A localization principle for the Basic Triangle Theorem.

BTT, monodromy interplay and a (tangled) tale of various subalgebras

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
C. Bui, Q.H. Ngô, S. Goodenough.

CAP days, 06-08 November 2019

Plan

Foreword: Goal of this talk

In this talk, I will show tools and sketch proofs about Noncommutative Differential (Evolution) Equations.

The main item of data (not to say the only one) is that of Noncommutative Formal Power Series with variable coefficients which allows to explore in a compact and effective (in the sense of computability, hence needing fraction fields or a localization process) way the Magnus group of proper exponentials and the Hausdorff group of Lie exponentials (i.e. group-like series for the dual of the shuffle product).

Parts of this work are connected with Dyson series and take place within the project: *Evolution Equations in Combinatorics and Physics*. Today we will focus on mathematical motivations and properties (namely localization).

This talk also prepares data structures used in forthcoming works.

Before reviewing the facts, let's recall the general setting.

Bits and pieces for the BTT

Theorem (DDMS [1])

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \; ; \; \langle S | 1_{X^*} \rangle = 1_{\mathcal{A}} \tag{1}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x \, x \in \mathcal{C}\langle\langle X \rangle\rangle \; . \tag{2}$$

[1] *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, **M. Deneufchâtel, GHED, V. Hoang Ngoc Minh and A. I. Solomon**, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

Bits and pieces for the BTT/2

Theorem (cont'd)

The following conditions are equivalent :

i) *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.

ii) *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.

iii) *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{3}$$

iv) *The family* $(u_x)_{x\in X}$ *is free over* k *and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{4}$$

Why BTT and NCDE ? : Review of the facts

- $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ $(\Re(s) > 1)$
- when one multiplies two of these, one gets quantities like

$$\zeta(s_1)\zeta(s_2) = \sum_{n_1,n_2\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} = \zeta(s_1,s_2) + \zeta(s_1+s_2) + \zeta(s_2,s_1)$$

- and, with several of them, we are led to the following definition of MultiZeta Values (MZV), converging in

$$\mathcal{H}_r = \{(s_1,\ldots,s_r) \in \mathbb{C}^r \,|\, \forall m = 1,\ldots,r, \Re(s_1) + \ldots + \Re(s_m) > m\} \ .$$

$$\zeta(s_1,\ldots,s_k) := \sum_{n_1>\ldots>n_k\geq 1} \frac{1}{n_1^{s_1}\ldots n_k^{s_k}} \tag{5}$$

- On the other hand, one has the classical polylogarithms defined, for $k \geq 1, |z| < 1$, by

$$-\log(1-z) = \mathrm{Li}_1 = \sum_{n\geq 1} \frac{z^n}{n^1};\ \mathrm{Li}_2 = \sum_{n\geq 1} \frac{z^n}{n^2};\ \ldots;\ \mathrm{Li}_k(z) := \sum_{n\geq 1} \frac{z^n}{n^k}$$

Why BTT and NCDE ? : Review of the facts/2

- The analogue of the classical polylogarithms for MZV reads

$$Li_{y_{s_1}\ldots y_{s_k}}(z) := \sum_{n_1>\ldots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\ldots n_k^{s_k}}\ ;\ |z|<1$$

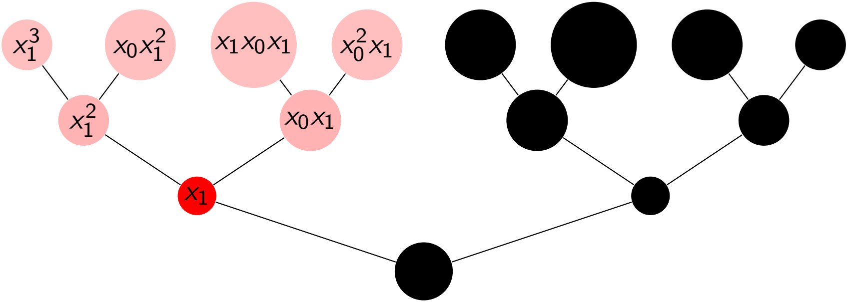
- They satisfy the recursion (ladder stepdown)

$$\begin{aligned} z\frac{d}{dz}Li_{y_{s_1}\ldots y_{s_k}} &= Li_{y_{s_1-1}\ldots y_{s_k}} \text{ if } s_1>1 \\ (1-z)\frac{d}{dz}Li_{y_1 y_{s_2}\ldots y_{s_k}} &= Li_{y_{s_2}\ldots y_{s_k}} \text{ if } k>1 \end{aligned} \qquad (6)$$

which, with $s_i \in \mathbb{N}_{\geq 1},\ k \geq 1$, ends at the "seed"

$$\mathrm{Li}_{y_1}(z) = \mathrm{Li}_1(z) = \log(\frac{1}{1-z}) \qquad (7)$$

- For the next step, we code the moves $z\frac{d}{dz}$ (resp. $(1-z)\frac{d}{dz}$) - or more precisely sections $\int_0^z \frac{f(s)}{s}ds$ (resp. $\int_0^z \frac{f(s)}{1-s}ds$) - with x_0 (resp. x_1).



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $*_0 = 0$

$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1^2n_2}$$

; above “cl. not.” stands for “classical notation”

Why BTT and NCDE ? : Review of the facts/3

- Calling S the prospective generating series

$$S = \sum_{w \in X^*} \underbrace{\langle S|w\rangle}_{\in \mathcal{H}(\Omega)} w \ ; \ X = \{x_0, x_1\} \tag{8}$$

V. Drinfel'd [1] indirectly proposed a way to complete the tree:

$$\begin{cases} \mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z}).S & (NCDE) \\ \lim_{\substack{z \to 0 \\ z \in \Omega}} S(z)e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X \rangle\rangle} & (Asympt.\ Init.\ Cond.) \end{cases} \tag{9}$$

from the general theory, this system has a unique solution which is precisely Li (called G_0 in [1]) ; $S \mapsto \mathbf{d}(S)$ being the term by term derivation of the coefficients.

- Minh [2] indicated a way to effectively compute this solution through (improper) iterated integrals.

1. V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.
2. H. N. Minh, *Summations of polylogarithms via evaluation transform*, Mathematics and Computers in Simulation, Vol. 42, 4-6, Nov. 1996, pp. 707-728

Explicit construction of Drinfeld's G_0

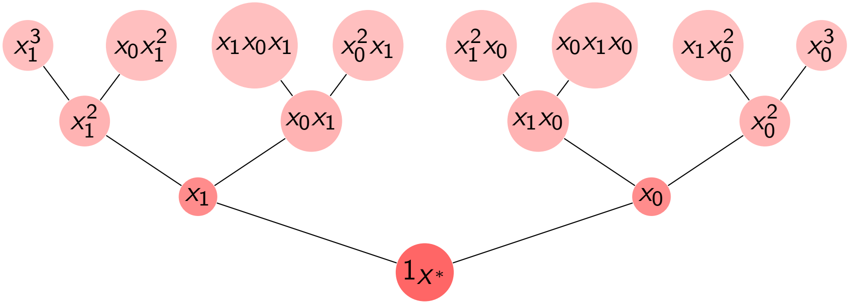
Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\alpha_0^z(w) = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_{0}^{z} \alpha_0^s(u) \frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_{1}^{z} \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \ (w \in x_0^*) \\ \int_{0}^{z} \alpha_0^s(u) \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \ (w \in x_0 X^* x_1 x_0^*) \end{cases}$$

The third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ is Li (G_0 in [1]).



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $t_0 = 0$

$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geq 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geq 1}\frac{z^{n_1}}{n_1^2n_2} \quad ; \quad \langle S|x_0^n\rangle = \frac{\log^n(z)}{n!}$$

General solution of NCDE and Picard's process

The series $S_{Pic}^{z_0}$ ($z_0 \in \Omega$) can be computed by Picard's process

$$S_0 = 1_{X^*} \ ; \ S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n$$

and its limit is $S_{Pic}^{z_0} := \lim_{n\to\infty} S_n$ ($= \sum_{w \in X^*} \alpha_{z_0}^{z}(w)\, w$ this afternoon). One has,

Proposition

i) Series $S_{Pic}^{z_0}$ is the unique solution of

$$\begin{cases} \mathbf{d}(S) &= M.S \text{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \\ S(z_0) &= 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} \end{cases} \quad (10)$$

ii) The complete set of solutions of $\mathbf{d}(S) = M.S$ is $S_{Pic}^{z_0}.\mathbb{C}\langle\langle X\rangle\rangle$.

About solutions of NCDE

1. The set $\mathcal{S}$ of series satisfying $(NCDE)$ has a lot of nice combinatorial properties.
   - Right $\mathbb{C}\langle\langle X\rangle\rangle$ module of rank one ($\mathcal{S} = S_0.\mathbb{C}\langle\langle X\rangle\rangle$, where S_0 is any solution with non-zero constant term, such a solution can be constructed by Picard process).
   - Linear independence of the coefficients (when non-zero).
2. The ones like Li or constructed through Picard's process (Chen series, i.e. limit of $S_0 = 1_{X^*}$; $S_{n+1} = 1_{X^*} + \int_{z_0}^{z} M.S_n$) have moreover
   - Shuffle property
   - Factorisation
   - Extension to rational functions (some of them for Li, all for $S_{Pic}^{z_0}$).

Now, as the lists are coded by words, it is possible to use the rich allowance of notations invented by algebraists, computer scientists, combinatorialists and physicists about NonCommutative Formal Power Series (NCFPS[1]).

[1]This was the initial intent of the series of conferences FPSAC.

What is so special with solutions like Li and $S_{Pic}^{z_0}$.

The following general theorem explains (a) why Li and $S_{Pic}^{z_0}$ have the shuffle property and (b) why Li is unique.

Theorem (Analyse et Géometrie, Cargèse, IESC, 21-24 Nov. 2017)

Let

$$(TSM) \qquad \mathbf{d}S = M_1 S + S M_2 \ . \tag{11}$$

with $S \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle,\ M_i \in \mathcal{H}(\Omega)_+\langle\langle X \rangle\rangle$

(i) *Solutions of* (TSM) *form a* $\mathbb{C}$*-vector space.*

(ii) *Solutions of* (TSM) *have their constant term (as coefficient of* 1_{X^*}*) which are constant functions (on* Ω*); there exists solutions with constant coefficient* 1_Ω *(hence invertible).*

(iii) *If two solutions coincide at one point* $z_0 \in \Omega$ *(or asymptotically), they coincide everywhere.*

What is so special with solutions like Li and $S_{Pic}^{z_0}$./2

Theorem (cont'd)

(iv) *Let be the following one-sided equations*

$$(LM_1) \quad \mathbf{d}S = M_1 S \qquad (RM_2) \quad \mathbf{d}S = SM_2. \tag{12}$$

and let S_1 (resp. S_2) be a solution of (LM_1) (resp. (LM_2)), then $S_1 S_2$ is a solution of (TSM). Conversely, every solution of (TSM) can be constructed so.

(v) *Let $S_{Pic,LM_1}^{z_0}$ (resp. $S_{Pic,RM_2}^{z_0}$) the unique solution of (LM_1) (resp. (RM_2)) s.t. $S(z_0) = 1_{\mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle}$ then, the space of all solutions of (TSM) is*

$$S_{Pic,LM_1}^{z_0}.\mathbb{C}\langle\langle X\rangle\rangle.S_{Pic,RM_2}^{z_0}$$

(vi) *If $M_i,\ i = 1,2$ are primitive for Δ_{III}[a] and if S, a solution of (TSM), is group-like at one point (or asymptotically), it is group-like everywhere (over Ω).*

[a] Δ_{III} is the canonical comultiplcation of $\mathbb{C}\langle X\rangle$ viewed as an enveloping algebra.

Solutions as $\shuffle$-characters with values in $\mathcal{H}(\Omega)$

We have seen that solutions of systems like (10) possess the shuffle property. Due to the special combinatorial for of $\Delta_{\shuffle}$, its dual law can be defined by the recursion

$$\begin{aligned} u \shuffle 1_{Y^*} &= 1_{Y^*} \shuffle u = u \text{ and} \\ au \shuffle bv &= a(u \shuffle bv) + b(au \shuffle v) \end{aligned}$$

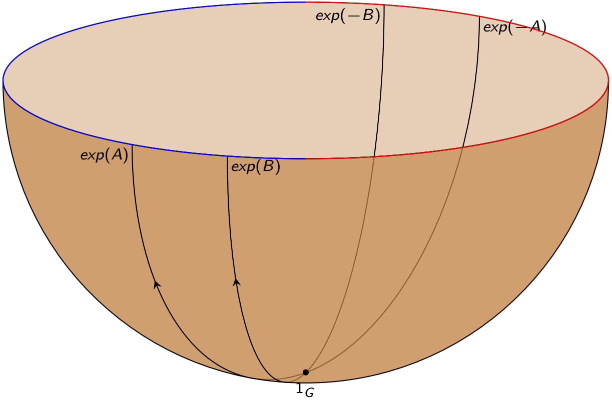
and one has

$$\langle S_{Pic}^{z_0} | u \shuffle v \rangle = \langle S_{Pic}^{z_0} | u \rangle \langle S_{Pic}^{z_0} | v \rangle \ ; \ \langle S | 1_{X^*} \rangle = 1_{\mathcal{H}(\Omega)} \qquad (13)$$

(product in $\mathcal{H}(\Omega)$).

Now it is not difficult to check that the characters of type (13) form a group (these are characters on a Hopf algebra). It is interesting to have at our disposal a system of local coordinates in order to perform estimates in neighbourhood of the singularities (CAP 18).

Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{Ш}$. These are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).

Lyndon words and factorizations

- Let $c = [2 \cdots n, 1]$ be the large cycle
- a Lyndon word is a word which is strict minimum of its conjugacy class (as a family) i.e. $(\forall 1 \leq k < n)(l \prec_{lex} l.\sigma^k)$
- Each word w factorizes uniquely as $w = l_1^{\alpha_1} \cdots l_n^{\alpha_n}$ with $l_i \in \mathcal{L}yn(X)$ and $l_1 \prec \cdots \prec l_n$ (strict). We write

$$X^* = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} l^* \tag{14}$$

- If $(P_l)_{l \in \mathcal{L}yn(X)}$ is any multihomogeneous basis of $Lie_R\langle X \rangle$ (R a $\mathbb{Q}$-algebra) then

$$\sum_{w \in X^*} w \otimes w = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} e^{S_l \otimes P_l}$$

where P_w is computed after eq. 14 and S_w is such that $\langle S_u | P_v \rangle = \delta_{u,v}$.

Domain of Li.

We now have an arrow of commutative algebras

$$(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) \xrightarrow{\mathrm{Li}_\bullet} (\mathcal{H}(\Omega), \times, 1_\Omega)$$

on the left $\mathbb{C}\langle X\rangle \hookrightarrow \mathbb{C}\langle\langle X\rangle\rangle$ is endowed with the Krull topology (coefficientwise stationary convergence) and, on the right $\mathcal{H}(\Omega)$ is endowed with the (Fréchet) topology of compact convergence. We are led to the following definition.

Definition [Domain of Li]

We define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S = \sum_{n\geq 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geq 0} Li_{S_n}(z)$ converges unconditionally[a] for the compact convergence in Ω. One then sets $Li_S(z) := \sum_{n\geq 0} Li_{S_n}(z)$.

[a] In order to use functional properties of $\mathcal{H}(\Omega)$.

Domain of Li/2

Examples and a diagram

$$Li_{x_0^*}(z) = z,\ Li_{x_1^*}(z) = (1-z)^{-1}\ ;\ Li_{(\alpha x_0+\beta x_1)^*}(z) = z^{\alpha}(1-z)^{-\beta} \quad (15)$$

$$\begin{array}{ccc} (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xhookrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\ \downarrow & & \downarrow \\ \mathbb{C}\langle\langle X\rangle\rangle \supset Dom(\mathrm{Li}) & \longrightarrow & \mathcal{H}(\Omega) \end{array}$$

Proposition

With this definition, we have

1. $Dom(Li)$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and then so is $Dom^{rat}(Li) := Dom(Li) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$
2. For $S, T \in Dom(Li)$, we have

$$\mathrm{Li}_{S ш T} = \mathrm{Li}_S \, . \, \mathrm{Li}_T$$

Continuing the ladder

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle \,ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \,ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
Dom(\mathrm{Li}) & \xrightarrow{\mathrm{Li}_\bullet^{(gen)}} & \mathcal{H}(\Omega)
\end{array}$$

We have, after a theorem by Leopold Kronecker,

$$\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \left\{\frac{P}{1-xQ}\right\}_{P,Q\in\mathbb{C}[x]} \; ; \; \left(CS \; : \; \frac{P}{1-xQ} = P(xQ)^*\right) \tag{16}$$

and, as $\mathbb{C}$ is algebraically closed, $\mathbb{C}^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_{\mathbb{C}}\{(ax)^* \,ш\, \mathbb{C}[x] | a \in \mathbb{C}\}$.

This proves that $\mathbb{C}\langle X\rangle \,ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \,ш\, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle \subset Dom(\mathrm{Li})$.

Further examples and counterexamples

For $|t| < 1$, one has $(tx_0)^* x_1 \in Dom(Li, D)$ (D is the open unit slit disc), whereas $x_0^* x_1 \notin Dom(Li, D)$.
Indeed, we have to examine the convergence of $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z)$, but, for $z \in]0,1[$, one has $0 < z < \mathrm{Li}_{x_0^n x_1}(z) \in \mathbb{R}$ and therefore, for these values $\sum_{n\geq 0} \mathrm{Li}_{x_0^n x_1}(z) = +\infty$.
In fact, in this case ($|t| < 1$),

$$\mathrm{Li}_{(tx_0)^* x_1}(z) = \sum_{n\geq 1} \frac{z^n}{n - t} \tag{17}$$

This last example opens the door of Hurwitz polyzetas.

Independence of characters w.r.t. polynomials (t).

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Independence of characters with respect to polynomials

5

1

I came across the following property :

Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U} \to k$ which vanishes on $\mathfrak{g}$.

Set $\mathcal{U}_+ = ker(\epsilon)$. We build the following filtrations $(N \geq 1)$

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots . \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

and

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^{\perp} = \{f \in \mathcal{U}^* | (\forall u \in \mathcal{U}_{N+1})(f(u) = 0)\} \qquad (2)$$

the first one is decreasing and the second one increasing. One shows easily that (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^* \diamond \mathcal{U}_q^* \subset \mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^* = \cup_{n \geq 1} \mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Now, we can state the

Theorem : The set of characters of $(\mathcal{U}, ., 1_{\mathcal{U}})$ is linearly free w.r.t. $\mathcal{U}_\infty^*$.

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Let $\mathfrak{g}$ be a Lie algebra over a ring k without zero divisors, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be its enveloping algebra. As such, $\mathcal{U}$ is a Hopf algebra and ϵ, its counit, is the only character of $\mathcal{U} \to k$ which vanishes on $\mathfrak{g}$.
Set $\mathcal{U}_+ = ker(\epsilon)$. We build the following filtrations ($N \geq 0$)

$$\mathcal{U}_N = \mathcal{U}_+^N = \underbrace{\mathcal{U}_+ \ldots . \mathcal{U}_+}_{N \text{ times}} \qquad (1)$$

(in fact $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_{N+1} = \mathcal{U}.\mathcal{U}_N$) and, for $N \geq -1$

$$\mathcal{U}_N^* = \mathcal{U}_{N+1}^{\perp} = \{f \in \mathcal{U}^* | (\forall u \in \mathcal{U}_{N+1})(f(u) = 0)\} \qquad (2)$$

the first one is decreasing and the second one increasing (in particular $\mathcal{U}_{-1}^* = \{0\}$, $\mathcal{U}_0^* = k.\epsilon$).
One shows easily that, for $p, q \geq 0$ (with $\diamond$ as the convolution product)

$$\mathcal{U}_p^* \diamond \mathcal{U}_q^* \subset \mathcal{U}_{p+q}^*$$

so that $\mathcal{U}_\infty^* = \cup_{n \geq 0} \mathcal{U}_n^*$ is a convolution subalgebra of $\mathcal{U}^*$.

Now, we can state the

Theorem

The set of characters of $(\mathcal{U}, ., 1_{\mathcal{U}})$ *is linearly free w.r.t.* $\mathcal{U}_{\infty}^{*}$.

Remark

i) $\mathcal{U}_{\infty}^{*}$ *is a commutative* k*-algebra.*
ii) The title comes from the fact that, with $(k\langle X\rangle, conc, 1)$ *(non commutative polynomials),* k *a* $\mathbb{Q}$*-algebra (without zero divisors) and one of the usual comultiplications (with* Δ_{+} *cocommutative and nilpotent) one takes* $\mathfrak{g}$ *as the space of primitive elements,* $\mathcal{U}^{*} = k\langle\langle X\rangle\rangle$ *(series) and* $\mathcal{U}_{\infty}^{*} = k\langle X\rangle$.

Proof.

Let $\Xi(\mathcal{U})$ be the set of characters of $(\mathcal{U}, ., 1_{\mathcal{U}})$. For non zero $\beta \in \mathcal{U}_\infty^*$, we set $\deg(\beta) = p$ as being the unique index p such that $\beta \in \mathcal{U}_p^*$ and $\beta \notin \mathcal{U}_{p-1}^*$. We consider linear relations between characters of the form

$$\sum_{i\in I} \beta_i \diamond f_i = 0 \ ; \ \beta_i \in \mathcal{U}_\infty^* \setminus \{0\} \text{ and } f_i \in \Xi(\mathcal{U}) \qquad (3)$$

Either all of them are trivial ($I = \emptyset$), or there are non trivial ones ($I \neq \emptyset$) among those we choose one with $\sum_{i\in I} \deg(\beta_i)$ minimal. WLOG[a] we can consider that $(\exists i_0 \in I)(f_{i_0} = \epsilon)$ (all characters being invertible we can multiply (3) by $f_{i_0}^{-1}$ for the law $\diamond$). Then the choosen relation becomes

$$\beta_{i_0} + \sum_{i\in I\setminus\{i_0\}} \beta_i \diamond f_i = 0 \qquad (4)$$

□

[a] Ohne Beschränkung der Allgemeinheit (for Antoine Derighetti :).

Proof cont'd (t)

Proof.

We now use the shift representation of $\mathcal{U}$ in $\mathcal{U}^*$ defined, for $\varphi \in \mathcal{U}^*$, $u, m \in \mathcal{U}$ by

$$\langle \varphi \triangleleft u | m \rangle = \langle \varphi | um \rangle$$

Remark that $\mathfrak{g}$ acts on $\mathcal{U}^*$ by derivations i.e. for $a, b \in \mathcal{U}^*$, $g \in \mathfrak{g}$

$$(a \diamond b) \triangleleft g = (a \triangleleft g) \diamond b + a \diamond (b \triangleleft g)$$

Observing that $I = \{i_0\}$ is impossible, we pick $i_1 \in I \setminus \{i_0\}$ and $g \in \mathfrak{g}$ such that $\langle f_{i_1} | g \rangle \neq 0$ (this is possible because $f_{i_1} \neq \epsilon$). We now shift (4) on the right by g and get

$$\beta_{i_0} \triangleleft g + \sum_{i \in I \setminus \{i_0\}} (\beta_i \triangleleft g + \beta_i \langle f_i | g \rangle) \diamond f_i = 0 \qquad (5)$$

Now by $\deg(\beta_{i_0} \triangleleft g) < \deg(\beta_{i_0})$, $\deg(\beta_i \triangleleft g + \beta_i \langle f_i | g \rangle) \leq \deg(\beta_i)$ and $(\beta_{i_1} \triangleleft g + \beta_{i_1} \langle f_{i_1} | g \rangle) \neq 0$ we get a contradiction w.r.t. minimality. □

Remarks about independence (t)

Remark

i) For $\mathfrak{g}$ *simple, we have* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ *and all the filtrations are stationary.*
ii) The conclusion does not hold true if k *has zero divisors as the characters are no longer independent even w.r.t.* k*. For example with* $X \neq \emptyset$ *an alphabet,* $\alpha.\beta = 0$ *with* $\alpha, \beta \in k \setminus \{0\}$*, we have for all* $x \in X$

$$\beta.(\alpha x)^* - \beta.\epsilon = 0$$

Abstract BTT theorem towards localisation

Theorem (DDMS.[1] "Linz")

Let $(\mathcal{A}, d)$ be a k-commutative associative differential algebra with unit ($\ker(d) = k$ is a field) and $\mathcal{C}$ be a differential subfield of $\mathcal{A}$ (i.e. $d(\mathcal{C}) \subset \mathcal{C}$). We suppose that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathbf{d}(S) = MS \ ; \ \langle S | 1_{X^*} \rangle = 1_{\mathcal{A}} \tag{18}$$

where the multiplier M is a homogeneous series (a polynomial in the case of finite X) of degree 1, i.e.

$$M = \sum_{x \in X} u_x x \in \mathcal{C}\langle\langle X \rangle\rangle \ . \tag{19}$$

The following conditions are equivalent :

[1] *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, **M. Deneufchâtel, GHED, V. Hoang Ngoc Minh and A. I. Solomon**, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

Abstract BTT theorem towards localisation/2

Theorem (cont'd)

i) *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.

ii) *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.

iii) *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$

$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{20}$$

iv) *The family* $(u_x)_{x\in X}$ *is free over* k *and*

$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{21}$$

Need for localization

In practical cases, we only have a differential subalgebra of $\mathcal{C}_0 \subset \mathcal{H}(\Omega)$ (as image, through Li, of a shuffle subalgebra of $Dom(\mathrm{Li})$).

- $\mathbb{C}[z]$
- $\mathbb{C}[z, z^{-1}, (1-z)^{-1}]$
- $\mathbb{C}[z^{\alpha}(1-z)^{-\beta}]_{\alpha,\beta\in\mathbb{C}} = \mathcal{C}_{\mathbb{C}}$

Realizing the fraction field $Fr(\mathcal{C}_0)$ as (differential) field of germs makes the computation difficult to handle. It is easier to check the freeness of the " basic triangle" directly with the algebra. For instance, for the polylogarithms, we just have to show that, given $P_i \in \mathcal{C}_{\mathbb{C}}$,

$$P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z})) = 0_\Omega \Longrightarrow P_i \equiv 0 \qquad (22)$$

which can be done using deck transformations (see below).

Localization

Theorem (Thm1 in "Linz", Localized form)

Let $(\mathcal{A}, d)$ be a commutative associative differential ring ($\ker(d) = k$ being a field) and $\mathcal{C}$ be a differential subring (i.e. $d(\mathcal{C}) \subset \mathcal{C}$) of $\mathcal{A}$ which is an integral domain containing the field of constants.
We suppose that, for all $x \in X$, $u_x \in \mathcal{C}$ and that $S \in \mathcal{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation (18) and that $(u_x)_{x \in X} \in \mathcal{C}^X$.
The following conditions are equivalent :

i) *The family $(\langle S|w\rangle)_{w \in X^*}$ of coefficients of S is free over $\mathcal{C}$.*

ii) *The family of coefficients $(\langle S|y\rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over $\mathcal{C}$.*

iii') *For all $f_1, f_2 \in \mathcal{C}$, $f_2 \neq 0$ and $\alpha \in k^{(X)}$, we have the property*

$$W(f_1, f_2) = f_2^2\Big(\sum_{x \in X} \alpha_x u_x\Big) \Longrightarrow (\forall x \in X)(\alpha_x = 0)\ . \qquad (23)$$

where $W(f_1, f_2)$, the wronskian, stands for $d(f_1)f_2 - f_1 d(f_2)$.

Discussion

In fact, in the localized form and with $\mathcal{C}$ not a differential field, (iii) is strictly weaker than (iii'), as shows the following family of counterexamples

1. $\Omega = \mathbb{C} \setminus (]-\infty, 0])$
2. $X = \{x_0\}$, $u_0 = z^\beta$, $\beta \notin \mathbb{Q}$
3. $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\} = \mathbb{C}.1_\Omega \oplus span_{\mathbb{C}}\{z^{(k+1)\beta - l}\}_{k,l \geq 0}$
4. $S = 1_\Omega + (\sum_{n\geq 1} \frac{z^{n(\beta+1)}}{(\beta+1)^n n!})$

Let us show that, for these data (iii) holds but not (i).
Firstly, we show that $\mathcal{C}_0 = \mathbb{C}\{\{z^\beta\}\}$ corresponds to the given direct sum.
We remark that the family $(z^\alpha)_{\alpha\in\mathbb{C}}$ is $\mathbb{C}$-linearly free (within $\mathcal{H}(\Omega)$), which is a consequence of the fact that they are eigenfunctions, for different eigenvalues, of the Euler operator $z\frac{d}{dz}$.

Then

$$\mathbb{C}\{\{z^{\beta}\}\} = \mathbb{C}1_{\Omega} \oplus span_{\mathbb{C}}\{z^{(k+1)\beta-l}\}_{k,l\geq 0} = span_{\mathbb{C}}\{z^{(k')\beta-l}\}_{k',l\geq 0}$$

comes from the fact that the RHS is a subset of the LHS as, for all, $k, l \geq 0$, $z^{(k+1)\beta-l} \in \mathbb{C}\{\{z^{\beta}\}\}$. Finally $1_{\Omega} \in \mathbb{C}\{\{z^{\beta}\}\}$ by definition ($\mathbb{C}\{\{X\}\}$ is a $\mathbb{C}$-AAU).

(iii) is fulfilled. Here
$u_0(z) = z^{\beta}$ is such that, for any $f \in \mathcal{C}_0$ and c_0 in $\mathbb{C}$, we have

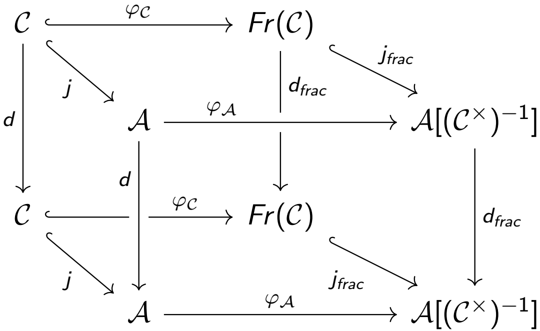
$$c_0 u_0 = \partial_z(f) \Longrightarrow (c_0 = 0) \tag{24}$$

But (i) is not Because we have the following relation

$$(\beta + 1)z^{\beta-1}\langle S|x_0\rangle - z^{2\beta}.1_{\Omega} = 0$$

Sketch of the proof

After some technicalities, we show that (18) can be transported in $\mathcal{A}[(\mathcal{C}^\times)^{-1}]$ by means of the following commutative diagram and back.



(25)

Proof that $[1_\Omega, \log(z), \log(\frac{1}{1-z})]$ is $\mathcal{C}_\mathbb{C}$-free.

Let us suppose $P_i, i = 1 \ldots 3$ such that

$$P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z})) = 0_\Omega$$

We first prove that $P_2 = \sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i}$ is zero using the deck transformation D_0 of index one around zero.
One has $D_0^n(\sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i}) = \sum_{i\in F} c_i z^{\alpha_i}(1-z)^{\beta_i} e^{2i\pi.n\alpha_i}$, the same calculation holds for all P_i which proves that all $D_0^n(P_i)$ are bounded. But one has $D_0^n(\log(z)) = \log(z) + 2i\pi.n$ and then

$$D_0^n(P_1(z) + P_2(z)\log(z) + P_3(z)(\log(\frac{1}{1-z}))) =$$
$$D_0^n(P_1(z)) + D_0^n(P_2(z))(\log(z) + 2i\pi.n) + D_0^n(P_3(z))\log(\frac{1}{1-z}) = 0$$

It suffices to build a sequence of integers $n_j \to +\infty$ such that $\lim_{j\to\infty} D_0^{n_j}(P_2(z)) = P_2(z)$ which is a consequence of the following lemma.

Lemma

Let us consider a homomorphism $\varphi : \mathbb{N} \to G$ where G is a compact (Hausdorff) group, then it exists $u_j \to +\infty$ such that

$$\lim_{j\to\infty} \varphi(u_j) = e$$

Proof.

First of all, due to the compactness of G, the sequence $\varphi(n)$ admits a subsequence $\varphi(n_k)$ convergent to some $\ell \in G$. Now one can refine the sequence as n_{k_j} such that

$$0 < n_{k_1} - n_{k_0} < \ldots < n_{k_{j+1}} - n_{k_j} < n_{k_{j+2}} - n_{k_{j+1}} < \ldots$$

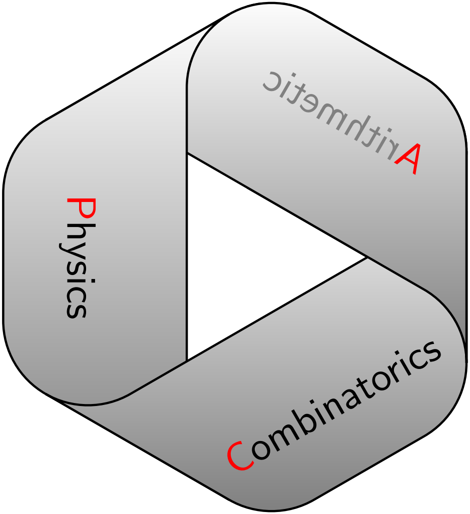
With $u_j = n_{k_{j+1}} - n_{k_j}$ one has $\lim_{j\to\infty} \varphi(u_j) = e$.
End of the proof One applies the lemma to the morphism

$$n \mapsto (e^{2i\pi.n\alpha_i})_{i\in F} \in \mathbb{U}^F$$

Conclusion

- For Series with variable coefficients, we have a theory of Noncommutative Evolution Equation sufficiently powerful to cover iterated integrals and multiplicative renormalisation
- Use of combinatorics on words gives a necessary and sufficient condition on the "inputs" to have linear independance of the solutions over higher function fields.
- Picard (Chen) solutions admit enlarged indexing w.r.t. compact convergence on Ω (polylogarithmic case) but Drinfeld's G_0 has a domain which includes only some rational series.
- Localization is possible (under certain conditions).
- Local BTT theorem allows to explore linear and algebraic independences w.r.t. subalgebras of $Dom(\mathrm{Li})$.

Thank you for your attention.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