

Compact packings with three discs

Thomas Fernique
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Sphere packings

Sphere packing: interior disjoint unit spheres.

Density: limsup of the proportion of $B(0, r)$ covered.

Question: densest packings?

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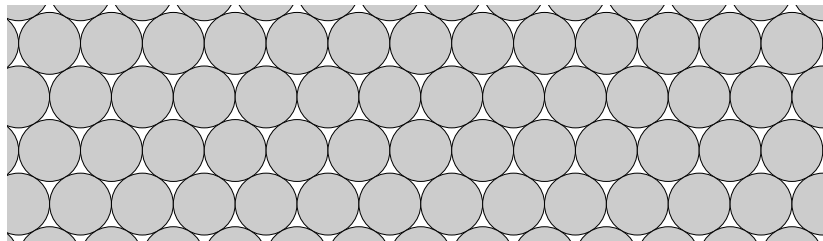
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Theorem (Thue, 1910)

The densest packing in \mathbb{R}^2 is the hexagonal compact packing.



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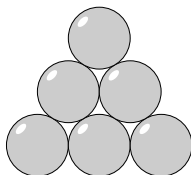
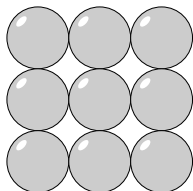
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The densest packings in \mathbb{R}^3 are the close-packings.



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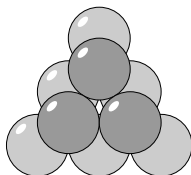
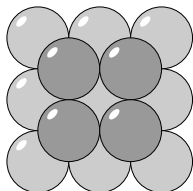
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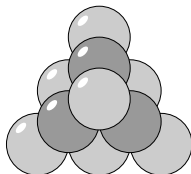
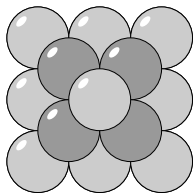
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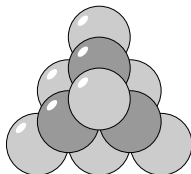
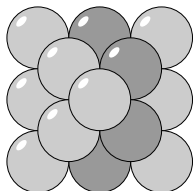
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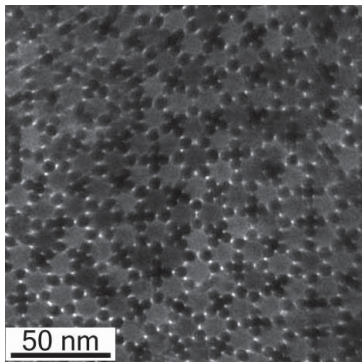
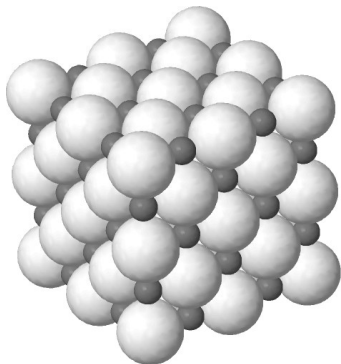
Question: densest packings?

Theorem (Vyazovska *et al.*, 2017)

The densest packings are known in \mathbb{R}^8 and \mathbb{R}^{24} .

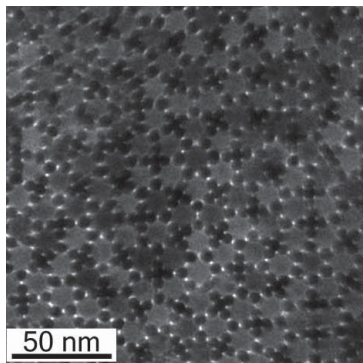
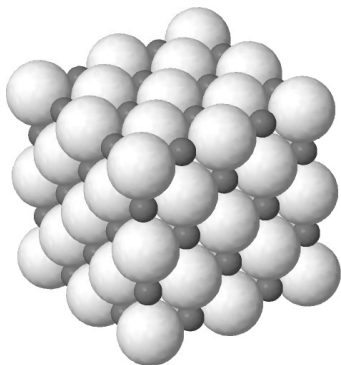
Unequal sphere packings

The density becomes parametrized by the ratios of sphere sizes.
Natural problem in materials science!



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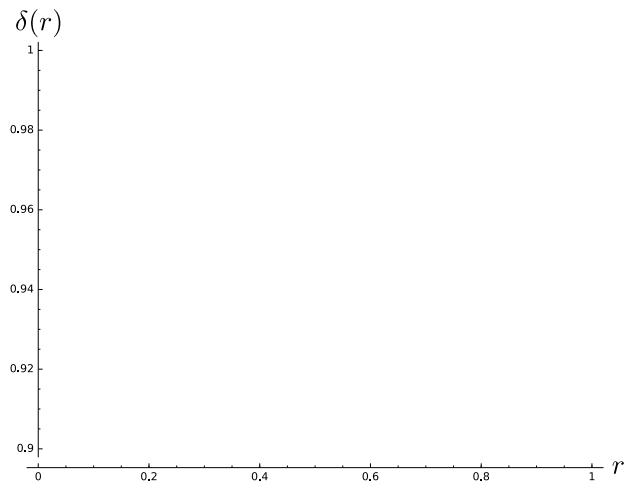
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Theorem (Heppes-Kennedy, 2004–2006)

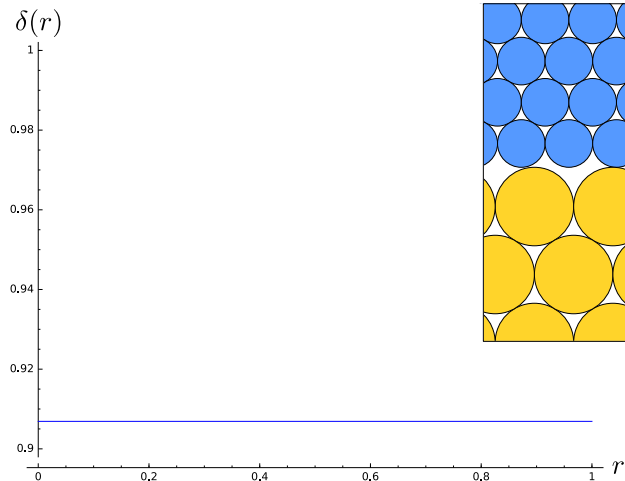
The densest packings with two discs are known for seven ratios.

Two discs



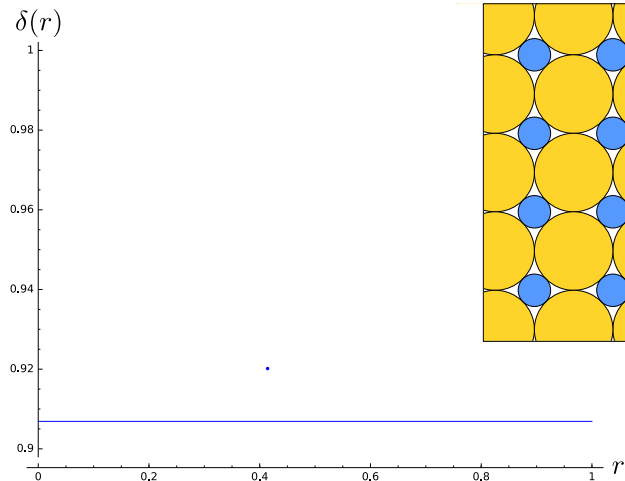
The maximal density is a function $\delta(r)$ of the ratio $r \in [0, 1]$.

Two discs



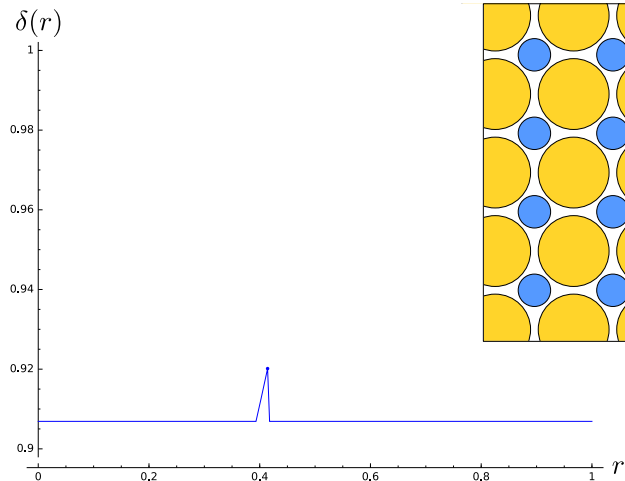
The hexagonal compact packing yields a uniform lower bound.

Two discs



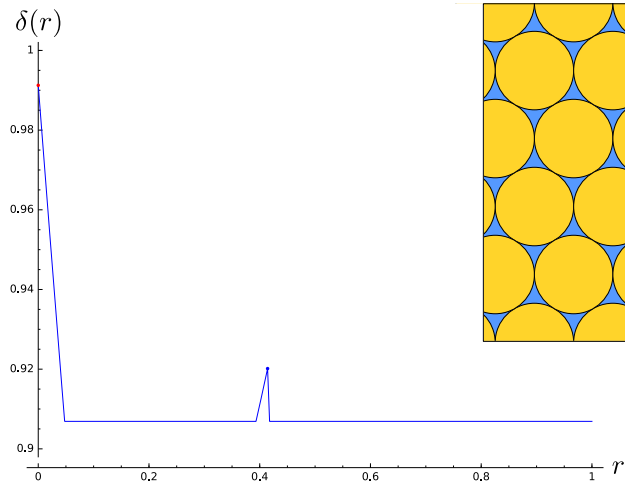
Any given packing yields a lower bound for a specific r .

Two discs



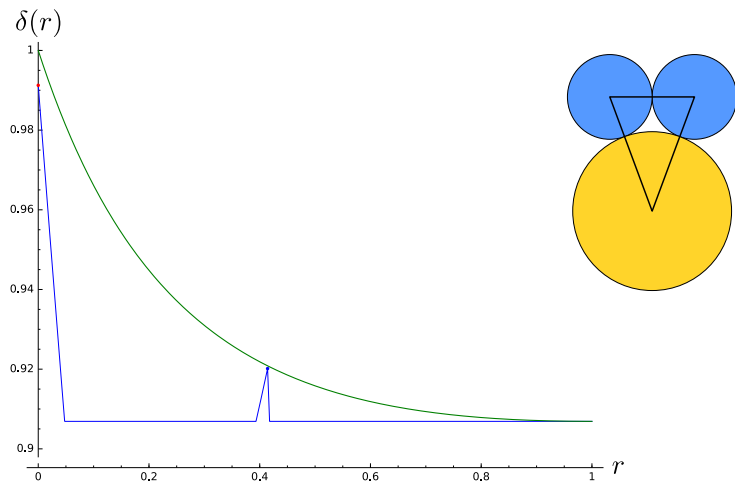
It actually yields a lower bound in a neighborhood of r .

Two discs



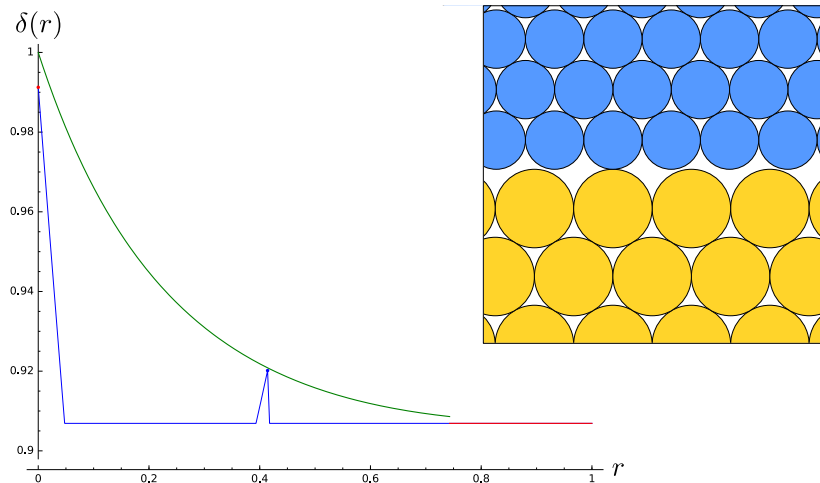
$$\lim_{r \rightarrow 0} \delta(r) = \delta(1) + (1 - \delta(1))\delta(1) \simeq 0.99133.$$

Two discs



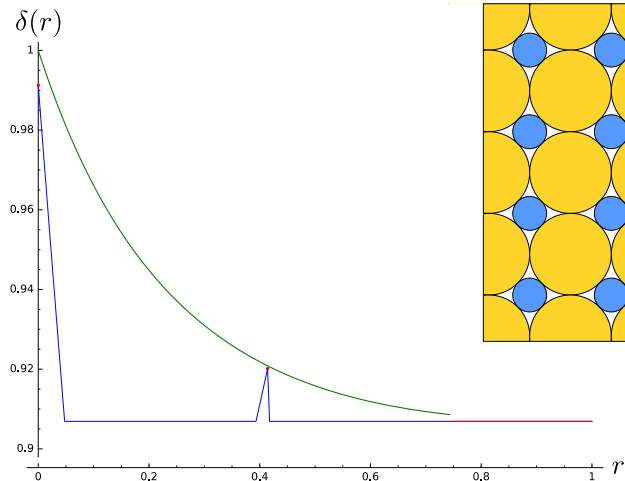
The density in the $r_1 r$ triangle is an upper bound (Florian, 1960).

Two discs



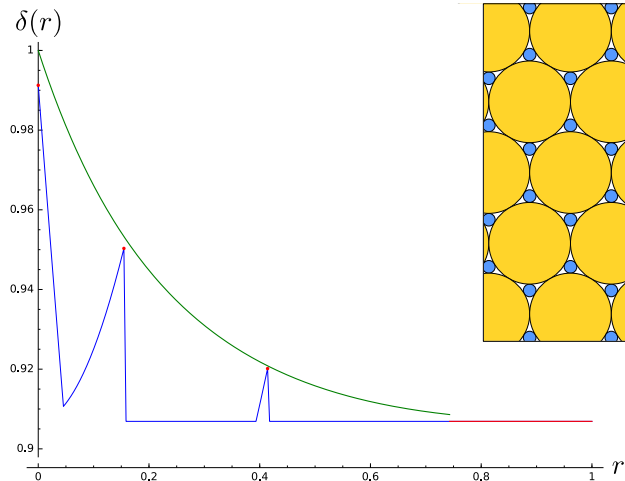
For $r \geq 0.74$, two discs do not pack better than one (Blind, 1969).

The seven "magic" ratios



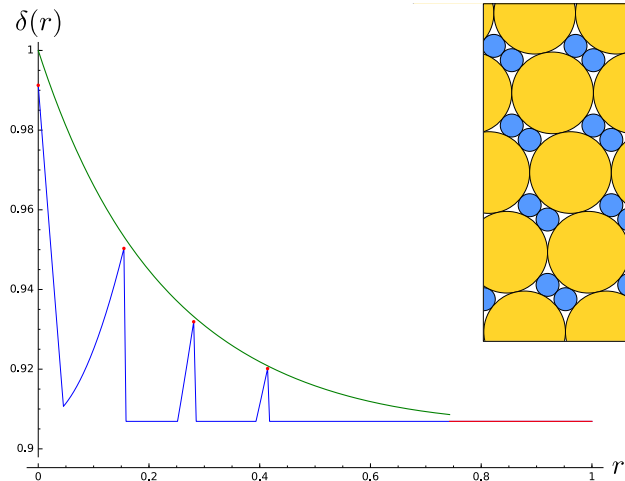
0.41, root of $X^2 + 2X - 1$.

The seven "magic" ratios



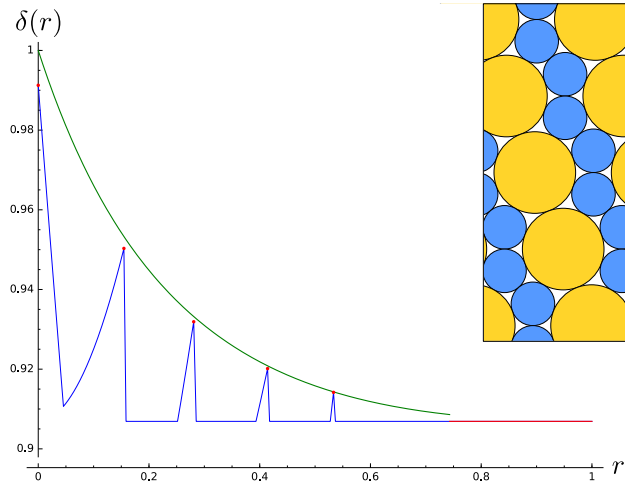
0.15, root of $3X^2 + 6X - 1$.

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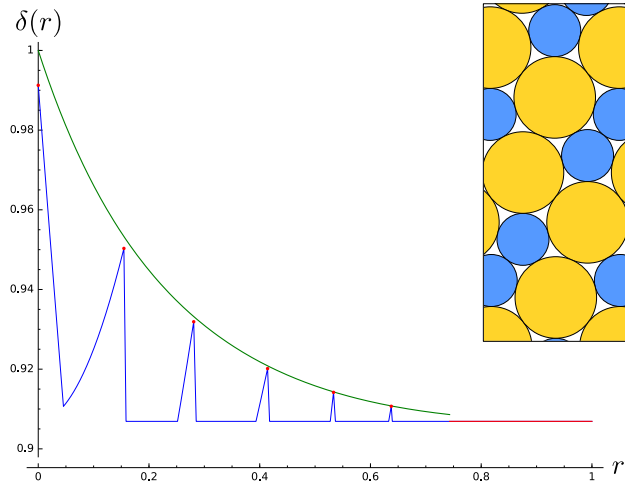
0.28, root of $2X^2 + 3X - 1$.

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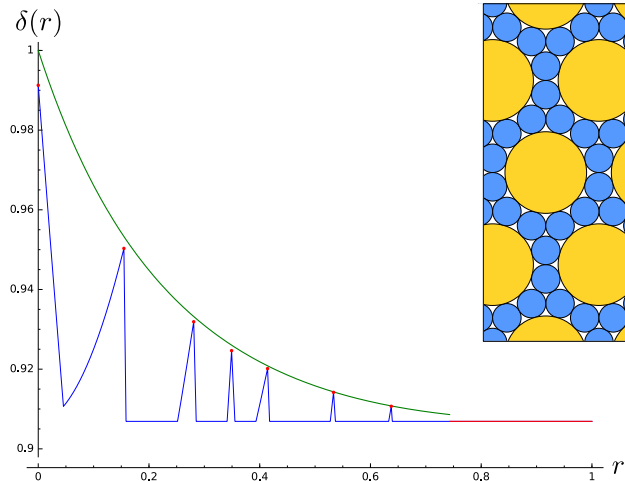
0.53, root of $8X^3 + 3X^2 - 2X - 1$.

The seven "magic" ratios



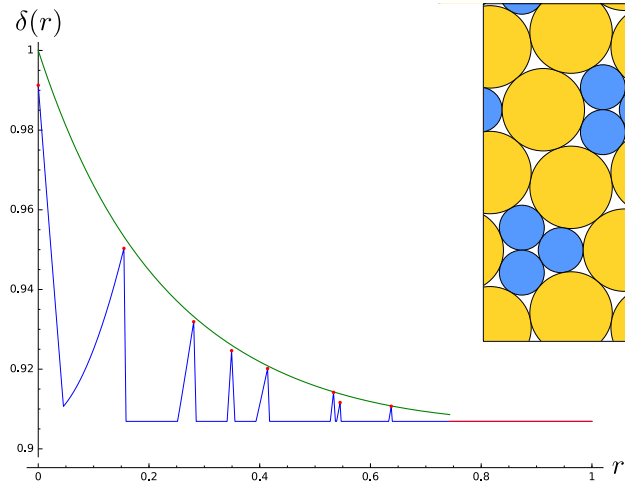
0.64, root of $X^4 - 10X^2 - 8X + 9$.

The seven "magic" ratios



0.35, root of $X^4 - 28X^3 - 10X^2 + 4X + 1$.

The seven "magic" ratios



0.55, root of $X^8 - 8X^7 - 44X^6 - 232X^5 - 482X^4 - 24X^3 + 388X^2 - 120X + 9$.

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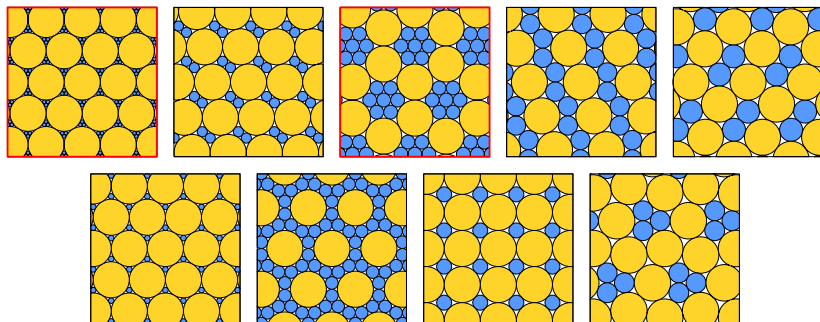
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Compact packings are candidates to provably maximize the density.

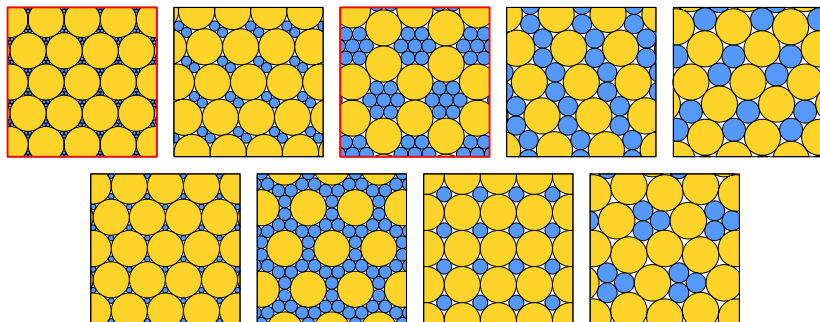
Compact packings with two discs



Theorem (Kennedy, 2006)

There are nine ratios allowing a compact packing with two discs.

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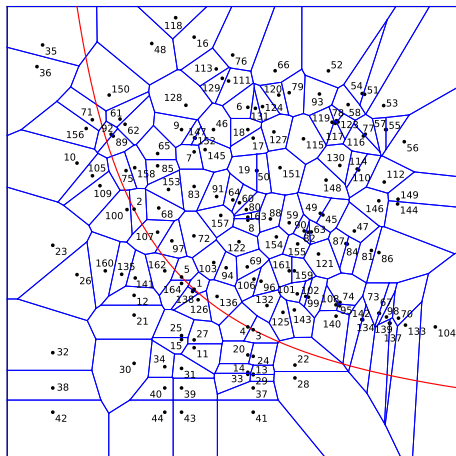


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Remark: two have (still?) not been proven to maximize the density.

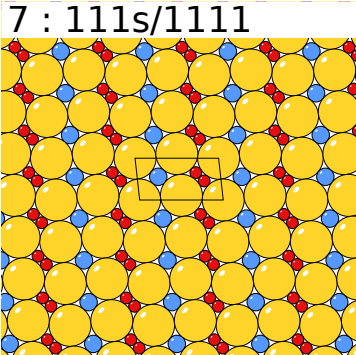
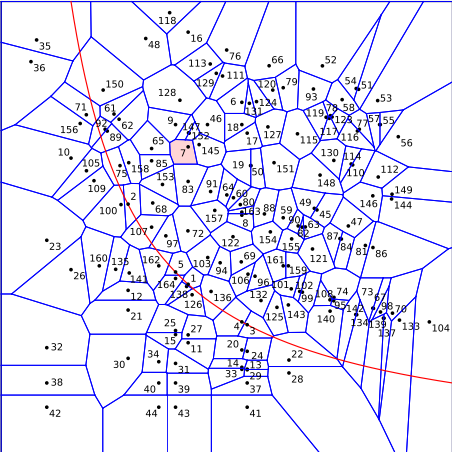
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Theorem (F.-Hashemi-Sizova)

There are 164 ratios allowing a compact packing with three discs.

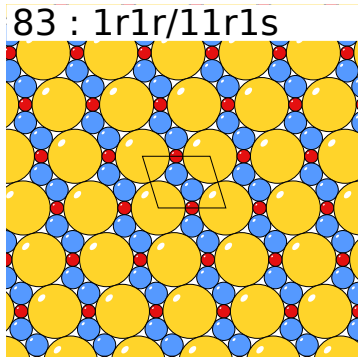
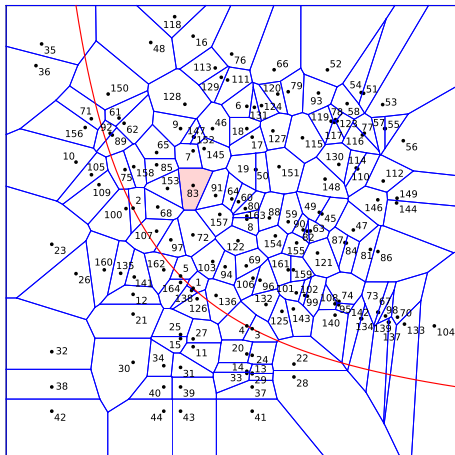
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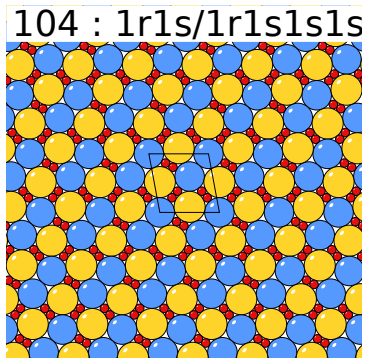
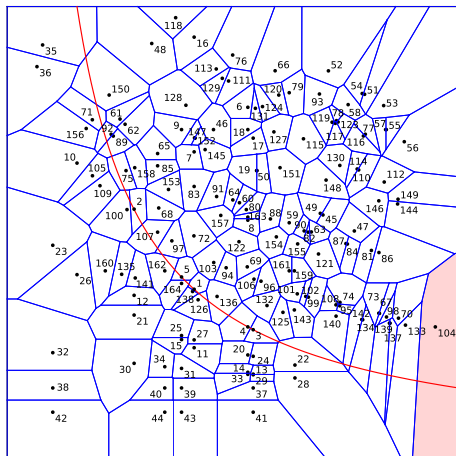
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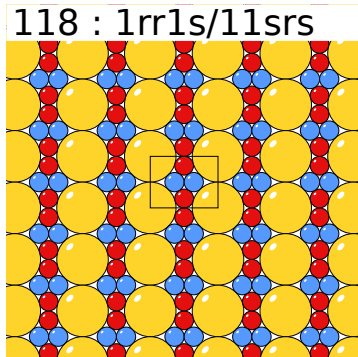
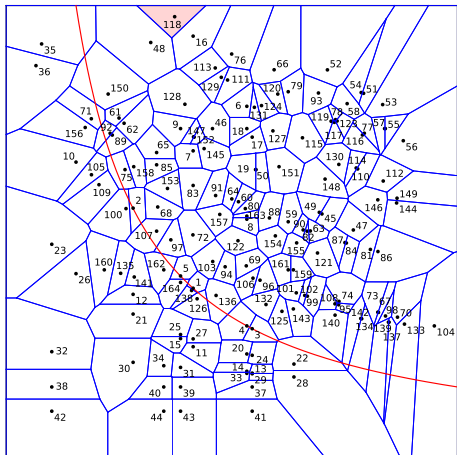
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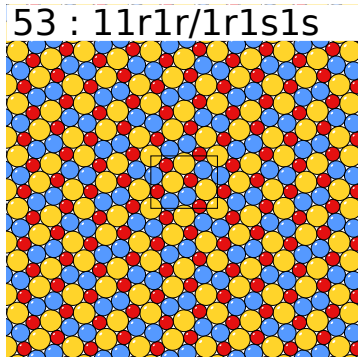
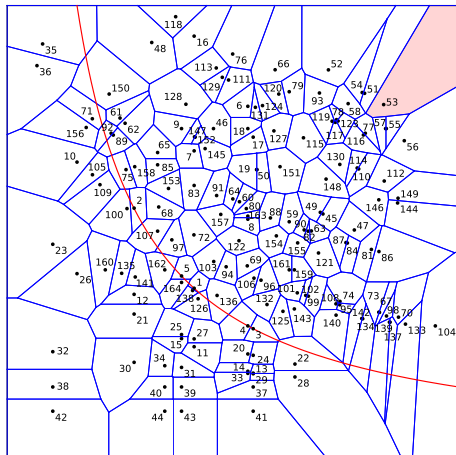
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Proof sketch

Let $s < r < 1$ be the three sizes of discs.

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Strategy

For each pair of s - and r -coronas, solve the polynomial system, then find all the possible coronas and finally find the packings.

Compact packing with spheres in \mathbb{R}^3

With two spheres, it is boring:

Theorem (Fernique, 2019)

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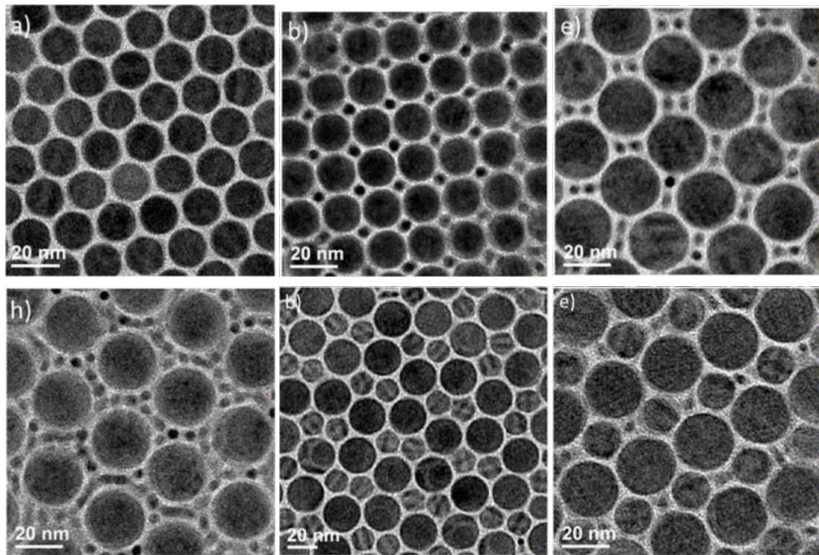
The compact packings by two sizes of spheres are exactly those obtained by filling the octahedral holes of a close-packing.

With three spheres, it is still boring:

Theorem (Fernique, expected 2019)

The compact packing by three sizes of spheres are exactly those obtained by filling one of the two types of tetrahedral holes of a compact packing by two sizes of spheres.

Back to material science

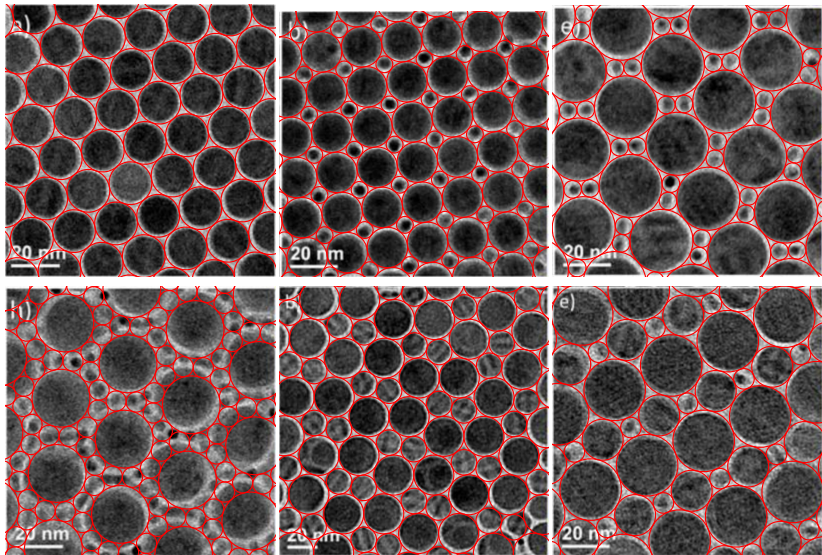


T. Paik, B. Diroll, C. Kagan, Ch. Murray

J. Am. Chem. Soc., 2015, 137

Binary and ternary superlattices self-assembled from colloidal nanodisks and nanorods

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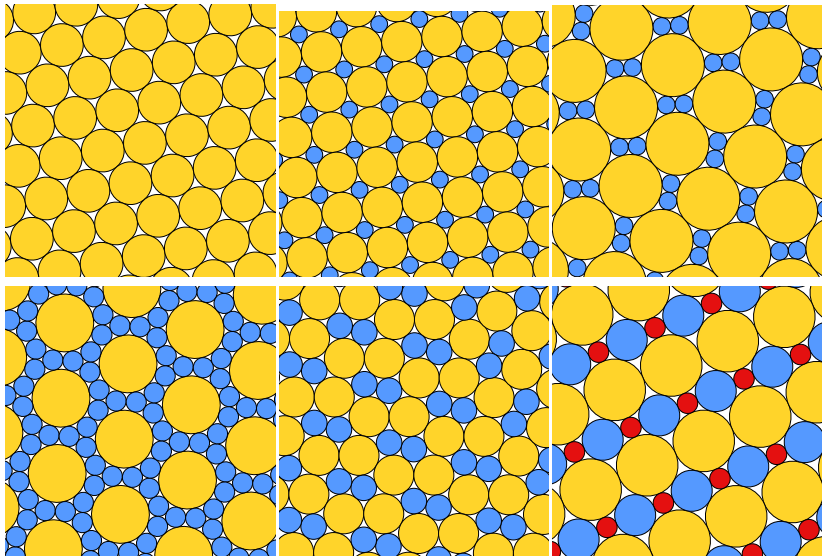


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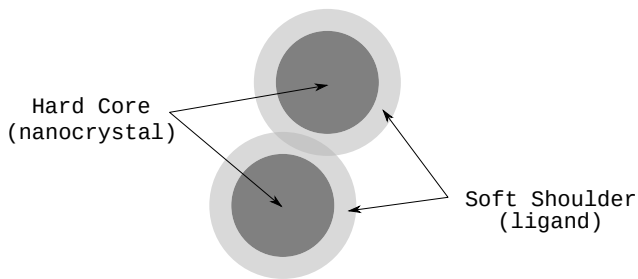


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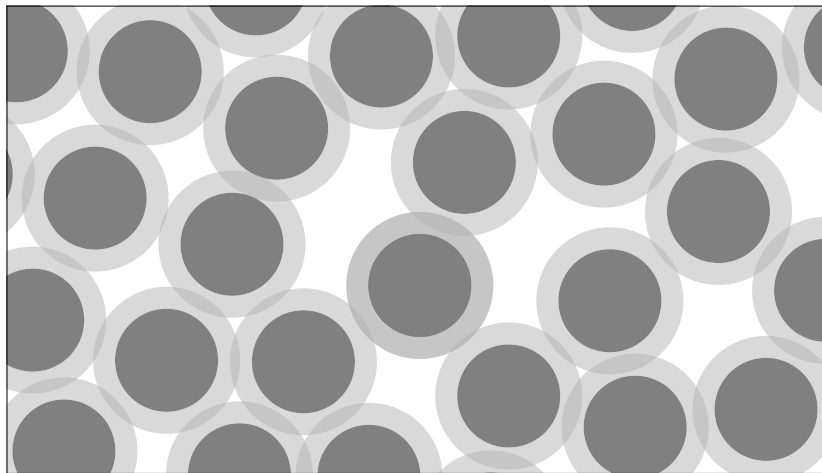
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Quasicompact packings



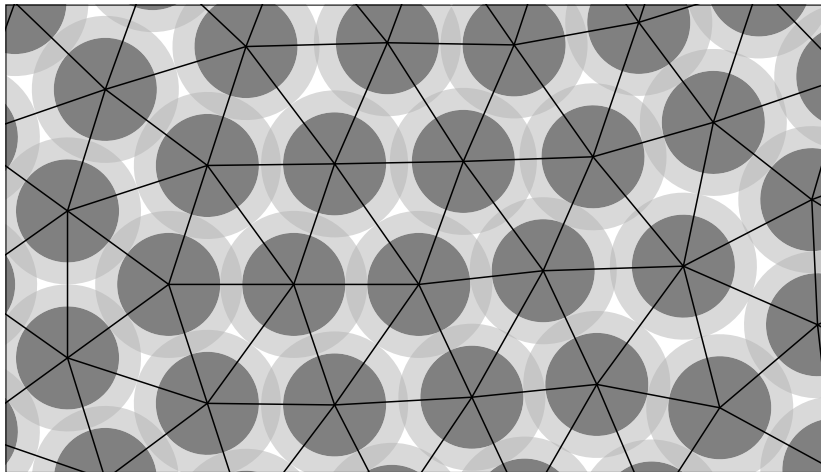
More realistic model: Hard Core + Soft Shoulder (HCSS).

Quasicompact packings



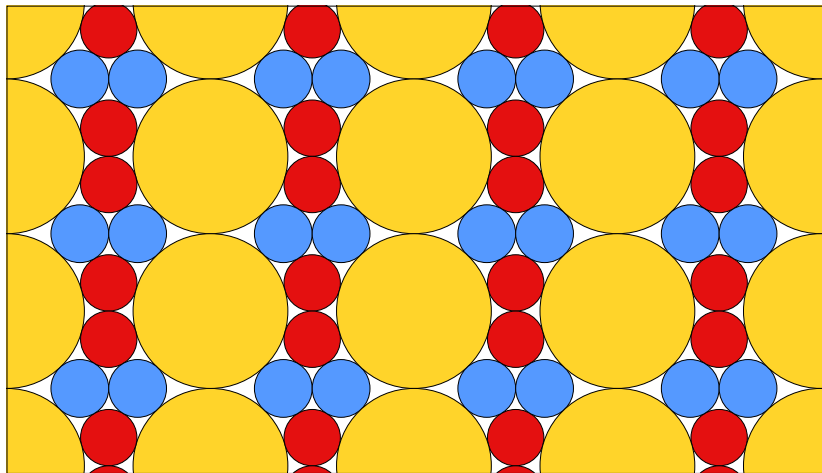
Which quantity has to be minimized or maximized (formally)?

Quasicompact packings



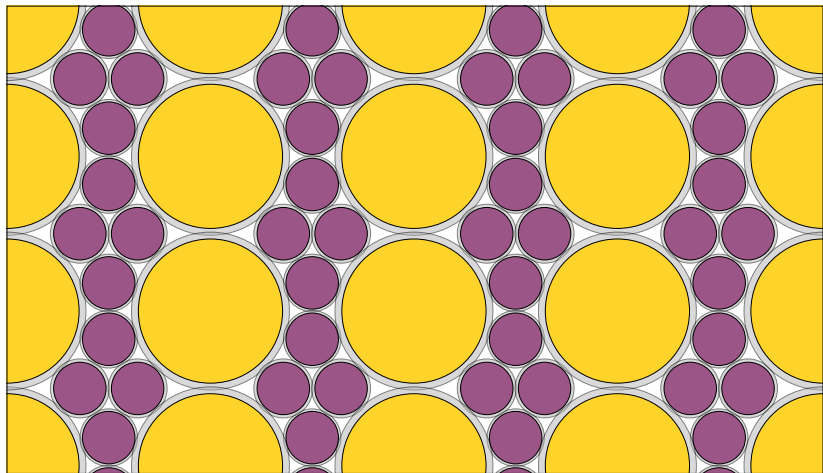
Compact packings naturally extend to *quasicompact* packings.

Quasicompact packings



Some compact packings with three sizes of discs...

Quasicompact packings



... can be seen as a quasicompact packings with two sizes of discs.

Project 80-PRIME CNRS 2019-2021

- ▶ INS2I: LIPN (Thomas Fernique)
- ▶ INC/INP: LPCNO (Simon Tricard)

Goals:

- ▶ experimental nanosynthesis of "compact supercrystals";
- ▶ new theoretical questions raised by experiences.