Quantum determinants

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Plan

Let V be a vector space over the field $\mathbb{C}$ and P be the usual flip acting in $V^{\otimes 2}$ or its matrix.
Also, let $M = (m_i^j)$ be a numerical $N \times N$ matrix. Consider the system

$$P\, M_1\, M_2 - M_1\, M_2\, P = 0, \;\; M_1 = M \otimes I, \;\; M_2 = I \otimes M.$$

Note that $M_2 = P\, M_1\, P$ and consequently, this system can be cast under the form

$$P\, M_1\, P\, M_1 - M_1\, P\, M_1\, P = 0.$$

This system written via the entries reads

$$m_i^j\, m_k^l = m_k^l\, m_i^j, \;\; \forall\, i, j, k, l,$$

i.e. the entries commute with each other.

Example $N = 2$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$M_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad M_2 = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

The corresponding system reads

$$ab = ba, \; ac = ca, \ldots$$

Let us introduce some symmetric polynomials of M (namely, elementary ones and power sums)

$$\det(M - t\,I) = \sum_{0}^{N} (-t)^{N-k} e_k(M), \quad p_k(M) = Tr\, M^k.$$

If M is a triangular matrix these elements are respectively elementary symmetric polynomials and power sums in the eigenvalues μ_i of M. Namely, we have

$$e_k = \sum_{i_1<\ldots<i_k} \mu_{i_1}\ldots\mu_{i_k}, \quad p_k(M) = \sum \mu_i^k.$$

Also, note that these symmetric polynomials of M are related by the Newton identities

$$k\, e_k - p_1\, e_{k-1} + p_2\, e_{k-2} + \cdots + (-1)^k p_k\, e_0 = 0.$$

Together with the initial system $P\,M_1\,P\,M_1 - M_1\,P\,M_1\,P = 0$ consider its inhomogeneous analog

$$P\,M_1\,P\,M_1 - M_1\,P\,M_1\,P = P\,M_1 - M_1\,P.$$

In terms of the entries we have the relations

$$m_i^j\,m_k^l - m_k^l\,m_i^j = m_i^l\,\delta_k^j - m_k^j\,\delta_i^l,$$

which define the enveloping algebra $U(gl(N))$.

Note that if in the homogeneous (inhomogeneous) system we replace P by the super-flip $P_{m|n}$, we get the defining relations of the super-commutative algebra $Sym(gl(m|n))$ (resp., the enveloping algebra $U(gl(m|n))$).

Now, deform $P \to R$ in the corresponding systems–homogeneous and not. And do the same with the super-flip $P_{m|n}$. Namely, take R as follows (here $N = 2, \quad m = n = 1$)

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

Note that for $q \to 1$ we respectively recover the flip P and the super-flip $P_{1|1}$.

There exist two ways of deforming the system $P\,M_1\,P\,M_1 - M_1\,P\,M_1\,P = 0$ and its inhomogeneous analog. If we deform P into R everywhere we get

$$R\,M_1\,R\,M_1 - M_1\,R\,M_1\,R = 0.$$

$$R\,M_1\,R\,M_1 - M_1\,R\,M_1\,R = R\,M_1 - M_1\,R.$$

The first one will be called Reflection Equation (RE) algebra. The second one–modified RE algebra.

If we only deform P on the outside positions, we get

$$R\,M_1\,M_2 - M_1\,M_2\,R = 0.$$

This algebra is called RTT algebra (or Leningrad one).

Note that all these algebras make sense for some other braidings R.

We call an invertible linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$ *braiding* if it satisfies the so-called *braid relation*

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R \otimes I,\ \ R_{23} = I \otimes R.$$

Then the operator $\mathcal{R} = R\, P$ where P is the usual flip is subject to the QYBE

$$\mathcal{R}_{12}\, \mathcal{R}_{13}\, \mathcal{R}_{23} = \mathcal{R}_{23}\, \mathcal{R}_{13}\, \mathcal{R}_{12}.$$

A braiding R is called *involutive symmetry* if $R^2 = I$.
A braiding is called *Hecke symmetry* if it is subject to the Hecke condition

$$(q\,I - R)(q^{-1}\,I + R) = 0,\ \ q \in \mathbb{C},\ \ q \neq 0,\ \ q \neq \pm 1.$$

In particular, such a symmetry comes from the QG $U_q(sl(N))$. For $N = 2$ it is just the example above.

We assume q to be generic. This means that $k_q \neq 0$ for any integer k.

In order to classify Hecke symmetries, consider "R-symmetric" and "R-skew-symmetric" algebras

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \ \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle,$$

where $T(V)$ is the free tensor algebra. Also, consider the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \ P_-(t) = \sum_k dim \bigwedge\nolimits_R^{(k)}(V)t^k,$$

where the upper index (k) labels homogenous components of these quadratic algebras.
If R is involutive, we put $q = 1$ in these formulae.

Example

Let us compare two symmetries. The first one is Hecke coming from $U_q(sl(2))$, the second one is involutive:

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the first (resp., second) symmetry we have

$$Sym_R = T(V)/ < xy - qyx >, \bigwedge_R = T(V)/ < x^2, y^2, qxy + yx > .$$

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Observe that the algebras $Sym_R(V)$ are similar, but $\bigwedge_R(V)$ are not.

The following holds $P_-(-t)P_+(t) = 1$. Also, the both series are rational functions. Thus,

$$P_-(t) = \frac{N(t)}{D(t)}$$

We call the couple $(r|s)$, where r is the degree of $N(t)$ and s is the degree of $D(t)$ the *bi-rank*. It plays the role of the super-dimension $(m|n)$ while R is a super-flip.

Let us present any braiding R in a basis $\{x_i\} \in V$:

$$R(x_i \otimes x_j) = R_{ij}^{kl} x_k \otimes x_l.$$

We say that a braiding R is *skew-invertible* if there exists an operator $\Psi : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_2 R_{12} \Psi_{23} = P_{13} \quad \Leftrightarrow \quad R_{ij}^{kl} \Psi_{lp}^{jq} = \delta_i^q \delta_p^k,$$

Below, we assume all braidings to be skew-invertible.

For any such braiding R there exist two operators B and C such that

$$B\,C = \alpha\, I$$

and C is involved in defining the R-trace

$$Tr_R\, A = Tr\, C\, A$$

for any $N \times N$ matrix A and $B = (B_i^j)$ is involved in the pairing

$$< x^j, x_i >= B_i^j$$

where $\{x^j\}$ is the right dual basis of the dual space V^*:

$$< x_i, x^j >= \delta_i^j.$$

Now, let R be a skew-invertible involutive or Hecke symmetry of bi-rank $(m|0)$ (in this case R is called *even*). Then the homogenous component

$$\bigwedge\nolimits_R^{(m)}(V)$$

of the skew-symmetric algebra is the highest non-trivial one and its dimension is 1. Moreover, there exist two tensors

$$\mathfrak{u} = \|u_{i_1\dots i_m}\| \quad \text{and} \quad \mathfrak{v} = \|v^{j_1\dots j_m}\|,$$

such that the projector of skew-symmetrization

$$A^{(m)}: \; V^{\otimes m} \to \bigwedge\nolimits_R^{(m)}(V)$$

can be cast as follows

$$A^{(m)}(x_{i_1} \otimes \dots \otimes x_{i_m}) = u_{i_1\dots i_m}\, v^{j_1\dots j_m} x_{j_1} \otimes \dots \otimes x_{j_m},$$
$$\langle \mathfrak{v}, \mathfrak{u} \rangle := v^{i_1\dots i_m}\, u_{i_1\dots i_m} = 1.$$

Definition

In the RTT algebra the element

$$\det_{RTT}(L) := \langle \mathfrak{v}|L_1 \ldots L_m|\mathfrak{u}\rangle := v^{i_1\ldots i_m}\,(L_1\ldots L_m)^{j_1\ldots j_m}_{i_1\ldots i_m}\,u_{j_1\ldots j_m}, \qquad (1)$$

is called *quantum determinant* of the generating matrix L.

In the RE algebra the formula is similar but all indices are overlined

$$\det_{RE}(L) := \langle \mathfrak{v}|L_{\overline{1}} \ldots L_{\overline{m}}|\mathfrak{u}\rangle := v^{i_1\ldots i_m}\,(L_{\overline{1}}\ldots L_{\overline{m}})^{j_1\ldots j_m}_{i_1\ldots i_m}\,u_{j_1\ldots j_m}, \qquad (2)$$

We use the following notions

$$L_{\overline{2}} = R_{12}\,L_1\,R_{12}^{-1}, \quad L_{\overline{3}} = R_{23}\,L_{\overline{2}}\,R_{23}^{-1} \ldots .$$

Again consider the symetries

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \qquad (3)$$

The former symmetry is involutive, the latter one is a Hecke symmetry.

For the involutive symmetry we have

$$\mathfrak{u} = (u_{11}, u_{12}, u_{21}, u_{22}) = \frac{1}{2}(0, 1, -q^{-1}, 0),$$

$$\mathfrak{v} = (v^{11}, v^{12}, v^{21}, v^{22}) = (0, 1, -q, 0).$$

For the Hecke symmetry we have

$$\mathfrak{u} = \frac{1}{2_q}(0, q^{-1}, -1, 0), \quad \mathfrak{v} = (0, 1, -q, 0).$$

Observe that the tensors $\mathfrak{v}$ corresponding to these symmetries coincide with each other and, consequently, the algebras

$$Sym_R(V) = T(V)/\langle \mathfrak{v} \rangle = T(V)/\langle xy - qyx \rangle,$$

called the quantum plan, are the same for the both symmetries.

Nevertheless, the tensors $\mathfrak{u}$ are different. Consequently, the canonical forms of the corresponding determinants differ from each other.

Let us compute the determinants of the matrix

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

in the RTT and RE algebras corresponding to the both symmetries.

The defining relations of the RTT algebra, corresponding to the involutive matrix are

$$ab = q^{-1}ba, \quad ac = qca, \quad ad = da,$$

$$bc = q^2cb, \quad bd = qdb, \quad cd = q^{-1}dc.$$

The quantum determinant in this algebra is

$$\frac{ad + da}{2} - \frac{q^{-1}bc + qcb}{2} = ad - q^{-1}bc = da - qcb.$$

The defining relations in the RTT algebra corresponding to the Hecke matrix are

$$ab = qba, \quad ac = qca, \quad ad - da = (q - q^{-1})bc,$$

$$bc = cb, \quad bd = qdb, \quad cd = qdc.$$

The corresponding quantum determinant is

$$\frac{q^{-1}ad + qda}{2_q} - \frac{bc + cb}{2_q} = ad - qbc = da - q^{-1}cb.$$

Thus, we see that to the same quantum plane $xy - qyx$ there correspond different RTT algebras and different quantum determinants in dependence of the tensor $\mathfrak{u}$.

A similar claim is valid for the RE algebras corresponding to the above symmetries.

Now, introduce the corresponding Generalized Yangians (GY).
The famous Yang braiding is $R(u,v) = P - \frac{I}{u-v}$.

Proposition.

1. If R is an involutive symmetry, then

$$R(u,v) = R - \frac{a\,I}{u-v}$$

is an R-matrix, i.e. it meets the quantum Yang-Baxter equation

$$R_{12}(u,v)\,R_{23}(u,w)\,R_{12}(v,w) = R_{23}(v,w)\,R_{12}(u,w)\,R_{23}(u,v).$$

2. If $R = R(q)$ is a Hecke symmetry, then the same is valid for

$$R(u,v) = R(q) - \frac{(q-q^{-1})u\,I}{u-v}.$$

The Drinfeld's Yangian $\mathbf{Y}(gl(N))$ is in fact an RTT algebra defined by

$$R(u,v)\,T_1(u)\,T_2(v) = T_1(v)\,T_2(u)\,R(u,v)$$

with the Yang braiding and under a assumption that $T(u)$ is a series

$$T(u) = \sum_{k\geq 0} T[k]\,u^{-k}$$

and $T[0] = I$.

Introduce two types of GY in a similar manner.

1. Generalized Yangians of RTT type are defined by

$$R(u,v)\,T_1(u)\,T_2(v) = T_1(u)\,T_2(v)\,R(u,v),$$

where $R(u,v)$ is one of the above current braidings.
2. GY of RE type (also called braided Yangians) are defined by

$$R(u,v)L_{\overline{1}}(u)\,L_{\overline{2}}(v) = L_{\overline{1}}(v)\,L_{\overline{2}}(u)\,R(u,v).$$

Here $L_{\overline{2}} = R\,L_{\overline{1}}\,R^{-1}$.

If a braiding $R(u,v)$ arises from an involutive symmetry R, the corresponding GY $Y(R,P)$ is called *rational*. If R is Hecke, then $Y(R,R)$ is called *trigonometrical*.

If R has the bi-rank $(m|0)$, we define quantum determinants in the rational (resp., trigonometrical) GY of RE type as follows

$$\det_{rat}(L(u)) = <\mathfrak{v}|L_{\overline{1}}(u)\, L_{\overline{2}}(u-1)\dots L_{\overline{m}}(u-m+1)|\mathfrak{u}>,$$

$$\det_{trig}(L(u)) = <\mathfrak{v}|L_{\overline{1}}(u)\, L_{\overline{2}}(q^{-2}u)\dots L_{\overline{m}}(q^{-2(m-1)}u)|\mathfrak{u}> .$$

Thus, the determinants are defined by formulae similar to those above but with shifts in arguments of the matrices $L(u)$, additive in the rational cases and multiplicative in the trigonometrical ones.

Now, we want to study the characteristic polynomials $ch(t)$, i.e. such that $ch(L) = 0$. Whether it is true that

$$ch(t) = \det(L - t\, I)\ ?$$

The answer is positive if the algebra is a RE one and R is an involutive symmetry. If R is Hecke then

$$ch(t) := t^m - qe_1\, t^{m-1} + q^2 e_2\, t^{m-2} + ... + (-q)^{m-1} e_{m-1}\, t + (-q)^m e_m.$$

Thus, by putting $t = L$ we get the Cayley-Hamilton identity for the matrix L:

$$L^m - qe_1\, L^{m-1} + q^2 e_2\, L^{m-2} + ... + (-q)^{m-1} e_{m-1}\, L + (-q)^m e_m\, I = 0.$$

In RTT algebras characteristic polynomials for the generating matrices do not exist.

Below, we deal with RE algebras.
Note that this Cayley-Hamilton identity for the generating matrices of these algebras provides us with a method of inverting the generating matrix L. Upon dividing the Cayley-Hamilton identity by the element $(-q)^m L e_m$ we can express the matrix L^{-1} as a linear combination of the powers sums L^k, $0 \le k \le m-1$ with coefficients e_k/e_m (observe that all elements e_k are central in the RE algebra).

Now, consider the case of general symmetries R (not necessary even).

If a given symmetry R has the bi-rank $(m|n)$ $n \neq 0$, the generating matrix L of the RE algebra also meets the Cayley-Hamilton identity

$$a_0\, L^{m+n} - a_1\, L^{m+n-1} + \dots (-1)^{m+n}\, a_{m+n}\, I = 0,$$

where all the coefficients a_k belong to the center of the RE algebra. Note that in this case the leading coefficient a_0 does not equal 1. Upon dividing this relation by $a_{m+n}\, L$, we can express the matrix L^{-1} as a linear combinations of the matrices L^k, $0 \le k \le m+n-1$ with coefficients $\pm\, a_k/a_{m+n}$.

Observe that for any Schur diagram (partition) $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ there exists an analog of the Schur functor $V \mapsto V_\lambda$ and the corresponding Schur polynomial p_λ. It is a polynomial in NC generators l_i^j. However, the Littlewood-Richardson table of multiplication is still valid:

$$p_\lambda \, p_\mu = \sum_\nu c_{\lambda,\mu}^\nu p_\nu$$

The coefficient a_0 is the Schur polynomial corresponding to the $m \times n$ rectangle, i.e. $\lambda = (n^m)$. Whereas the coefficient a_{m+n} is the Schur polynomial corresponding to the partition $((n+1)^m, n)$. Then we assign the meaning of the quantum determinant to the fraction a_{m+n}/a_0 (up to a sign).
Note that the quantum determinant differs from the quantum Berezinian, which also can be realized as a fraction of two Schur polynomials.

In order to exhibit the quantum determinant and Berezinian in a more explicit form we introduce "eigenvalues" of the characteristic polynomial.

Observe that after having multiplied the CH identity by a_0 we can factorize the CH in two factors: "even" and "odd". Denote $\mu_1, ..., \mu_m$ the roots of the even factor and $\nu_1, ...\nu_n$ these of the odd one. Then the quantum determinant is (up to a factor)

$$\prod_{k=1}^{m} \mu_k \prod_{k=1}^{n} \nu_k$$

whereas the quantum Berezinian is

$$\frac{\prod_{k=1}^{m} \mu_k}{\prod_{k=1}^{n} \nu_k}.$$

Let us express the power sums via the eigenvalues:

$$Tr_R L^k = \sum_{i=1}^{m} d_i \mu_i^k + \sum_{j=1}^{n} \tilde{d}_j \nu_j^k$$

where the coefficients d_i and $\tilde{d}_j$ explicitly read

$$d_i = q^{-1} \prod_{\substack{p=1 \\ p\neq i}}^{m} \frac{\mu_i - q^{-2}\mu_p}{\mu_i - \mu_p} \prod_{j=1}^{n} \frac{\mu_i - q^2\nu_j}{\mu_i - \nu_j}, \qquad (4)$$

$$\tilde{d}_j = -q \prod_{i=1}^{n} \frac{\nu_j - q^{-2}\mu_i}{\nu_j - \mu_i} \prod_{\substack{p=1 \\ p\neq j}}^{n} \frac{\nu_j - q^2\nu_p}{\nu_j - \nu_p}. \qquad (5)$$

Stress that our quantum determinants are defined only for the generating matrices of some QMA. A more general method of defining an analog of the determinant is based on using the so-called quasideterminants [GR].

Again, consider a matrix

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which generate the RE or RTT algebra, corresponding to the Hecke symmetry above.

In this case there exist four different quasideterminants, associated with L. We consider two of them:

$$\text{Q-det}_{11}(L) = a - bd^{-1}c, \qquad \text{Q-det}_{22}(L) = d - ca^{-1}b.$$

We get the corresponding candidates for the role of determinants if we respectively multiply these quasideterminants by d and a from the left or from the right. Then we get the determinants which are equal to the determinant $\det_{RTT}(L)$ in the RTT algebra, provided R is the Hecke matrix above.

As for the corresponding RE algebra, only $\text{Q-det}_{22}(L)$ leads to the determinant $\det_{RE}(L)$ above, whereas the determinant, arising in a similar way from $\text{Q-det}_{11}(L)$, is not equal to $\det_{RE}(L)$.

Many thanks