On the (non)-freeness of operads considered as pre-Lie algebras

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Outline

Pre-Lie algebras on operads

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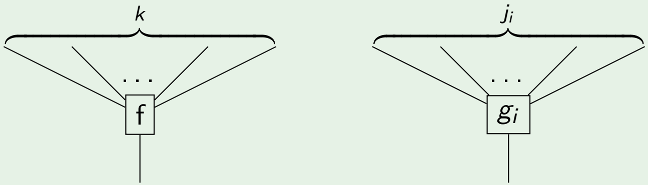
1. Pre-Lie algebras on operads
   - Operads
   - Pre-Lie algebras
   - Pre-Lie product on operads
2. A bi-Pre-Lie algebra on planar trees

What is an operad ? [May, Boardman-Vogt, 70s]

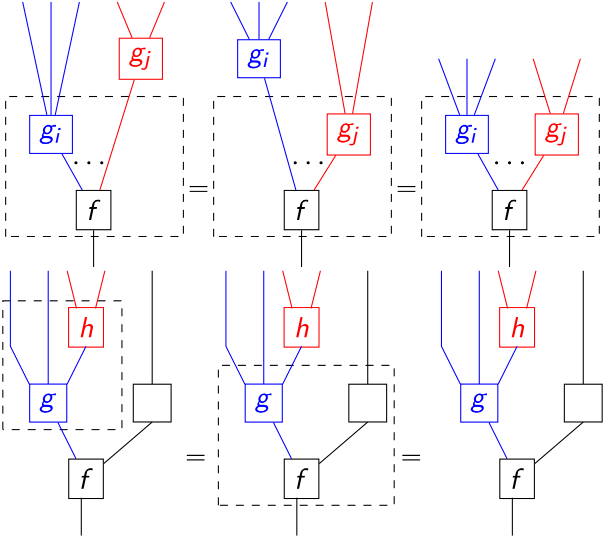
Let V be a vector space. Consider the space of multilinear endomorphisms:

$$(\mathrm{End})(V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$$



endowed with the composition of endomorphisms.

What is an operad ?



What is an operad ?

Definition (May, Boardman-Vogt, 70s)

A (symmetric) operad $\mathcal{P}$ is a pair formed by:

- a family $\{\mathcal{P}(n)\}_{n\geqslant 1}$ of finite dimensional $\mathfrak{S}_n$-modules (=vector spaces endowed with an action of the symmetric group),

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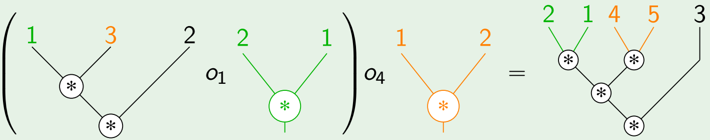
What is an operad for ?

It encodes products in an algebra.

Examples of operads

First example: Magmatic algebras and operads

- $\mathcal{P}(n) = \mathrm{PBT}_n$ with PBT_n the vector space of planar binary trees on n leaves labelled by $\{1, \ldots, n\}$ and γ the grafting on leaves (magmatic operad Mag)



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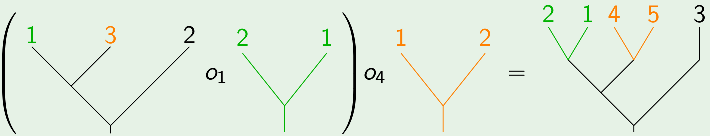
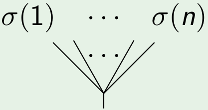
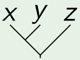
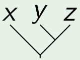


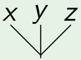
Example :

- $\mathcal{P}(n) = \mathbb{K}.\{$ 

$|\sigma \in \mathfrak{S}_n\}$ with composition given by concatenation

(Associative operad List : 

$=$ 

$=:$ 

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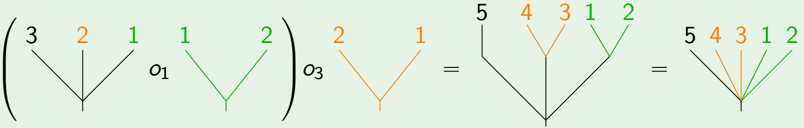

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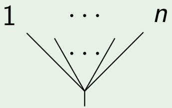
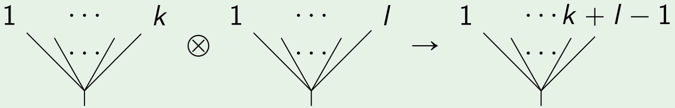
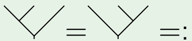
- $\mathcal{P}(n) = \mathbb{K}.$ 

  with o_i : 

  (Commutative operad Set$^+$  and 

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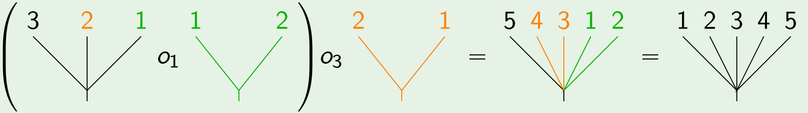


Presentation of an operad

For any operad $\mathcal{P}$,

$$\mathcal{P}(n) = \mathrm{PT}_n^G \,/(\text{relations}),$$

where PT_n^G is the vector space of planar trees with inner nodes decorated by the generating operations of the operad.

Especially, if $G =$ ,

$$\mathcal{P}(n) = \mathrm{PBT}_n \,/(\text{relations}).$$

Algebra over an operad

Definition

A $\mathcal{P}$-algebra is a vector space V endowed with

$$\mu_n : \mathcal{P}(n) \otimes_{\mathfrak{S}_n} V^{\otimes n} \to V.$$

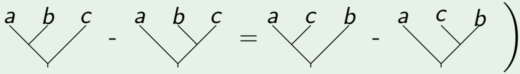
Definition

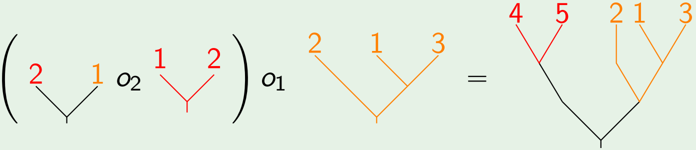
The $\mathcal{P}$-free algebra over V is

$$\mathcal{P}(V) = \oplus_{n \geqslant 1} \mathcal{P}(n) \otimes_{\mathfrak{S}_n} V^{\otimes n}.$$

Last example: Pre-Lie operad [Chapoton-Livernet, 2001]

- $\mathcal{P}(n) = \mathrm{RT}_n$ the vector spaces of rooted trees with γ the composition of trees inside nodes (Pre-Lie operad PreLie :





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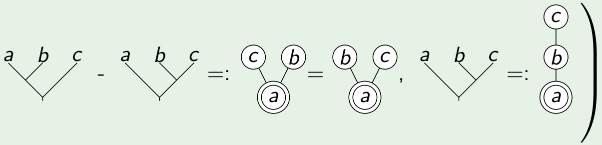
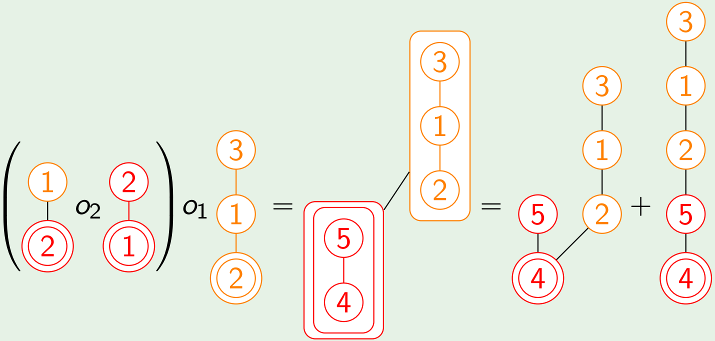


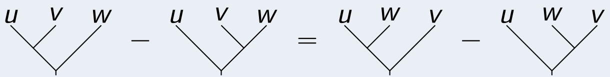


Pre-Lie algebras

Definition (Gerstenhaber, 1963 ; Vinberg, 1963 ; Matsushima, 1968)

A pre-Lie algebra is a vector space V endowed with a product $\curvearrowleft$ satisfying for any u, v and w in V:

$$(u \curvearrowleft v) \curvearrowleft w - u \curvearrowleft (v \curvearrowleft w) = (u \curvearrowleft w) \curvearrowleft v - u \curvearrowleft (w \curvearrowleft v)$$



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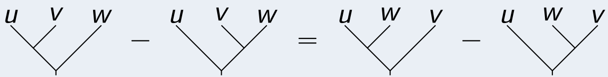

Example :

- Hypertrees
- Fat trees
- Algebra of derivations

$$P(x_1, \ldots, x_n)\partial_{x_i} \leftharpoonup Q(x_1, \ldots, x_n)\partial_{x_j} = P(x_1, \ldots, x_n)\,(\partial_{x_i} Q)\,(x_1, \ldots, x_n)\partial_{x_j}$$

Pre-Lie products on operads

Given an operad $\mathcal{P}$,
pre-Lie product $\curvearrowleft$ on $\oplus_{n\geqslant 2}\mathcal{P}(n)$, defined on any $\mu \in \mathcal{P}(n)$, $\nu \in \mathcal{P}(m)$ by:

$$\mu \curvearrowleft \nu = \sum_{i=1}^{n} \mu \circ_i \nu.$$

Examples of operads

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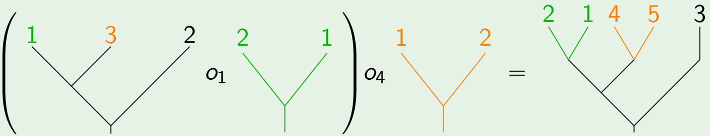

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Given an operad $\mathcal{P}$,
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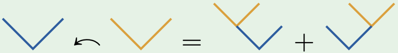

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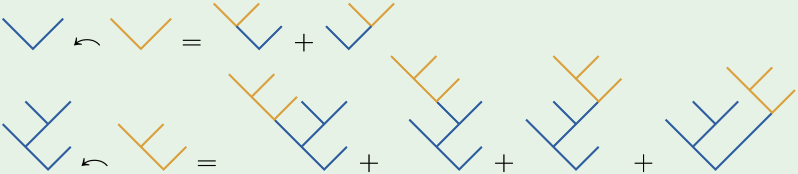

Main problem

Are operads free as pre-Lie algebras ?

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First answer :

No !

Relations on operads

Definition

The brace products are defined on t of arity (=nb of inputs of the box) l by:

$$\begin{aligned} t \leftharpoondown (s_1, \ldots, s_n) &= \sum_{m_1,\ldots,m_n} (\ldots((t \circ_{m_1} s_1) \circ_{m_2} s_2) \ldots) \circ_{m_n} s_n, \\ &= (t \leftharpoondown (s_1, \ldots, s_{n-1})) \leftharpoondown s_n \\ &- \sum_{i=1}^{n-1} t \leftharpoondown (\ldots, s_{i-1}, s_i \leftharpoondown s_n, s_{i+1}, \ldots), \end{aligned}$$

where the sum runs over any n-tuples $(m_1, \ldots, m_n)$ of elements in $\{1, \ldots, l\}$.

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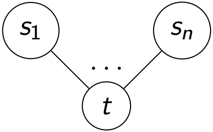
$t \curvearrowleft (s_1, \ldots, s_n) =$ 

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$$- \vee \curvearrowleft (\vee, \vee \curvearrowleft \vee)$$

$= \Big(2$  $\Big) \curvearrowleft \vee - \vee \curvearrowleft ($  $+$  $, \vee)$

$- \vee \curvearrowleft (\vee,$  $+$  $)$

$$= 0$$

Proposition (Burgunder - D.O. - Manchon)

For any operad $\mathcal{P}$, if $t \in \mathcal{P}(i)$, $t \curvearrowleft (s_1, \ldots, s_n) = 0$ for any s_i and $n > i$. (Hence the pre-Lie algebra $\oplus_{n \geqslant 2} \mathcal{P}(n)$ is not free).

1

New problem

Are brace relations the only relations of these pre-Lie algebras ?

A bi-Pre-Lie algebra on planar trees

Outline

[Recall] Presentation of an operad

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where PT_n^G is the vector space of planar trees with inner nodes decorated by the generating operations of the operad.
Especially, if $G =$ ,

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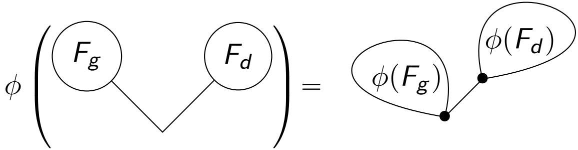
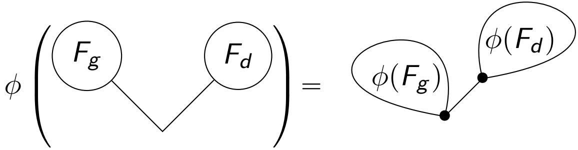
Bijection between planar binary trees and planar trees : Knuth's rotation correspondence

Planar trees on n nodes and planar binary trees on n leaves are both counted by Catalan numbers and linked by the following recursively defined bijection ϕ:

$\phi(\vee) =$ 





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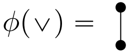

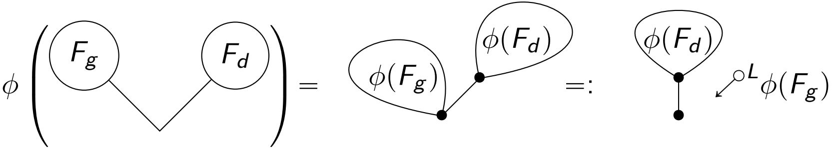

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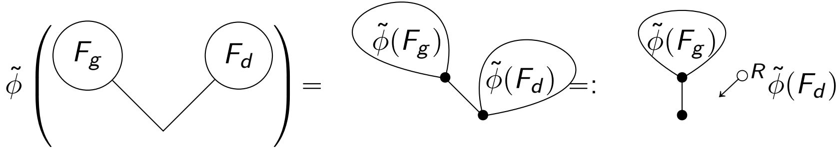

Each bijection gives a different pre-Lie product on planar trees :

- $t \curvearrowleft^{L} s$ which is the sum over all the way to graft the root of s on the left of the root of t
- and $t \curvearrowleft^{R} s$ which is the sum over all the way to graft the root of s on the right of the root of t.

New problem :

What happens if we consider at the same time both products ?

Bi-pre-Lie algebras

Definition

A bi-pre-Lie algebra is a vector space V endowed with two pre-Lie products $\leftharpoondown^L$ and $\leftharpoondown^R$ satisfying:

$$(u \leftharpoondown^L v) \leftharpoondown^L w - u \leftharpoondown^L (v \leftharpoondown^L w) = (u \leftharpoondown^L w) \leftharpoondown^L v - u \leftharpoondown^L (w \leftharpoondown^L v)$$

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Proposition (Burgunder-D.O.-Manchon)

The vector space of planar trees PT, *with pre-Lie products* $\leftharpoondown^L$ *and* $\leftharpoondown^R$, *is a bi-pre-Lie algebra.*

New problem :

What happens if we consider at the same time both products ?

Theorem (Burgunder-D.O.-Manchon)

The bi-pre-Lie algebra $(\mathrm{PT}, \curvearrowleft^{L}, \curvearrowleft^{R})$ *is generated by*  *and* 

Sketch of the proof of the theorem

Theorem (Burgunder-D.O.-Manchon)

The bi-pre-Lie algebra $(\mathrm{PT}, \curvearrowleft^L, \curvearrowleft^R)$ *is generated by*  *and* 

Proof.

By induction on the following well-founded partial order on trees : $S \leqslant T$ if

- either $|S| < |T|$,
- or $|S| = |T|$ and $\mathrm{height}(S) > \mathrm{height}(T)$,
- or $\big(|S|, \mathrm{height}(S)\big) = \big(|T|, \mathrm{height}(T)\big)$ and $\sum_{v \in V(S)} \mathrm{height}(v) < \sum_{v' \in V(T)} \mathrm{height}(v')$,
- or $S = T$.

$\square$

Proof.

- Initialization : E_2
- Induction step :
  Goal : rewrite t as a product of smaller terms
  Two cases : t planted or not

$\square$

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Thank you for your attention !