrow \mathcal{P}(k + l - 1)$$



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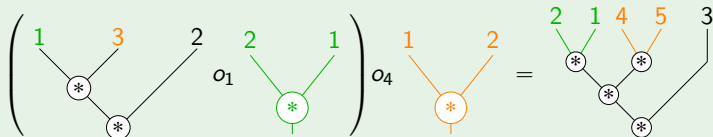
### What is an operad for ?

It encodes **products** in an algebra.

## Examples of operads

### First example: Magmatic algebras and operads

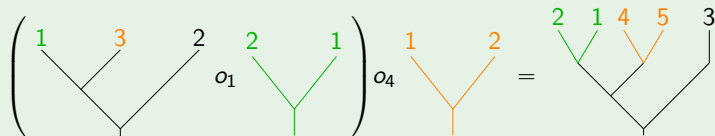
- $\mathcal{P}(n) = \text{PBT}_n$  with  $\text{PBT}_n$  the vector space of planar binary trees on  $n$  leaves labelled by  $\{1, \dots, n\}$  and  $\gamma$  the grafting on leaves (magmatic operad  $\text{Mag}$ )



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## Example :

- $\mathcal{P}(n) = \mathbb{K}\{ \begin{array}{c} \sigma(1) \quad \dots \quad \sigma(n) \\ \diagdown \quad \dots \quad / \\ \dots \\ \diagup \end{array} \mid \sigma \in \mathfrak{S}_n \}$  with composition given by concatenation

(Associative operad List :  $\begin{array}{c} x \ y \ z \\ \diagdown \quad / \\ \diagup \end{array} = \begin{array}{c} x \ y \ z \\ \diagdown \quad / \\ \diagup \end{array} =: \begin{array}{c} x \ y \ z \\ \diagdown \quad / \\ \diagup \end{array}$  )

$$\left( \begin{array}{c} 3 \quad 2 \quad 1 \\ \diagdown \quad | \quad / \\ \diagup \end{array} \sigma_1 \begin{array}{c} 1 \quad 2 \\ \diagdown \quad / \\ \diagup \end{array} \right) \sigma_3 \begin{array}{c} 2 \quad 1 \\ \diagdown \quad / \\ \diagup \end{array} = \begin{array}{c} 5 \quad 4 \quad 3 \quad 1 \quad 2 \\ | \quad \diagdown \quad / \quad \diagdown \quad / \\ \diagup \end{array} = \begin{array}{c} 5 \quad 4 \quad 3 \quad 1 \quad 2 \\ \diagdown \quad | \quad / \quad \diagdown \quad / \\ \diagup \end{array}$$

Example :

•  $\mathcal{P}(n) = \mathbb{K}$ .

with  $\circ_i$  :

(Commutative operad  $\text{Set}^+$   $\vee = \vee =:$   $\vee$  and  $\begin{matrix} 1 & 2 \\ \vee & \\ & \vee \\ & 1 & 2 \end{matrix} = \begin{matrix} 2 & 1 \\ \vee & \\ & \vee \\ & 2 & 1 \end{matrix}$ )

## Presentation of an operad

For any operad  $\mathcal{P}$ ,

$$\mathcal{P}(n) = \text{PT}_n^G / (\text{relations}),$$

where  $\text{PT}_n^G$  is the vector space of planar trees with inner nodes decorated by the generating operations of the operad.

Especially, if  $G = \begin{array}{c} \diagup \quad \diagdown \\ \quad \vee \end{array}$ ,

$$\mathcal{P}(n) = \text{PBT}_n / (\text{relations}).$$

# Algebra over an operad

## Definition

A  $\mathcal{P}$ -algebra is a vector space  $V$  endowed with

$$\mu_n : \mathcal{P}(n) \otimes_{\mathfrak{S}_n} V^{\otimes n} \rightarrow V.$$

## Definition

The  $\mathcal{P}$ -free algebra over  $V$  is

$$\mathcal{P}(V) = \bigoplus_{n \geq 1} \mathcal{P}(n) \otimes_{\mathfrak{S}_n} V^{\otimes n}.$$

## Last example: Pre-Lie operad [Chapoton-Livernet, 2001]

- $\mathcal{P}(n) = \text{RT}_n$  the vector spaces of rooted trees with  $\gamma$  the composition of trees inside nodes (Pre-Lie operad  $\text{PreLie}$  :

$$\left( \begin{array}{c} a & b & c \\ \diagdown & / & / \\ & & \diagdown \end{array} - \begin{array}{c} a & b & c \\ \diagdown & / & / \\ & & \diagdown \end{array} = \begin{array}{c} a & c & b \\ \diagdown & / & / \\ & & \diagdown \end{array} - \begin{array}{c} a & c & b \\ \diagdown & / & / \\ & & \diagdown \end{array} \right)$$

$$\left( \begin{array}{c} 2 & 1 \\ \diagdown & / \\ & \diagdown \end{array} \circ_2 \begin{array}{c} 1 & 2 \\ \diagdown & / \\ & \diagdown \end{array} \right) \circ_1 \begin{array}{c} 2 & 1 & 3 \\ \diagdown & / & / \\ & & \diagdown \end{array} = \begin{array}{c} 4 & 5 & 2 & 1 & 3 \\ \diagdown & / & / & / & / \\ & & \diagdown & / & / \\ & & & & \diagdown \end{array}$$



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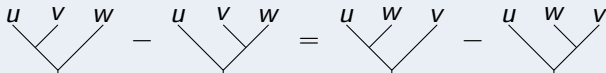
$$\left( \begin{array}{c} \textcircled{1} \\ | \\ \textcircled{2} \end{array} \sigma_2 \begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} \end{array} \right) \sigma_1 \begin{array}{c} \textcircled{3} \\ | \\ \textcircled{1} \\ | \\ \textcircled{2} \end{array} = \begin{array}{c} \textcircled{5} \\ | \\ \textcircled{4} \end{array} = \begin{array}{c} \textcircled{3} \\ | \\ \textcircled{1} \\ | \\ \textcircled{2} \end{array} = \begin{array}{c} \textcircled{5} \\ | \\ \textcircled{4} \end{array} + \begin{array}{c} \textcircled{3} \\ | \\ \textcircled{1} \\ | \\ \textcircled{2} \\ | \\ \textcircled{5} \\ | \\ \textcircled{4} \end{array} + \begin{array}{c} \textcircled{3} \\ | \\ \textcircled{1} \\ | \\ \textcircled{2} \\ | \\ \textcircled{5} \\ | \\ \textcircled{4} \end{array}$$

# Pre-Lie algebras

Definition (Gerstenhaber, 1963 ; Vinberg, 1963 ; Matsushima, 1968)

A **pre-Lie algebra** is a vector space  $V$  endowed with a product  $\leftarrow$  satisfying for any  $u, v$  and  $w$  in  $V$ :

$$(u \leftarrow v) \leftarrow w - u \leftarrow (v \leftarrow w) = (u \leftarrow w) \leftarrow v - u \leftarrow (w \leftarrow v)$$

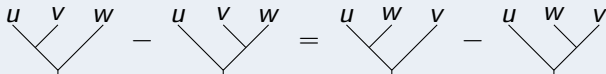


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Example :

- Hypertrees
- Fat trees
- Algebra of derivations

$$P(x_1, \dots, x_n) \partial_{x_i} \leftarrow Q(x_1, \dots, x_n) \partial_{x_j} = P(x_1, \dots, x_n) (\partial_{x_i} Q) (x_1, \dots, x_n) \partial_{x_j}$$

## Pre-Lie products on operads

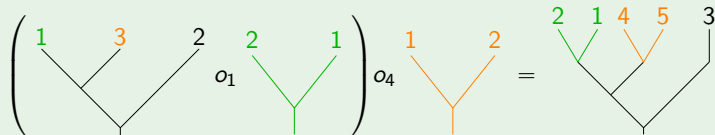
Given an operad  $\mathcal{P}$ ,  
pre-Lie product  $\leftarrow$  on  $\bigoplus_{n \geq 2} \mathcal{P}(n)$ , defined on any  $\mu \in \mathcal{P}(n)$ ,  $\nu \in \mathcal{P}(m)$  by:

$$\mu \leftarrow \nu = \sum_{i=1}^n \mu \circ_i \nu.$$

# Examples of operads

## First example:

- $\mathcal{P}(n) = \text{PBT}_n$  with  $\text{PBT}_n$  the vector space of planar binary trees on  $n$  leaves labelled by  $\{1, \dots, n\}$  and  $\gamma$  the grafting on leaves (magmatic operad  $\text{Mag}$ )



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$$\leftarrow = +$$

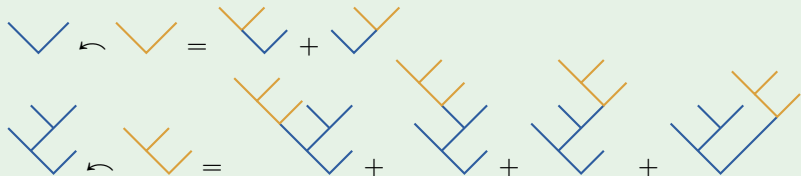
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Example on Mag operad :





## Main problem

Are operads free as pre-Lie algebras ?

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First answer :

No !

## Relations on operads

### Definition

The brace products are defined on  $t$  of arity (=nb of inputs of the box)  $l$  by:

$$\begin{aligned}
 t \leftarrow (s_1, \dots, s_n) &= \sum_{m_1, \dots, m_n} (\dots ((t \circ_{m_1} s_1) \circ_{m_2} s_2) \dots) \circ_{m_n} s_n, \\
 &= (t \leftarrow (s_1, \dots, s_{n-1})) \leftarrow s_n \\
 &- \sum_{i=1}^{n-1} t \leftarrow (\dots, s_{i-1}, s_i \leftarrow s_n, s_{i+1}, \dots),
 \end{aligned}$$

where the sum runs over any  $n$ -tuples  $(m_1, \dots, m_n)$  of elements in  $\{1, \dots, l\}$ .

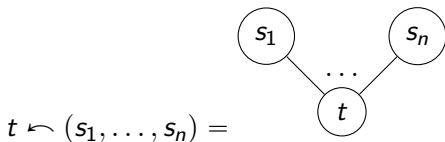
## Relations on operads

### Definition

The brace products are defined on  $t$  of arity ( $=$ nb of inputs of the box)  $l$  by:

$$t \leftarrow (s_1, \dots, s_n) = (t \leftarrow (s_1, \dots, s_{n-1})) \leftarrow s_n - \sum_{i=1}^{n-1} t \leftarrow (\dots, s_{i-1}, s_i \leftarrow s_n, s_{i+1}, \dots),$$

where the sum runs over any  $n$ -tuples  $m_1 > \dots > m_n$  of elements in  $\{1, \dots, l\}$ .



### Example on Mag operad :

$$v \leftarrow (v, v) = 2 \quad \checkmark \checkmark$$

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$$v \leftarrow (v, v) = 2 \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$$

$$\begin{aligned} v \leftarrow (v, v, v) &= (v \leftarrow (v, v)) \leftarrow v - v \leftarrow (v \leftarrow v, v) \\ &\quad - v \leftarrow (v, v \leftarrow v) \\ &= \left( 2 \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right) \leftarrow v - v \leftarrow \left( \begin{array}{c} \diagdown \\ \diagup \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array}, v \right) \\ &\quad - v \leftarrow \left( v, \begin{array}{c} \diagdown \\ \diagup \end{array} + \begin{array}{c} \diagup \\ \diagdown \end{array} \right) \\ &= 0 \end{aligned}$$

## Proposition (Burgunder - D.O. - Manchon)

For any operad  $\mathcal{P}$ , if  $t \in \mathcal{P}(i)$ ,  $t \leftarrow (s_1, \dots, s_n) = 0$  for any  $s_i$  and  $n > i$ .  
(Hence the pre-Lie algebra  $\bigoplus_{n \geq 2} \mathcal{P}(n)$  is not free).

## New problem

Are brace relations the only relations of these pre-Lie algebras ?

# A bi-Pre-Lie algebra on planar trees



# Outline

- 1 Pre-Lie algebras on operads
- 2 A bi-Pre-Lie algebra on planar trees

## [Recall] Presentation of an operad

For any operad  $\mathcal{P}$ ,

$$\mathcal{P}(n) = \text{PT}_n^G / (\text{relations}),$$

where  $\text{PT}_n^G$  is the vector space of planar trees with inner nodes decorated by the generating operations of the operad.

Especially, if  $G = \begin{array}{c} \diagup \quad \diagdown \\ \text{Y} \end{array}$ ,

$$\mathcal{P}(n) = \text{PBT}_n / (\text{relations}).$$

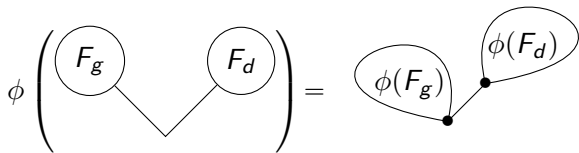
## Bijection between planar binary trees and planar trees : Knuth's rotation correspondence

Planar trees on  $n$  nodes and planar binary trees on  $n$  leaves are both counted by Catalan numbers and linked by the following recursively defined bijection  $\phi$ :

$$\phi(v) = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$

$$\phi \left( \begin{array}{c} \text{---} \left( \begin{array}{c} \text{---} \text{---} \end{array} \right) \text{---} \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \text{---} \\ \text{---} \end{array}$$

The diagram illustrates the mapping of a planar tree to a planar binary tree. On the left, a planar tree with root node  $v$  is shown, where the left child is a subtree  $F_g$  and the right child is a subtree  $F_d$ . This tree is enclosed in large parentheses with a  $\phi$  symbol to the left. An equals sign follows, leading to the corresponding planar binary tree on the right. In this binary tree, the root node has a left child (represented by a bubble labeled  $\phi(F_g)$ ) and a right child (represented by a bubble labeled  $\phi(F_d)$ ).



Examples :

# Bijection between planar binary trees and planar trees : Knuth's rotation correspondance

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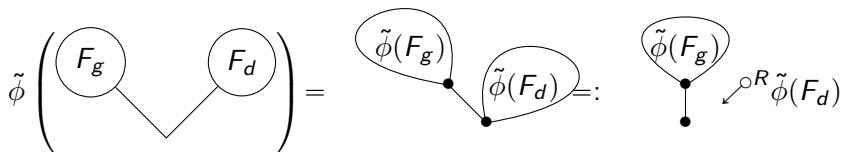
$$\phi \left( \begin{array}{c} \text{---} \left( \begin{array}{c} \text{---} \text{---} \end{array} \right) \text{---} \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \text{---} \\ \text{---} \end{array} =: \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}$$

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## Bijection between planar binary trees and planar trees : Knuth's rotation correspondance

Planar trees on  $n$  nodes and planar binary trees on  $n$  leaves are both counted by Catalan numbers and linked by the following recursively defined bijection  $\tilde{\phi}$ :

$$\tilde{\phi}(v) = \begin{array}{c} \bullet \\ | \\ \bullet \end{array}$$



Each bijection gives a different pre-Lie product on planar trees :

- $t \leftarrow^L s$  which is the sum over all the way to graft the root of  $s$  on the left of the root of  $t$
- and  $t \leftarrow^R s$  which is the sum over all the way to graft the root of  $s$  on the right of the root of  $t$ .

**New problem :**

What happens if we consider at the same time both products ?

# Bi-pre-Lie algebras

## Definition

A **bi-pre-Lie algebra** is a vector space  $V$  endowed with two pre-Lie products  $\leftarrow^L$  and  $\leftarrow^R$  satisfying:

$$\begin{aligned} (u \leftarrow^L v) \leftarrow^L w - u \leftarrow^L (v \leftarrow^L w) &= (u \leftarrow^L w) \leftarrow^L v - u \leftarrow^L (w \leftarrow^L v) \\ (u \leftarrow^R v) \leftarrow^R w - u \leftarrow^R (v \leftarrow^R w) &= (u \leftarrow^R w) \leftarrow^R v - u \leftarrow^R (w \leftarrow^R v) \\ (u \leftarrow^L v) \leftarrow^R w - u \leftarrow^L (v \leftarrow^R w) &= (u \leftarrow^R w) \leftarrow^L v - u \leftarrow^R (w \leftarrow^L v) \end{aligned}$$



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### Proposition (Burgunder-D.O.-Manchon)

*The vector space of planar trees  $PT$ , with pre-Lie products  $\leftarrow^L$  and  $\leftarrow^R$ , is a **bi-pre-Lie algebra**.*

## New problem :



What happens if we consider at the same time both products ?

## Theorem (Burgunder-D.O.-Manchon)

The bi-pre-Lie algebra  $(PT, \leftarrow^L, \leftarrow^R)$  is generated by  and 

## Sketch of the proof of the theorem

### Theorem (Burgunder-D.O.-Manchon)

The bi-pre-Lie algebra  $(PT, \leftarrow^L, \leftarrow^R)$  is generated by  and .

### Proof.

By induction on the following well-founded partial order on trees :  $S \leq T$  if

- either  $|S| < |T|$ ,
- or  $|S| = |T|$  and  $\text{height}(S) > \text{height}(T)$ ,
- or  $(|S|, \text{height}(S)) = (|T|, \text{height}(T))$  and  $\sum_{v \in V(S)} \text{height}(v) < \sum_{v' \in V(T)} \text{height}(v')$ ,
- or  $S = T$ .



## Proof.

- Initialization :  $E_2$
- Induction step :  
Goal : rewrite  $t$  as a product of smaller terms  
Two cases :  $t$  planted or not



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Goal : rewrite  $t$  as a product of smaller terms  
Two cases :  $t$  planted or not



Thank you for your attention !