Z_3-GRADED EXTENSION OF LORENTZ-POINCARE ALGEBRA

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- We propose a modification of standard QCD description of the colour triplet of quarks by introducing a 12-**component** colour generalization of Dirac spinor, with built-in Z_3 **grading** playing an important algebraic role in quark confinement.
- In "**colour Dirac equations**" the $SU(3)$ colour symmetry is entangled with the Z_3-**graded** generalization of Lorentz symmetry, containing three 6-**parameter** sectors related by Z_3-**graded** maps.
- The generalized Lorentz covariance requires simultaneous presence of 12 **colour Dirac multiplets**, which lead to the description of all internal symmetries of quarks: besides $SU(3) \times SU(2) \times U(1)$, the flavour symmetries and three quark families.

- **Let us denote by j and j^2 the two complex third roots of unity, given by**

$$j = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad j^2 = e^{\frac{4\pi i}{3}} = -\frac{1}{2} - \frac{i\sqrt{3}}{2} \qquad (1)$$

satisfying obvious identities $1 + j + j^2 = 0$, so that $j + j^2 = -1, \; j - j^2 = i\sqrt{3}$,

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- The six S_3 symmetry transformations contain the identity, two rotations, one by 120^o, another one by 240^o, and three reflections, in the x-axis, in the j-axis and in the j^2-axis. The Z_3 subgroup contains only the three rotations. Odd permutations must be represented by idempotents, whose square is the identity operation.

Let us recall briefly the properties of the cyclic (Z_3) and the permutation (S_3) groups of three elements. Their representation in terms of rotations and reflections in the complex plane are shown in the following Figure 1:

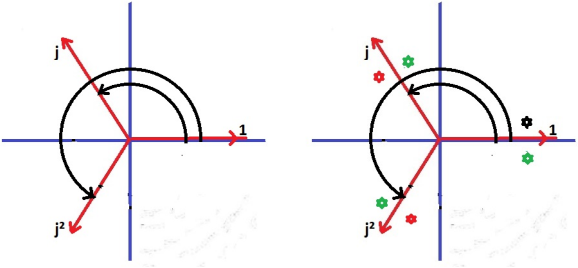

Figure: Rotations (Z_3-group) and reflections added (S_3 group)

- In what follows, we shall use the Z_3 **group** for grading of linear spaces and matrix algebras. The Z_3-**graded** algebras are composed of three vector subspaces, one of which (of Z_3-**grade zero**) constitutes a subalgebra in the ordinary sense:

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \tag{2}$$

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- The multiplication in the graded algebra **(2)** obeys the following scheme:

$$\overset{(r)}{\mathcal{A}} \cdot \overset{(s)}{\mathcal{A}} \subset \overset{(r+s)|_3}{\mathcal{A}}, \qquad (3)$$

with $r, s, .. = 0, 1, 2, \;\; (r+s)\,|_3 = (r+s) \text{ modulo } 3.$

- **The Z_3 symmetry can be combined with the Z_2 symmetry; 3 and 2 being prime numbers, the Cartesian product of the two is isomorphoic with another cyclic group,**

$$Z_3 \times Z_2 = Z_6$$

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- **The generalized Dirac equation is invariant under the discrete group $Z_3 \times Z_2 \times Z_2 \simeq Z_6 \times Z_2$ (which is not isomorphic with Z_{12} because 6, being divisible by 2 and by 3, is not a prime number).**

The cyclic group Z_6 is represented in the complex plane by its generator $q = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}}$, and its powers from 1 to 6. In terms of the Z_3 group generated by j and Z_2 group generated by -1, we have $q = -j^2, \;\; q^2 = j, \; q^3 = -1, \; q^4 = j^2, \; q^5 = -j, \; q^6 = 1$, as shown in the figure (2) below.

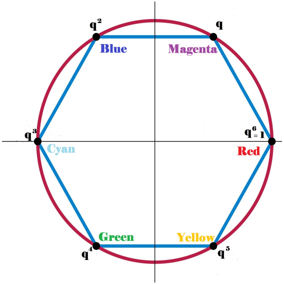

Figure: The six complex numbers q^k can be put into correspondence with three colours and three anti-colours.

- In analogy with colours labeling quark fields, if the **"white"** combination is represented by 0, then we have *two* linear colourless sums of three powers of q, namely

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- and *three* white combinations of colour with its anti-colour,

$$q + q^4 = 0, \; q^2 + q^5 = 0, \; q^3 + q^6 = 0,$$

just like a fermion and its antiparticle, or three bosons (like e.g. mesons π^0, π^+ and π^-).

A Z_3-graded analog of Pauli's exclusion principle and the Z_3-graded Dirac's equation ere introduced in our papers in 2017, 2018, 2019.

R. Kerner, *Ternary generalization of Pauli's principle and the Z_6-graded algebras*, *Physics of Atomic Nuclei*, **80** (3), pp. 529-531 (2017). also: arXiv:1111.0518, arXiv:0901.3961
R. Kerner, *Ternary $Z_2 \times Z_3$ graded algebras and ternary Dirac equation*, *Physics of Atomic Nuclei* **81** (6), pp. 871-889 (2018), also: arXiv:1801.01403
R. Kerner, *The Quantum nature of Lorentz invariance*, *Universe*, **5** (1), p.1, (2019). https://doi.org/10.3390/universe5010001 (2019).
R. Kerner and J. Lukierski, *Z_3-graded colour Dirac equation for quarks, confinementt and generalized Lorentz symmetries*, *Phys. Letters B*, Vol. 792, pp. 233-237 (2019), also: arXiv:1901.10936 [hep-th]

- After the discovery of spin of the electron (the Stern-Gerlach experiment,), Pauli understood that a Schroedinger equation involving only one complex-valued wave function is not enough to take into account this new degree of freedom.

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- He proposed then to describe the dichotomic spin variable by introducing a two-component function forming a column on which hermitian matrices can act as linear operators.

- **The basis of complex traceless 2×2 hermitian matrices contains just three elements since then known as** ***Pauli matrices*****:**

- **The basis of complex traceless 2×2 hermitian matrices contains just three elements since then known as *Pauli matrices*:**
- $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3].$$

- **The three Pauli matrices multiplied by $\frac{i}{2}$ span the three dimensional Lie algebra: let $\tau_k = \frac{i}{2}\sigma_k$, then**

$$[\tau_1, \tau_2] = \tau_3, \quad [\tau_2, \tau_3] = \tau_1, \quad [\tau_3, \tau_1] = \tau_2.$$

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- On the other hand, the three Pauli matrices form the Clifford algebra related to the Euclidean 3-dimensional metric:

$$\sigma_i \sigma_k + \sigma_k \sigma_i = 2\delta_{ik} \mathbb{1}_2$$

ensuring that

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mid \mathbf{p} \mid^2 \mathbb{1}_2.$$

The simplest linear relation between the operators of energy, mass and momentum acting on a column vector (called *a Pauli spinor*) would read then:

$$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & mc^2 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} + c\, \boldsymbol{\sigma} \cdot \mathbf{p} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \qquad (4)$$

where

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \sigma_1\, p^1 + \sigma_2\, p^2 + \sigma_3\, p^3 = \begin{pmatrix} p^3 & p^1 - i\, p^2 \\ p^1 + i\, p^2 & -p^3 \end{pmatrix}.$$

We can write (4) in a simplified manner, denoting the Pauli spinor by one letter ψ and treating the unit matrix symbolically like a number:

$$E\,\psi = mc^2\,\psi + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi. \tag{5}$$

This equation is not invariant under Lorentz transformations. Indeed, by iterating, i.e. taking the square of this operator, we arrive at the following relation between the operators of energy and momentum and the mass of the particle:

$$E^2 = m^2 c^4 + 2\,mc^3\mid \mathbf{p}\mid^2 \boldsymbol{\sigma}\cdot\mathbf{p} + c^2\mathbf{p}^2, \tag{6}$$

instead of the relativistic relation

$$E^2 - c^2\,\mathbf{p}^2 = m^2 c^4. \tag{7}$$

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- So let us denote the first Pauli spinor by ψ_+ and the second one by ψ_-, and let them satisfy the following coupled system of equations:

$$E\,\psi_+ = mc^2\,\psi_+ + \boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,$$

$$E\,\psi_- = -mc^2\,\psi_- + \boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+, \tag{8}$$

(by the way, here $-1 = e^{i\pi}$, a complex number!)

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- which coincides with the relativistic equation for the electron found by Dirac a few years later.

Therefore the standard Dirac equation for the electron (or any **spin $\frac{1}{2}$** particle with non-zero mass m) may be interpreted as a pair of coupled equations involvng two Pauli spinors,

$$\psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix} \quad \text{and} \quad \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix},$$

$$E\psi_+ = mc^2\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\;\psi_-,$$
$$E\psi_- = -mc^2\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\;\psi_+,$$

where as usual

$$E = -i\hbar\,\partial_t, \quad \mathbf{p} = -i\hbar\,\textbf{grad}$$

- The relativistic invariance is now manifest: due to the negative mass term in the second equation, the iteration leads to the separation of variables, and all the components satisfy the desired relation

$$[E^2 - c^2\,\mathbf{p}^2]\psi_+ = m^2 c^4\ \psi_+, \quad [E^2 - c^2\,\mathbf{p}^2]\,\psi_- = m^2 c^4\ \psi_-.$$

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- In a more appropriate basis the Dirac equation becomes manifestly relativistic: $[\gamma^\mu p_\mu - mc]\;\psi = 0$, with $p_0 = \frac{E}{c}$,

$$\gamma^0 = \sigma_3 \otimes \mathbb{1}_2 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = (i\sigma_2)\otimes\sigma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}.$$

It can be written in a compact way as follows:

$$\gamma^{\mu}\, p_{\mu}\, \psi = mc\, \psi \quad \text{with} \;\; \psi = (\psi_+, \psi_-)^T \,, \tag{9}$$

where $p_{\mu}{=}{-}i\hbar\partial_{\mu}$, $\psi_{\pm}$ are two complex 2-component Pauli spinors, and as Dirac matrices γ^{μ} one can choose

$$\gamma^0{=}\sigma_3 \otimes \mathbb{1}_2, \qquad \gamma^k{=}(i\sigma_2) \otimes \sigma^k, \tag{10}$$

where $\sigma_0 = \mathbb{1}_2$, and σ^k $(k{=}1,2,3)$ are Pauli matrices. The Dirac matrices realize the 4-dimensional Clifford algebra

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}{=}2\,\eta^{\mu\nu}\mathbb{1}_4, \quad \eta^{\mu\nu}{=}\mathrm{diag}(+,-,-,-). \tag{11}$$

- **Under the Lorentz transformation**

$$x^{\mu} \to x^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, x^{\nu} \tag{12}$$

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- **the spinor field** $\psi = \psi^A$ $(A = 1, 2, 3, 4)$ **transforms as follows:**

$$\psi'(x^{\rho'}) = \psi'(\Lambda^{\rho'}_{\ \mu} x^{\mu}) = S\psi(x^{\mu})\ . \tag{13}$$

In order to ensure the standard Lorentz covariance, the condition relating the vectorial and spinorial realizations of the Lorentz group $O(3,1) \simeq SL(2,\mathbf{C})$ is:

$$S\gamma^{\mu'}S^{-1} = \Lambda^{\mu'}{}_{\nu}(S)\gamma^{\nu} \; . \tag{14}$$

The spinorial representation S is given by the formula

$$S = exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) , \tag{15}$$

- where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and the corresponding infinitesimal vectorial representation is given by the formula

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \text{ where} \tag{16}$$

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$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \text{ where} \tag{16}$$

- $$\omega_{\mu\nu} = \eta_{\mu\lambda}\, \omega^\lambda{}_\nu = -\omega_{\nu\mu}. \tag{17}$$

with three independent Lorentz boosts $(\omega_{0k} = -\omega_{k0})$ and three independent spatial rotations $(\omega_{ij} = -\omega_{ji})$).

The generalized Dirac equation incorporating colour degrees of freedom in a Z_3-symmetric way was proposed in publications cited above; after introducing three pairs of independent Pauli spinors

$$\varphi_+ = \begin{pmatrix} \varphi_+^1 \\ \varphi_+^2 \end{pmatrix}, \;\; \varphi_- = \begin{pmatrix} \varphi_-^1 \\ \varphi_-^2 \end{pmatrix}, \;\; \chi_+ = \begin{pmatrix} \chi_+^1 \\ \chi_+^2 \end{pmatrix},$$

$$\chi_- = \begin{pmatrix} \chi_-^1 \\ \chi_-^2 \end{pmatrix}, \;\; \psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix}, \;\; \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix}. \tag{18}$$

with Pauli sigma-matrices acting on them in a natural way.

- These three Pauli spinors φ_+, χ_+ and ψ_+ are conventionally named "red", "blue" and "green", while their antiparticle counterparts φ_-, χ_- and ψ_- are called, respectively, "cyan", "yellow" and "magenta".

- These three Pauli spinors φ_+, χ_+ and ψ_+ are conventionally named **"red"**, **"blue"** and **"green"**, while their antiparticle counterparts φ_-, χ_- and ψ_- are called, respectively, **"cyan"**, **"yellow"** and **"magenta"**.
- The cyclic group Z_3 is represented on the complex plane by multiplicative group of three complex numbers, generated by powers of $j = e^{\frac{2\pi i}{3}}$, namely:

$$j = e^{\frac{2\pi i}{3}}, \;\; j^2 = e^{\frac{4\pi i}{3}}, \;\; j^3 = 1, \;\; 1 + j + j^2 = 0. \qquad (19)$$

The resulting system of equation is as follows:

$$
\begin{aligned}
E\,\varphi_+ &= mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_-,\\
E\,\chi_- &= -j\;mc^2\,\chi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+,\\
E\,\psi_+ &= j^2\;mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_-,\\
E\,\varphi_- &= -mc^2\,\varphi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_+\\
E\,\chi_+ &= j\;mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,\\
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\end{aligned}
\tag{20}
$$

The color content is better seen in the following alternative basis:

$$
\begin{aligned}
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E\,\chi_+ &= j\;mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,\\
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E\,\psi_+ &= j^2\;mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_-,\\
E\,\psi_- &= -j^2\;mc^2\,\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,.\varphi_+
\end{aligned}
\tag{21}
$$

- **The particle-antiparticle Z_2-symmetry appears as $m \to -m$ and simultaneously $(\varphi_+, \chi_+, \psi_+) \to (\varphi_-, \chi_-, \psi_-)$ and vice versa; the Z_3-colour symmetry is realized by multiplication of mass m by j each time the colour changes, i.e. more explicitly, Z_3 symmetry is realized as follows:**

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-

$$m \to jm, \quad \varphi_\pm \to \chi_\pm \to \psi_\pm \to \varphi_\pm, \tag{22}$$

$$m \to j^2 m, \quad \varphi_\pm \to \psi_\pm \to \chi_\pm \to \varphi_\pm, \tag{23}$$

- The energy operator is obviously diagonal, and its action on the spinor-valued column-vector can be represented as a 6×6 operator valued unit matrix. The mass operator is diagonal, too, but its elements represent all powers of the **sixth root of unity** $q = e^{\frac{2\pi i}{6}}$, which are

$$q = -j^2, \; q^2 = j, \; q^3 = -1, \; q^2 = j^2, \; q^5 = -j \text{ and } q^6 = 1.$$

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$$q = -j^2,\ q^2 = j,\ q^3 = -1,\ q^2 = j^2,\ q^5 = -j \text{ and } q^6 = 1.$$

- The system (21) was formulated in a basis in which the "coloured" Pauli spinors alternate with their antiparticles; however, if we want to put forward the colour content, it is better to choose an alternative basis in the space of spinors arranged as follows:

$$(\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-)^T. \tag{24}$$

Then the mass and momentum operators take on the following form:

$$M = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & -m & 0 & 0 & 0 & 0 \\ 0 & 0 & jm & 0 & 0 & 0 \\ 0 & 0 & 0 & -jm & 0 & 0 \\ 0 & 0 & 0 & 0 & j^2 m & 0 \\ 0 & 0 & 0 & 0 & 0 & -j^2 m \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 \\ 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ 0 & 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 & 0 \\ \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- **The dimension of the two matrices M and P displayed above is 12×12: all the entries in the first one are proportional to the 2×2 identity matrix, so that in the definition one should read $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ instead of m, $\begin{pmatrix} jm & 0 \\ 0 & jm \end{pmatrix}$ instead of $j\,m$, etc.**

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- **The entries in the second matrix P contain 2×2 Pauli's sigma-matrices, so that P is also a 12×12 matrix. The energy operator E is proportional to the 12×12 identity matrix.**

- Only even powers of σ-matrices are proportional to $\mathbb{1}_2$, and only the powers of 3×3 circulant matrix that are multiplicities of 3 are proportional to $\mathbb{1}_3$.
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  The diagonalization of the system is achieved only at the sixth iteration. The final result is extremely simple: all the components satisfy the same sixth-order equation,
- $$E^6 \, \varphi_+ = m^6 c^{12} \, \varphi_+ + c^6 \mid \mathbf{p} \mid^6 \, \varphi_+,$$
  $$E^6 \, \varphi_- = m^6 c^{12} \, \varphi_- + c^6 \mid \mathbf{p} \mid^6 \, \varphi_-. \tag{25}$$

  and similarly for all other components.

Using a more rigorous approach the three operators can be expressed in terms of tensor products of matrices of lower dimensions. Let us introduce two following 3×3 matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad \text{and} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{26}$$

whose products and powers generate the $U(3)$ Lie group algebra, or the $SU(3)$ algebra if we remove the unit matrix.

The 12×12 matrices M and P can be represented as the following tensor products:

$$M = m\, B \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad P = Q_3 \otimes \sigma_1 \otimes (\boldsymbol{\sigma} \cdot \mathbf{p}) \tag{27}$$

with as usual,

$$\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Let us rewrite the matrix operator generating the system **(21)** when it acts on the column vector containing twelve components of three "colour" fields, in the basis **(24)** $[\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-]$:

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-

$$E \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 = mc^2 \; B \otimes \sigma_3 \otimes \mathbb{1}_2 + Q_3 \otimes \sigma_1 \otimes c \, \boldsymbol{\sigma} \cdot \mathbf{p}$$

with energy and momentum operators on the left hand side, and the mass operator on the right hand side:

$$E\mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_2 - Q_3 \otimes \sigma_1 \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p} = mc^2\; B \otimes \sigma_3 \otimes \mathbb{1}_2 \tag{28}$$

Like with the standard Dirac equation, let us transform this equation so that the mass operator becomes proportional the the unit matrix. To do so, we multiply the equation (28) from the left by the matrix $B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$.

- Now we get the following equation which enables us to interpret the energy and the momentum as the components of a Minkowskian four-vector $c\, p^{\mu} = [E,\ c\mathbf{p}]$:

$$E\ B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes c\,\boldsymbol{\sigma} \cdot \mathbf{p} = mc^2 \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \quad (29)$$

where we used the fact that under matrix multiplication, $\sigma_3 \sigma^3 = \mathbb{1}_2$, $B^{\dagger} B = \mathbb{1}_3$ and $B^{\dagger} Q_3 = Q_2$.

- Now we get the following equation which enables us to interpret the energy and the momentum as the components of a Minkowskian four-vector $c\, p^\mu = [E,\ c\mathbf{p}]$:

$$E\ B^\dagger \otimes \sigma_3 \otimes 1\!\!1_2 - Q_2 \otimes (i\sigma_2) \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p} = mc^2 1\!\!1_3 \otimes 1\!\!1_2 \otimes 1\!\!1_2, \quad (29)$$

where we used the fact that under matrix multiplication, $\sigma_3\sigma^3 = 1\!\!1_2$, $B^\dagger B = 1\!\!1_3$ and $B^\dagger Q_3 = Q_2$.

- The sixth power of this operator gives the same result as before,

$$\left[E\ B^\dagger \otimes \sigma_3 \otimes 1\!\!1_2 - Q_2 \otimes (i\sigma_2) \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p}\right]^6 = \left[E^6 - c^6 \mathbf{p}^6\right] 1\!\!1_{12}$$
$$= m^6 c^{12} 1\!\!1_{12} \quad (30)$$

- The equation **(29)** can be written in a concise manner using the Minkowskian indices and the usual pseudo-scalar product of two four-vectors as follows:

$$\Gamma^{\mu} p_{\mu} \, \Psi = mc \mathbb{1}_{12} \, \Psi, \quad \text{with} \;\; p^0 = \frac{E}{c}, \;\; p^k = [\, p^x, p^y, p^z \,]. \tag{31}$$

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- with 12×12 matrices $\Gamma^{\mu}, \ \ (\mu = 0, 1, 2, 3)$ defined as follows:

$$\Gamma^0 = B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad \Gamma^k = Q_2 \otimes (i\sigma_2) \otimes \sigma^k \tag{32}$$

where Ψ is the generalized 12-component spinor made of 6 Pauli spinors where

$$B^{\dagger} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \tag{33}$$

The two traceless matrices B and Q_2 are both cubic roots of unit 3×3 matrix. They generate the entire Lie algebra of the $SU(3)$ **group**.

The set of six matrices Q_A and $Q^\dagger_B$, $A, B = 1, 2, 3$, together with two diagonal traceless matrices B and $B^\dagger$ generated by B and Q_3 form a special basis of the $SU(3)$ algebra V.G.Kac in 1994. They can be obtained by iteration, using the following multiplication table:

$$\begin{gathered} BQ_A{=}j^2 Q_A B{=}Q_{A+1}, \quad B^\dagger Q_A{=}jQ_A B^\dagger{=}Q_{A-1}, \\ Q^\dagger B{=}j^2 BQ^\dagger_A{=}Q^\dagger_{A-1}, \quad Q^\dagger_A B^\dagger{=}B^\dagger Q^\dagger_A{=}Q^\dagger_{A+1}, \\ Q_A Q_{A-1}{=}jQ^\dagger_{A+1}, \quad Q^\dagger_{A-1} Q^\dagger_A{=}j^2 Q_{A+1}, \\ Q_A Q^\dagger_{A+1}{=}B^\dagger, \quad Q_A Q^\dagger_{A-1}{=}B, \quad Q^\dagger_A Q_{A-1}{=}jB^\dagger, \quad Q^\dagger_A Q_{A+1}{=}j^2 B. \end{gathered} \tag{34}$$

Also $Q_A Q^\dagger_A = Q^\dagger_a Q_A = \mathbb{1}_3$. **where the indices** $A,\ A+1,\ A-1$ **are always taken** modulo 3, **so that e.g.** $3+1\ |_{modulo\ 3} = 4\ |_{modulo\ 3} = 1$, **etc., and the cube of each of the eight matrices in** (34) **is the unit** 3×3 **matrix**.

- **The 12×12 matrices Γ^μ appearing in the coloured Dirac equation do not span 4-dimensional Clifford algebra. In fact, the $Z_3 \otimes Z_2$ structure of Γ^μ-matrices implies that only their sixth powers are proportional to the unit matrix $\mathbb{1}_{12}$**

- The 12×12 matrices Γ^μ appearing in the coloured Dirac equation do not span 4-dimensional Clifford algebra. In fact, the $Z_3 \otimes Z_2$ structure of Γ^μ-matrices implies that only their sixth powers are proportional to the unit matrix$\mathbb{1}_{12}$
- Thus, in order to obtain the realization of $D = 4$ **Lorentz algebra generators** one can not use just two standard commutators

$$J_i = \frac{i}{2}\, \epsilon_{ijk} \left[\Gamma^j, \Gamma^k\right], \quad K_l = \frac{1}{2}\ [\Gamma_l, \Gamma_0]\ . \qquad (35)$$

However, the generators $\left(J_i^{(0)},\ K_l^{(0)}\right)$ satisfying the standard Lorentz algebra relations for $r = 0, s = 0$) can be defined by triple commutators:

$$\begin{aligned} [J_i,[J_j,J_k]] &= (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})\ J_l^{(0)}, \\ [K_i,[K_j,K_k]] &= (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})\ K_l^{(0)}\ . \end{aligned} \tag{36}$$

Indeed, substituting in (36) the explicit form of Γ^μ we get

$$\begin{aligned} J_i &= -\frac{i}{2}\, Q_2^\dagger \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_l = -\frac{1}{2}\, Q_1 \otimes \sigma_1 \otimes \sigma_l, \\ J_i^{(0)} &= -\frac{i}{2} \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_l^{(0)} = -\frac{1}{2} \mathbb{1}_3 \otimes \sigma_1 \otimes \sigma_l. \end{aligned} \tag{37}$$

In order to close the generalized Lorentz algebra where $L^{(0)}{=}(J_i^{(0)}, K_j^{(0)}), \quad L^{(1)}{=}(J_i^{(1)}, K_j^{(1)}), \quad L^{(2)}{=}(J_i^{(2)}, K_j^{(2)})$, one should supplement (36) by two missing triple commutators:

$$[J_i, [J_j, K_k]] = (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) \; K_l^{(2)}, \qquad (38)$$
$$[K_i, [K_j, J_k]] = (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl}) \; J_l^{(1)},$$

where using the representation **(37)** we get

$$J_l^{(1)} = -\frac{i}{2}\, Q_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K_i^{(2)} = -\frac{1}{2}\, Q_3^{\dagger} \otimes \sigma_1 \otimes \sigma_i. \tag{39}$$

The full set of Z_3-graded relations defining the algebra ($r, s,\ r+s$ are modulo 3) are:

$$\left[J_i^{(r)}, J_k^{(s)}\right] = \epsilon_{ikl} J_l^{(r+s)}, \quad \left[J_i^{(r)}, K_k^{(s)}\right] = \epsilon_{ikl} K_l^{(r+s)}, \\ \left[K_i^{(r)}, K_k^{(s)}\right] = -\epsilon_{ikl} J_l^{(r+s)}. \tag{40}$$

From the commutators $[K_i^{(1)}, K_m^{(1)}] \simeq J^{(2)}$ and $[J^{(1)}, J^{(1)}] \simeq J^{(2)}$ **one gets the remaining generators of** $\mathcal{L}$:

$$J_i^{(2)} = -\frac{i}{2}\, Q_3^{\dagger} \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_m^{(1)} = -\frac{1}{2}\, Q_3 \otimes \sigma_1 \otimes \sigma_m. \qquad (41)$$

These formulae describe the realization of $\mathcal{L}$ **which follows from the choice of matrices** Γ^{μ}.

- **Let us introduce the following notation:**

$$\Gamma^{\mu}_{(A;\alpha)} = I_A \otimes \sigma_\alpha \otimes \sigma^\mu, \; A = 0, 1, .., 8; \; \alpha = 2, 3; \; \mu = 0, 1, 2, 3 \; . \tag{42}$$

Let the 3×3 "colour matrices" I_A appearing as the first factor be defined as follows:

$I_0 = \mathbb{1}_3, \; I_r = Q_r, \; I_{r+3} = Q_r^\dagger, \; I_7 = B, \; I_8 = B^\dagger.$

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- Then the original Γ-matrices encoded as $\Gamma^0_{(8,3)} = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$ and $\Gamma^i_{(2;2)} = Q_2 \otimes (i\sigma_2) \otimes \sigma^i$. The eight matrices with $A = 1,2,...8$ with the multiplication rules given above span the ternary basis, generated by the cyclic Z_3-automorphism of the $SU(3)$ algebra.

- **In order to get a closed formula for the action** $\mathcal{S}^{(0)}\Gamma^{\mu}[\mathcal{S}^{(0)}]^{-1}$ **of classical spinorial Lorentz symmetries generated by** $L^{(0)}$**, we should introduce the pairs of** Γ^{μ}-matrices $\Gamma^{\mu} = (\Gamma^{i}_{(A;2)},\ \Gamma^{0}_{(B;3)})$ and $\tilde{\Gamma}^{\mu} = (\Gamma^{i}_{(B;2)},\ \Gamma^{0}_{(A;3)})$, $A \neq B$.

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$\Gamma^{\mu} = (\Gamma^{i}_{(A;2)},\ \Gamma^{0}_{(B;3)})$ and $\tilde{\Gamma}^{\mu} = (\Gamma^{i}_{(B;2)},\ \Gamma^{0}_{(A;3)})$, $A \neq B$.
- **For any choice of** Γ^{μ}**'s in (42) we get:**

$$\left[J_i^{(0)}, \Gamma^{j}_{(A;\alpha)}\right] = \epsilon_{ijk}\Gamma^{k}_{(A;\alpha)}, \quad \left[J_i^{(0)}, \Gamma^{0}_{(A;\alpha)}\right] = 0, \tag{43}$$

The boosts $K_i^{(0)}$ act covariantly on doublets $\left(\Gamma^\mu, \tilde{\Gamma}^\mu\right)$ as follows:

$$\left[K_i^{(0)}, \Gamma^j_{(A;2)}\right] = \delta_i^j \, \Gamma^0_{(A;3)}, \quad \left[K_i^{(0)}, \Gamma^0_{(B;3)}\right] = \Gamma^i_{(B;2)},$$

$$\left[K_i^{(0)}, \Gamma^j_{(B;2)}\right] = \delta_i^j \, \Gamma^0_{(B;3)}, \quad \left[K_i^{(0)}, \Gamma^0_{(A;3)}\right] = \Gamma^i_{(A;2)}, \qquad (44)$$

(with $A \neq B$), i.e. the standard Lorentz covariance requires the *doublet* **of coloured Dirac spinors;)**

- **In particular, the Γ^{μ} matrices should be supplemented by:**

$$\tilde{\Gamma}^0 = \Gamma^0_{(2;3)} = Q_2 \otimes (\sigma_3) \otimes \mathbb{1}_2, \quad \tilde{\Gamma}^i = \Gamma^k_{(8;2)} = B^{\dagger} \otimes i\sigma_2 \otimes \sigma^k. \tag{45}$$

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- **Conjecture: the pairs of Γ-matrices generated by the standard Lorentz covariance requirement can be used for the introduction of weak isospin doublets of the $SU(2) \times U(1)$ electroweak symmetry.**
  We conclude that the internal symmetries $SU(3) \times SU(2) \times U(1)$ of the Standard Model follow from the imposition of Lorentz covariance on colour Dirac multiplets.

It follows that in order to obtain the closure of the faithful action of generators $(J_k^{(s)},\ K_m^{(s)})$ $(s = 0, 1, 2)$ on matrices Γ^μ, one should introduce two sets $\Gamma^\mu_{(a)}, \Gamma^\mu_{\dot{a}} = (\Gamma^\mu_{(a)})^\dagger$ $(a = 1, 2, ...6)$ of coloured 12×12 Dirac matrices supplemented by Lorentz doublet partners $(\tilde{\Gamma}^\mu_{(a)}, \tilde{\Gamma}^\mu_{(\dot{a})})$.

Choosing $(J_k^{(1)}, K_m^{(1)})$ as given by **Eqs. (39), (41)** , and assuming that $\Gamma^\mu_{(1)}$ are given by the same formula, by calculating the multicommutators of $\left(J_i^{(1)}, K_l^{(1)}\right)$ with the set $\Gamma^\mu_{(a)}$, $(a = 1, 2...6)$, we get the following six Γ-matrices closed under the action of $L^{(1)}$:

$$\begin{aligned}
\Gamma^\mu_{(1)} &= \left(\Gamma^0_{(8;3)},\ \Gamma^i_{(2;2)}\right);\quad \Gamma^\mu_{(4)} = \left(\Gamma^0_{(8;2)},\ \Gamma^i_{(2;3)}\right);\\
\Gamma^\mu_{(2)} &= \left(\Gamma^0_{(2;2)},\ \Gamma^i_{(4;3)}\right);\quad \Gamma^\mu_{(5)} = \left(\Gamma^0_{(2;3)},\ \Gamma^i_{(4;2)}\right);\\
\Gamma^\mu_{(3)} &= \left(\Gamma^0_{(4;3)},\ \Gamma^i_{(8;2)}\right);\quad \Gamma^\mu_{(6)} = \left(\Gamma^0_{(4;2)},\ \Gamma^i_{(8;3)}\right).
\end{aligned} \tag{46}$$

- **The six matrices (46) form three pairs, each of which transforms in itself under the action of the 0-grade subalgebra ($\overset{(0)}{J}_i$, $\overset{(0)}{K}_l$).**

  The Z_3-graded components of $\mathcal{L}_{Z_3}$, $\overset{(1)}{\mathcal{L}}$ and $\overset{(2)}{\mathcal{L}}$, act on these pairs transforming them into other pairs, with conjugate Q and B matrices.

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  The Z_3-graded components of $\mathcal{L}_{Z_3}$, $\overset{(1)}{\mathcal{L}}$ and $\overset{(2)}{\mathcal{L}}$, act on these pairs transforming them into other pairs, with conjugate Q and B matrices.
- The realization of $L^{(2)}$ sector is obtained by introducing the Hermitean-conjugate sextet $\Gamma^{\mu}_{(\dot{a})} = (\Gamma^{\mu}_{(a)})^{\dagger}$; further one should add $\tilde{\Gamma}^{\mu}_{(\dot{a})} = (\tilde{\Gamma}^{\mu}_{(a)})^{\dagger}$ due to standard Lorentz covariance.

The generalized Lorentz transformations of 24 matrices $\Gamma^{\mu}_{(F)} = (\Gamma^{\mu}_{(a)}, \Gamma^{\mu}_{(\dot{a})};\ \tilde{\Gamma}^{\mu}_{(a)}, \tilde{\Gamma}^{\mu}_{(\dot{a})})$ will be expressed by the following generalization of the formula (14)

$$\mathcal{S}\Gamma^{\mu}_{(F)}\mathcal{S}^{-1} = \Lambda^{\mu\ (G)}_{\ \nu\ (F)}\ \Gamma^{\nu}_{(G)}, \quad \mu,\nu = 0,1,2,3; \quad F,G = 1,2,...,24. \tag{47}$$

- The Z_3-graded extension of full Poincaré algebra is quite obvious. Whatever the representation we choose (spinorial or "orbital"), the commutation relations for the **Z_3-graded Lorentz subalgebra** remain the same.

- The Z_3-graded extension of full Poincaré algebra is quite obvious. Whatever the representation we choose (spinorial or "orbital"), the commutation relations for the Z_3-graded Lorentz subalgebra remain the same.
- Let us denote the generators of the extended Z_3-graded Poincaré algebra by $\overset{(r)}{\mathcal{K}_i}$ (generalized Lorentz boosts) and $\overset{(r)}{\mathcal{J}_i}$ (the generalized spatial rotations), where the superscript $r = 0, 1, 2$ refers to the Z_3-grade of one of the three components of Z_3-graded extended Lorentz algebra, and $i, k = 1, 2, 3$ are the 3-space indices.

The commutation rules of the Lorentz algebra:

$$[\overset{(r)}{\mathcal{K}_i}, \overset{(s)}{\mathcal{K}_k}] = -\epsilon_{ikl} \overset{(r+s)}{\mathcal{J}_l}, \quad [\overset{(r)}{\mathcal{J}_i}, \overset{(s)}{\mathcal{K}_k}] = \epsilon_{ikl} \overset{(r+s)}{\mathcal{K}_l}, \tag{48}$$

$$[\overset{(r)}{\mathcal{J}_i}, \overset{(s)}{\mathcal{J}_k}] = \epsilon_{ikl} \overset{(r+s)}{\mathcal{J}_l}. \tag{49}$$

must be now supplemented by set of commutation rules between Lorentz generators and the generators of 4-translations, which should also form a Z_3-graded extension of usual 4-dimensional Minkowskian translations P_μ.

Denoting them by

$$(\overset{(0)}{\mathcal{P}}_\mu, \quad \overset{(1)}{\mathcal{P}}_\mu, \quad \overset{(2)}{\mathcal{P}}_\mu), \tag{50}$$

with $r = 0, 1, 2$ and $\mu, \nu = 0, 12, 3$, we impose the following Z_3-graded extra commutation relations:

$$[\overset{(r)}{\mathcal{P}}_0, \overset{(s)}{\mathcal{P}}_k] = 0; \qquad [\overset{(r)}{\mathcal{P}}_i, \overset{(s)}{\mathcal{P}}_j] = 0, \tag{51}$$

$$[\overset{(r)}{\mathcal{J}}_k, \overset{(s)}{\mathcal{P}}_0] = 0; \qquad [\overset{(r)}{\mathcal{J}}_i, \overset{(s)}{\mathcal{P}}_k] = \epsilon_{ikl} \overset{(r+s)}{\mathcal{P}}_l \,, \tag{52}$$

$$[\overset{(r)}{\mathcal{K}}_i, \overset{(s)}{\mathcal{P}}_0] = \overset{(r+s)}{\mathcal{P}}_i \,, \qquad [\overset{(r)}{\mathcal{K}}_i, \overset{(s)}{\mathcal{P}}_k] = -\delta_{ik} \overset{(r+s)}{\mathcal{P}}_0 \,. \tag{53}$$

In all the above relations the grades $r, s = 0, 1, 2$ add up modulo 3.

To realize these commutation relations in terms of differential operators, the ordinary 4-dimensional Minkowskian space cannot suffice; it must be extended so as to accomodate the Z_3-grading. We denote by M_4 the standard 4-dimensional real vector space endowed with pseudo-Euclidean (Minkowskian) metric $\eta_{\mu\nu} = \text{diag}[+,-,-,-]$. A spacetime vector is given by its coordinates in a chosen orthonormal frame:

$$k^\mu = [k^0, \mathbf{k}] = [k^0, k^x, k^y, k^z] \tag{54}$$

often replaced by a more practical notation with small Greek indices running from 0 to 3:

$$k^\mu = [k^0, \mathbf{k}] = [k^0, k^1, k^2, k^3] \tag{55}$$

The three replicas of a 4-vector k^μ will be labeled with the superscripts relative to the elements of the Z_3-group as follows:

$$\overset{(0)}{k^\mu} = (\overset{(0)}{k^0}, \overset{(0)}{\mathbf{k}}), \quad \overset{(1)}{k^\mu} = (\overset{(1)}{k^0}, \overset{(1)}{\mathbf{k}}), \quad \overset{(2)}{k^\mu} = (\overset{(2)}{k^0}, \overset{(2)}{\mathbf{k}}). \quad (56)$$

In each of the three sectors the specific quadratic form is given, defining the group of transformations keeping it invariant:

$$(\overset{(0)}{k^0})^2 - (\overset{(0)}{\mathbf{k}})^2 = m^2, \quad (\overset{(1)}{k^0})^2 - j\,(\overset{(1)}{\mathbf{k}})^2 = j\,m^2, \quad (\overset{(2)}{k^0})^2 - j^2\,(\overset{(2)}{\mathbf{k}})^2 = j^2\,m^2, \tag{57}$$

leading to the following explicit expressions of $(\overset{(r)}{k^0})$ as functions of $\overset{(r)}{\mathbf{k}}$ and m $(r = 0, 1, 2)$:

$$\overset{(0)}{k^0} = \pm\sqrt{\overset{(0)}{\mathbf{k}^2} + m^2}, \quad \overset{(1)}{k^0} = \pm j^2\sqrt{\overset{(1)}{\mathbf{k}^2} + m^2}, \quad \overset{(2)}{k^0} = \pm j\sqrt{\overset{(2)}{\mathbf{k}^2} + m^2}, \tag{58}$$

Let us denote the three quadratic forms, one real and two mutually complex conjugate ones, by the following three tensors

$$\overset{(0)}{\eta}_{\mu\nu} = \mathrm{diag}[+1, -1, -1, -1], \quad \overset{(1)}{\eta}_{\mu\nu} = \mathrm{diag}[+1, -j, -j, -j],$$

$$\overset{(2)}{\eta}_{\mu\nu} = \mathrm{diag}[+1, -j^2, -j^2, -j^2] \qquad (59)$$

defined on each of the subspaces of the generalized Minkowskian space

$$\overset{(Z_3)}{M}_{12} = \overset{(0)}{M}_4 \oplus \overset{(1)}{M}_4 \oplus \overset{(2)}{M}_4 \qquad (60)$$

The superscripts $(r) = (0), (1), (2)$ refer to the Z_3-grades attributed to each of the three subspaces.

The three "replicas" are to be treated as really independent components of the resulting 12-dimensional manifold. For convenience, we shall use the same letters designing three types of space-time components, labeling them with an extra index as follows:

$$x_r^\mu = (x_0^\mu,\ x_1^\mu,\ x_2^\mu) = [\tau_0, x_0, y_0, z_0;\ \tau_1, x_1, y_1, z_0;\ \tau_2, x_2, y_2, z_2]. \tag{61}$$

Idempotent operators projecting on one of the three subspaces of the generalized Minkowskian space-time $\overset{(Z_3)}{M}_{12}$ can be constructed using the 3×3 matrices B **and** $B^\dagger$ as follows. Let us define two 12×12 matrices acting on $\overset{(Z_3)}{M}_{12}$:

$$\mathcal{B} = B \otimes 1\!\!1_4, \quad \mathcal{B}^\dagger = B^\dagger \otimes 1\!\!1_4,$$

Then the following three projection operators can be formed:

$$\overset{(0)}{\Pi} = \frac{1}{3}\,(\mathbb{1}_{12} + \mathcal{B} + \mathcal{B}^{\dagger}), \quad \overset{(1)}{\Pi} = \frac{1}{3}\,(\mathbb{1}_{12} + j^2\,\mathcal{B} + j\,\mathcal{B}^{\dagger}), \tag{62}$$

$$\overset{(2)}{\Pi} = \frac{1}{3}\,(\mathbb{1}_{12} + j\,\mathcal{B} + j^2\,\mathcal{B}^{\dagger}), \tag{63}$$

One checks easily that $[\overset{(r)}{\Pi}]^2 = \overset{(r)}{\Pi},\ r = 0, 1, 2$ and $\overset{(r)}{\Pi}\,\overset{(s)}{\Pi} = 0$ for $r \neq s$.

- **The quadratic Minkowskian square of the 4 vector k^μ, $(k^0)^2 - \mathbf{k}^2$ is invariant under the transformations of the Lorentz group. The space rotations touching only the 3-dimensional vector k leave all the three quadratic expressions invariant, because they depend only on its 3-dimensional Euclidean square $\mathbf{k}^2$; therefore we can fix our attention at the Lorentzian** boosts.

- The quadratic Minkowskian square of the 4 **vector** k^μ, $(k^0)^2 - \mathbf{k}^2$ is invariant under the transformations of the Lorentz group. The space rotations touching only the 3-dimensional vector k leave all the three quadratic expressions invariant, because they depend only on its 3-dimensional Euclidean square $\mathbf{k}^2$; therefore we can fix our attention at the Lorentzian boosts.
- As we can always align the relative velocity along one of the orthonormal axes of the chosen inertial frame, say $0x$, those boosts can be considered only between the time and the x coordinates. Here are the three 2×2 **matrices** representing the same Lorentz boost (with real parameter u equal to $\tanh \frac{v}{c}$) leaving invariant one of the three quadratic invariants given in **(57)**:

$$\overset{(0)}{L}_{00} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad \overset{(0)}{L}_{11} = \begin{pmatrix} \cosh u & j^2 \sinh u \\ j \sinh u & \cosh u \end{pmatrix}, \tag{64}$$

$$\overset{(0)}{L}_{22} = \begin{pmatrix} \cosh u & j \sinh u \\ j^2 \sinh u & \cosh u \end{pmatrix}, \tag{65}$$

The three matrices are self-adjoint:

$$\overset{(0)\dagger}{L}_{00} = \overset{(0)}{L}_{00}\,, \quad \overset{(0)\dagger}{L}_{11} = \overset{(0)}{L}_{11}\,, \quad \overset{(0)\dagger}{L}_{22} = \overset{(0)}{L}_{22}\,, \tag{66}$$

The above matrices transform each of the three sectors of the Z_3-Minkowski space into itself, which founds its reflection in the lower indices is quite transparent: L_{00} **transforms a vector belonging to the 0-th sector of the Z_3-graded Minkowskian space into a 4-vector belonging to the same sector, and similarly for the matrix operators** L_{11} and L_{22}.

- **Each set is a representation of a one-parameter subgroup representing a particular Lorentz boost, here between the time variable (hereafter always represented by $\tau = ct$) and one cartesian coordinate, say x. For example, the product of two Lorentz boosts acting on the sector (1), is a boost of the same type:**

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-

$$\overset{(0)}{L}_{11}(u) \cdot \overset{(0)}{L}_{11}(v) = \overset{(0)}{L}_{11}(u+v), \tag{67}$$

and similarly for a product of two boosts acting on the sector (2),

$$\overset{(0)}{L}_{22}(u) \cdot \overset{(0)}{L}_{22}(v) = \overset{(0)}{L}_{22}(u+v), \tag{68}$$

The full set of three independent "classical" (i.e. belonging to the subgroup denoted by $\overset{(0)}{L}_{00}$) Lorentz boosts is given by three 4×4 matrices, with independent parameters u, v, w:

$$\begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cosh v & 0 & \sinh v & 0 \\ 0 & 1 & 0 & 0 \\ \sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{69}$$

$$\begin{pmatrix} \cosh w & 0 & 0 & \sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{70}$$

- **To make the extension of the Lorentz boosts complete we need also two sets of complementary matrix operators transforming one sector into another.**

- To make the extension of the Lorentz boosts complete we need also two sets of complementary matrix operators transforming one sector into another.
- There are two types of such operators, one raising the Z_3 index of each subspace, another lowering the Z_3 index by 1. It is quite easy to find out their matrix representation.

The matrices lowering the Z_3 index by 1 are:

$$\overset{(1)}{L}_{01} = \begin{pmatrix} j\cosh u & \sinh u \\ j\sinh u & \cosh u \end{pmatrix}, \quad \overset{(1)}{L}_{12} = \begin{pmatrix} j\cosh u & j^2\sinh u \\ j^2\sinh u & \cosh u \end{pmatrix}, \tag{71}$$

$$\overset{(1)}{L}_{20} = \begin{pmatrix} j\cosh u & j\sinh u \\ \sinh u & \cosh u \end{pmatrix}, \tag{72}$$

The determinant of each of these matrices is equal to j .

The matrices raising the Z_3 **index** by 1 (or decreasing it by 2, which is equivalent from the point of view of the Z_3-**grading**) are:

$$\overset{(2)}{L}_{10} = \begin{pmatrix} j^2 \cosh u & j^2 \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad \overset{(2)}{L}_{21} = \begin{pmatrix} j^2 \cosh u & j \sinh u \\ j \sinh u & \cosh u \end{pmatrix}, \tag{73}$$

$$\overset{(2)}{L}_{02} = \begin{pmatrix} j^2 \cosh u & \sinh u \\ j^2 \sinh u & \cosh u \end{pmatrix}, \tag{74}$$

The determinant of each of these matrices is equal to j^2.

The above sets of three matrices each, decreasing and raising the Z_3 **index**, are mutually hermitian adjoint:

$$\overset{(1)\dagger}{L_{01}} = \overset{(2)}{L_{10}}\,, \quad \overset{(1)\dagger}{L_{12}} = \overset{(2)}{L_{21}}\,, \quad \overset{(1)\dagger}{L_{20}} = \overset{(2)}{L_{02}}\,, \tag{75}$$

Here again, the logic of the lower indices is quite transparent: a matrix labeled L_{12} transforms a 4-**vector** belonging to the sector (2) into a 4-**vector** belonging to the sector (1), and so forth, e.g.:

$$L_{01}\,\overset{(1)}{k^{\mu}} = \overset{(0)}{k^{\mu'}}, \quad L_{20}\,\overset{(0)}{k^{\mu}} = \overset{(2)}{k^{\mu'}}, \quad L_{12}\,\overset{(2)}{k^{\mu}} = \overset{(1)}{k^{\mu'}}, \quad \text{etc.} \tag{76}$$

The matrices raising or lowering the Z_3-**grade** of the particular type of the 4-**vector** they are acting on do not form a group, because most of the products of two such matrices produce new matrices not belonging to the set defined above.

However, inside each of one-parameter families corresponding to a given choice of the single space direction concerned by the Lorentz boost, $0x$, $0y$ or $0z$ displays the group property if the products are taken according to the chain rule, with second index of the first factor equal to the first index of the second, like in the following examples:

$$\overset{(1)}{L}_{12}(\tau, x;\ u)\overset{(1)}{L}_{20}(\tau, x;\ v) = \overset{(2)}{L}_{10}(\tau, x;\ (u+v)), \tag{77}$$

$$\overset{(2)}{L}_{21}(\tau, y;\ u)\overset{(1)}{L}_{12}(\tau, y;\ v) = \overset{(0)}{L}_{22}(\tau, y;\ (u+v)), \text{ etc.} \tag{78}$$

- The above 2×2 matrices represent a reduced version of Lorentz boosts with relative velocity aligned on the axis Ox.

- The above 2×2 **matrices** represent a reduced version of Lorentz boosts with relative velocity aligned on the axis Ox.
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- As in the previous case, the full 4×4 versions are given by the following three matrices corresponding to the three independent Lorentz boosts.
- The boosts of the increasing type, transforming 4-**vectors** from sector 2 **to** 0, from sector 1 **to** 2 and from sector 0 **to** 1, respectively, are as follows:

- **the three boosts** $\overset{(1)}{L}_{20}(\tau, x)$, $\overset{(1)}{L}_{20}(\tau, y)$, $\overset{(1)}{L}_{20}(\tau, z)$ **are given by:**

$$\begin{pmatrix} j\cosh u & j\sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j\cosh v & 0 & j\sinh v & 0 \\ 0 & 1 & 0 & 0 \\ \sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{79}$$

$$\begin{pmatrix} j\cosh w & 0 & 0 & j\sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{80}$$

- the three boosts $\overset{(1)}{L}_{12}(\tau,x)$, $\overset{(1)}{L}_{12}(\tau,y)$, $\overset{(1)}{L}_{12}(\tau,z)$ are given by:

$$\begin{pmatrix} j\cosh u & j^2\sinh u & 0 & 0 \\ j^2\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j\cosh v & 0 & j^2\sinh v & 0 \\ 0 & 1 & 0 & 0 \\ j^2\sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{81}$$

$$\begin{pmatrix} j\cosh w & 0 & 0 & j^2\sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ j^2\sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{82}$$

and the three boosts $\overset{(1)}{L_{01}}(\tau, x)$, $\overset{(1)}{L_{01}}(\tau, y)$, $\overset{(1)}{L_{01}}(\tau, z)$ **are given by:**

$$\begin{pmatrix} j\cosh u & \sinh u & 0 & 0 \\ j\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j\cosh v & 0 & \sinh v & 0 \\ 0 & 1 & 0 & 0 \\ j\sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{83}$$

$$\begin{pmatrix} j\cosh w & 0 & 0 & \sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ j\sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{84}$$

The boosts of the decreasing type, transforming 4-vectors from sector 1 to 0, from sector 2 to 1 and from sector 0 to 2, respectively, are as follows:

- the three boosts $\overset{(2)}{L}_{10}(\tau, x)$, $\overset{(2)}{L}_{10}(\tau, y)$, $\overset{(2)}{L}_{10}(\tau, z)$ **are given by:**

$$\begin{pmatrix} j^2\cosh u & j^2\sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j^2\cosh v & 0 & j^2\sinh v & 0 \\ 0 & 1 & 0 & 0 \\ \sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{85}$$

$$\begin{pmatrix} j^2\cosh w & 0 & 0 & j^2\sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{86}$$

- **the three boosts** $\overset{(2)}{L}_{21}(\tau,x)$, $\overset{(2)}{L}_{21}(\tau,y)$, $\overset{(2)}{L}_{21}(\tau,z)$ **are given by:**

$$\begin{pmatrix} j^2\cosh u & j\sinh u & 0 & 0 \\ j\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j^2\cosh v & 0 & j\sinh v & 0 \\ 0 & 1 & 0 & 0 \\ j\sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{87}$$

$$\begin{pmatrix} j^2\cosh w & 0 & 0 & j\sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ j\sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{88}$$

and the three boosts

$\overset{(2)}{L}_{02}(\tau, x)$, $\overset{(2)}{L}_{02}(\tau, y)$, $\overset{(2)}{L}_{02}(\tau, z)$ **are given by:**

$$\begin{pmatrix} j^2\cosh u & \sinh u & 0 & 0 \\ j^2\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j^2\cosh v & 0 & \sinh v & 0 \\ 0 & 1 & 0 & 0 \\ j^2\sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{89}$$

$$\begin{pmatrix} j^2\cosh w & 0 & 0 & \sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ j^2\sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{90}$$

- **The nine 4×4 matrices $\overset{(r)}{L}_{st}$, $r, s, t = 0, 1, 2$ act on the Z_3-extended Minkowskian vector in a specifically ordered way.**

- **The nine** 4×4 **matrices** $\overset{(r)}{L}_{st}$ **,** $r, s, t = 0, 1, 2$ **act on the** Z_3**-extended Minkowskian vector in a specifically ordered way.**
- **Let us write a** Z_3**-extended vector as a column with** 12 **entries, composed of three 4-vectors belonging each to one of the** Z_3**-graded sectors**

$$\left(\overset{(0)}{k^{\mu}},\ \overset{(1)}{k^{\mu}},\ \overset{(2)}{k^{\mu}} \right)$$

$$\overset{(0)}{\Lambda} = \begin{pmatrix} \overset{(0)}{L}_{00} & 0 & 0 \\ 0 & \overset{(0)}{L}_{11} & 0 \\ 0 & 0 & \overset{(0)}{L}_{22} \end{pmatrix} \quad \overset{(1)}{\Lambda} = \begin{pmatrix} 0 & \overset{(1)}{L}_{01} & 0 \\ 0 & 0 & \overset{(1)}{L}_{12} \\ \overset{(1)}{L}_{20} & 0 & 0 \end{pmatrix}$$

$$\overset{(2)}{\Lambda} = \begin{pmatrix} 0 & 0 & \overset{(2)}{L}_{02} \\ \overset{(2)}{L}_{10} & 0 & 0 \\ 0 & \overset{(2)}{L}_{21} & 0 \end{pmatrix} \tag{91}$$

It is easy to see that the so defined matrices display not only the group property, but also the Z_3 grading in the following sense:

$$\overset{(0)}{\Lambda} \cdot \overset{(0)}{\Lambda} \subset \overset{(0)}{\Lambda}, \quad \overset{(0)}{\Lambda} \cdot \overset{(1)}{\Lambda} \subset \overset{(1)}{\Lambda}, \quad \overset{(0)}{\Lambda} \cdot \overset{(2)}{\Lambda} \subset \overset{(2)}{\Lambda},$$

$$\overset{(1)}{\Lambda} \cdot \overset{(1)}{\Lambda} \subset \overset{(2)}{\Lambda}, \quad \overset{(2)}{\Lambda} \cdot \overset{(2)}{\Lambda} \subset \overset{(1)}{\Lambda}, \quad \overset{(1)}{\Lambda} \cdot \overset{(2)}{\Lambda} = \overset{(2)}{\Lambda} \cdot \overset{(1)}{\Lambda} \subset \overset{(0)}{\Lambda}. \tag{92}$$

- **The elements of three subsets of the Z_3-graded group of boosts behave under associative matrix multiplication as folows:**

$$\overset{(r)}{\Lambda} \cdot \overset{(s)}{\Lambda} \subset \overset{(r+s)|_3}{\Lambda}, \quad \text{with} \;\; r, s, .. = 0, 1, 2, \;\; (r+s)\,|_3 = (r+s) \text{ modulo } 3. \tag{93}$$

- The elements of three subsets of the Z_3-graded group of boosts behave under associative matrix multiplication as folows:

$$\overset{(r)}{\Lambda} \cdot \overset{(s)}{\Lambda} \subset \overset{(r+s)|_3}{\Lambda}, \quad \text{with } r, s, .. = 0, 1, 2, \ (r+s)\,|_3 = (r+s) \text{ modulo } 3. \tag{93}$$

- The three sets of matrices ordered in particular blocks **(91)** form a three-parameter family which can be considered as the extension of the set of three independent Lorentz boosts. To obtain the extension of the entire Lorentz group including the 3-parameter subgroup of space rotations we shall first investigate the Z_3-graded infinitesimal generators of the Lorentz boosts, then, taking their commutators, define the Z_3-graded extension of space rotations.

The construction of differential operators representing the Z_3-graded Poincaré algebra (53) follows the prescription given above with 12×12 matrices introduced in previous section, and 12-component generalizations of Minkowskian 4-vectors and co-vectors. Let us introduce the following notation for generalized vectors in triple Minkowskian space-time:

$$[\tau_0, x_0, y_0, z_0; \ \tau_1, x_1, y_1, z_1; \ \tau_2, x_2, y_2, z_2], \tag{94}$$

The notations are obvious: the lower index "0" refers to standard Minkowskian component (graded 0), while the indices "1" and "2" refer to mutually conjugate complex extensions of Z_3 grades 1 and 2, respectively.

Partial derivatives with respect to these variables are represented by the following 12-component column vector (written here as a horizontal co-vector transposed, in order to spare the space):

$$[\, \partial_{\tau_0}, \partial_{x_0}, \partial_{y_0}, \partial_{z_0};\ \partial_{\tau_1}, \partial_{x_1}, \partial_{y_1}, \partial_{z_1};\ \partial_{\tau_2}, \partial_{x_2}, \partial_{y_2}, \partial_{z_2};\ ]^T \qquad (95)$$

- Now we compute the results of contraction of the co-vector **(94)** with the 12-component generator of generalized translations **(95)** with one of the eighteen 12×12 **matrices** representing the generalized Lorentz algebra **(40)** sandwiched in between.

- **Now we compute the results of contraction of the co-vector (94) with the 12-component generator of generalized translations (95) with one of the eighteen 12×12 matrices representing the generalized Lorentz algebra (40) sandwiched in between.**
- **This will produce 18 generators of the Z_3-graded Poincaré algebra represented in form of linear differential operators. With twelve translations we shall get the 30-dimensional Z_3-graded extended algebra; its 10-dimensional subalgebra is the standard Poincaré algebra.**

The results are a bit cumbersome, but their construction and symmetry properties are quite clear.

Let us start with the nine generalized Lorentz boosts $\overset{(r)}{\mathcal{K}}_i$. We have explicitly:

$$\overset{(0)}{\mathcal{K}}_x = (\tau_0 \partial_{x_0} + x_0 \partial_{\tau_0}) + (j^2\ \tau_1 \partial_{x_1} + j\ x_1 \partial_{\tau_1}) + (j\ \tau_2 \partial_{x_2} + j^2\ x_2 \partial_{\tau_2}),$$

$$\overset{(0)}{\mathcal{K}}_y = (\tau_0 \partial_{y_0} + y_0 \partial_{\tau_0}) + (j^2\ \tau_1 \partial_{y_1} + j\ y_1 \partial_{\tau_1}) + (j\ \tau_2 \partial_{y_2} + j^2\ y_2 \partial_{\tau_2}),$$

$$\overset{(0)}{\mathcal{K}}_z = (\tau_0 \partial_{z_0} + z_0 \partial_{\tau_0}) + (j^2\ \tau_1 \partial_{z_1} + j\ z_1 \partial_{\tau_1}) + (j\ \tau_2 \partial_{z_2} + j^2\ z_2 \partial_{\tau_2}); \tag{96}$$

$$\overset{(1)}{\mathcal{K}}_x = (\tau_0 \partial_{x_1} + j\ x_0 \partial_{\tau_1}) + (j^2\ \tau_1 \partial_{x_2} + j^2\ x_1 \partial_{\tau_2}) + (j\ \tau_2 \partial_{x_0} + x_2 \partial_{\tau_0}),$$

$$\overset{(1)}{\mathcal{K}}_y = (\tau_0 \partial_{y_1} + j\ y_0 \partial_{\tau_1}) + (j^2\ \tau_1 \partial_{y_2} + j^2\ y_1 \partial_{\tau_2}) + (j\ \tau_2 \partial_{y_0} + y_2 \partial_{\tau_0}),$$

$$\overset{(1)}{\mathcal{K}}_z = (\tau_0 \partial_{z_1} + j\ z_0 \partial_{\tau_1}) + (j^2\ \tau_1 \partial_{z_2} + j^2\ z_1 \partial_{\tau_2}) + (j\ \tau_2 \partial_{z_0} + z_2 \partial_{\tau_0}); \tag{97}$$

$$\overset{(2)}{\mathcal{K}}_x = (\tau_0 \partial_{x_2} + j^2\ x_0 \partial_{\tau_2}) + (j\ \tau_2 \partial_{x_1} + j\ x_2 \partial_{\tau_1}) + (j^2\ \tau_1 \partial_{x_0} + x_1 \partial_{\tau_0}),$$

$$\overset{(2)}{\mathcal{K}}_y = (\tau_0 \partial_{y_2} + j^2\ y_0 \partial_{\tau_2}) + (j\ \tau_2 \partial_{y_1} + j\ y_2 \partial_{\tau_1}) + (j^2\ \tau_1 \partial_{y_0} + y_1 \partial_{\tau_0}),$$

$$\overset{(2)}{\mathcal{K}}_z = (\tau_0 \partial_{z_2} + j^2\ z_0 \partial_{\tau_2}) + (j\ \tau_2 \partial_{z_1} + j\ z_2 \partial_{\tau_1}) + (j^2\ \tau_1 \partial_{z_0} + z_1 \partial_{\tau_0}).. \tag{98}$$

- **The Z_3-graded generalized differential operators representing the Lorentz boosts display remarkable symmetry properties.**

  The "diagonal" generators $\overset{(0)}{\mathcal{K}}_i$ are hermitian: they are invariant under the simultaneous complex conjugation, replacing j by j^2 and vice versa, and switching the indices $1 \to 2, \; 2 \to 1.$

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  The "diagonal" generators $\overset{(0)}{\mathcal{K}}_i$ are hermitian: they are invariant under the simultaneous complex conjugation, replacing j by j^2 and vice versa, and switching the indices $1 \rightarrow 2, \; 2 \rightarrow 1.$

- Under the same hermitian symmetry operation the Z_3-graded boosts $\overset{(1)}{\mathcal{K}}_i$ and $\overset{(2)}{\mathcal{K}}_i$ transform into each other, so that we have

$$\overset{(1)\dagger}{\mathcal{K}_i} = \overset{(2)}{\mathcal{K}}_i, \quad \overset{(2)\dagger}{\mathcal{K}_i} = \overset{(1)}{\mathcal{K}}_i.$$

- **The commutation relations between the generalized Lorentz boosts given by (96, 97) and (98) define the differential representation of Z_3-graded extension of pure rotations, $\overset{(r)}{\mathcal{J}_k}$, with $r = 0, 1, 2$ and $i, j, .. = 1, 2, 3.$**

- The commutation relations between the generalized Lorentz boosts given by **(96, 97)** and **(98)** define the differential representation of Z_3-graded extension of pure rotations, $\overset{(r)}{\mathcal{J}_k}$, with $r = 0, 1, 2$ and $i, j, .. = 1, 2, 3.$
- By tedious (but not too sophisticated) calculation we can check that the commutation relations between the Z_3-graded Lorentz boosts imposed as hypothesis in **(40)** :

$$[\overset{(r)}{\mathcal{K}_i}, \overset{(s)}{\mathcal{K}_k}] = -\epsilon_{ikl} \overset{(r+s)}{\mathcal{J}_l} ,$$

lead indeed to the following expressions for spatial rotations $\overset{(s)}{\mathcal{J}_i}$:

$$\overset{(0)}{\mathcal{J}_x} = (z_0\partial_{y_0} - y_0\partial_{z_0}) + (z_1\partial_{y_1} - y_1\partial_{z_1}) + (z_2\partial_{y_2} - y_2\partial_{z_2}),$$

$$\overset{(0)}{\mathcal{J}_y} = (x_0\partial_{z_0} - z_0\partial_{x_0}) + (x_1\partial_{z_1} - z_1\partial_{x_1}) + (x_2\partial_{z_2} - z_2\partial_{x_2}),$$

$$\overset{(0)}{\mathcal{J}_z} = (y_0\partial_{x_0} - x_0\partial_{y_0}) + (y_1\partial_{x_1} - x_1\partial_{y_1}) + (y_2\partial_{x_2} - x_2\partial_{y_2}), \quad (99)$$

Note that the above generators are sums of classical expressions for J_k , each of them acting in its own sector of the Z_3-graded extension of Minkowskian space-time.

- **The grade 1 generators of rotations $\overset{(1)}{\mathcal{J}_i}$ have the same form, but mix up coordinates with derivatives from different sectors, in cyclical order, symbolically** $0 \to 1,\ 1 \to 2,\ 2 \to 0$:

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-

$$\overset{(1)}{\mathcal{J}_x} = (z_0 \partial_{y_1} - y_0 \partial_{z_1}) + (z_1 \partial_{y_2} - y_1 \partial_{z_2}) + (z_2 \partial_{y_0} - y_2 \partial_{z_0}),$$

$$\overset{(1)}{\mathcal{J}_y} = (x_0 \partial_{z_1} - z_0 \partial_{x_1}) + (x_1 \partial_{z_2} - z_1 \partial_{x_2}) + (x_2 \partial_{z_0} - z_2 \partial_{x_0}),$$

$$\overset{(1)}{\mathcal{J}_z} = (y_0 \partial_{x_1} - x_0 \partial_{y_1}) + (y_1 \partial_{x_2} - x_1 \partial_{y_2}) + (y_2 \partial_{x_0} - x_2 \partial_{y_0}), \quad (100)$$

Finally, the grade 2 generators of spatial rotations, $\overset{(2)}{\mathcal{J}_i}$, repeat the same scheme, but in reverse (anti-cyclic) order, i.e.
$0 \to 2,\ 1 \to 0,\ 2 \to 1$:

$$\overset{(2)}{\mathcal{J}_x} = (z_0\partial_{y_2} - y_0\partial_{z_2}) + (z_1\partial_{y_0} - y_1\partial_{z_0}) + (z_2\partial_{y_1} - y_2\partial_{z_1}),$$

$$\overset{(2)}{\mathcal{J}_y} = (x_0\partial_{z_2} - z_0\partial_{x_2}) + (x_1\partial_{z_0} - z_1\partial_{x_0}) + (x_2\partial_{z_1} - z_2\partial_{x_1}),$$

$$\overset{(2)}{\mathcal{J}_z} = (y_0\partial_{x_2} - x_0\partial_{y_2}) + (y_1\partial_{x_0} - x_1\partial_{y_0}) + (y_2\partial_{x_1} - x_2\partial_{y_1}), \quad (101)$$

- It can be checked that these differential operators correspond to what we would get by direct construction using the matrix representation given before.

- It can be checked that these differential operators correspond to what we would get by direct construction using the matrix representation given before.
- The 18 differential operators acting on the Z_3-graded extension of Minkowskian space-time; the 9 generalized Lorentz boosts $\overset{(r)}{\mathcal{K}_i}$ and the 9 generalized space rotations $\overset{(s)}{\mathcal{J}_k}$, with $r, s = 0, 1, 2$ and $i, j = 1, 2, 3$, define the faithful representation of the Z_3-graded generalization of the Lorentz group.

- **In order to introduce the extension to full Poincaré group we have to add three 4-component generators of translations each one acting on its own sector of the generalized Z_3-graded Minkowskian space-time. It turns out that in order to satisfy the Z_3-graded set of standard commutation relations given by (53), the three differential operators**

- In order to introduce the extension to full Poincaré group we have to add three 4-component generators of translations each one acting on its own sector of the generalized Z_3-graded Minkowskian space-time. It turns out that in order to satisfy the Z_3-graded set of standard commutation relations given by **(53)**, the three differential operators
- $\overset{(0)}{\mathcal{P}}_\mu, \quad \overset{(1)}{\mathcal{P}}_\mu, \quad \overset{(2)}{\mathcal{P}}_\mu$ must be defined as follows:

$$\overset{(0)}{\mathcal{P}}_\mu = [\, \partial_{\tau_0},\ -\partial_{x_0},\ -\partial_{y_0},\ -\partial_{z_0} \,] \tag{102}$$

$$\overset{(1)}{\mathcal{P}}_\mu = [\, j\partial_{\tau_1},\ -\partial_{x_1},\ -\partial_{y_1},\ -\partial_{z_1} \,] \tag{103}$$

$$\overset{(2)}{\mathcal{P}}_\mu = \left[\, j^2\partial_{\tau_2},\ -\partial_{x_2},\ -\partial_{y_2},\ -\partial_{z_2} \,\right] \tag{104}$$

- **The eighteen generators $\overset{(r)}{\mathcal{K}_i}$ and $\overset{(s)}{\mathcal{J}_k}$ together with the twelve generalized Z_3-graded translations defined above by (102, 103, 104) satisfy the full set of Z_3-graded extension of the Poincaré algebra.**

- The eighteen generators $\overset{(r)}{\mathcal{K}_i}$ and $\overset{(s)}{\mathcal{J}_k}$ together with the twelve generalized Z_3-**graded** translations defined above by **(102, 103, 104)** satisfy the full set of Z_3-**graded** extension of the Poincaré algebra.
- Its total dimension is $3 \times 10 = 30$, corresponding to three replicas of the classical Poincaré group, one "diagonal", acting on three components of the Z_3-graded Minkowskian space-time separately, and two other replicas acting on all three components transforming them into one another. The commutations relations are given by the set defined in 50, 52) and (53).

Classical Poincaré algebra admits two Casimir operators which commute with all generators. These are the **4-square** of the translation **4-vector** $P_\mu P^\mu$, and the 4-square of the **Pauli-Lubanski 4-vector** $W_\mu W^\mu$, where

$$W^\mu = \frac{1}{2}\,\varepsilon^{\mu\nu\lambda\rho}\,J_{\nu\lambda}P_\rho, \quad J_{0i} = K_i, \quad J_{ik} = \epsilon_{ikl}J^l. \tag{105}$$

In terms of more familiar generators K_i and J_l the Pauli-Lubanski vector takes on the following form:

$$W_0 = J_i P^i = \mathbf{J} \cdot \mathbf{P}, \qquad W_i = P_0 \, J_i - \epsilon_{ijk} P^j K^k, \ \text{ or } \ \mathbf{W} = P^0 \, \mathbf{J} - \mathbf{P} \wedge \mathbf{K}. \tag{106}$$

The following relations are easily verified:

$$W_\mu P^\mu = 0, \quad \left[W^\mu, P^\lambda \right] = 0, \quad \left[J^{\mu\lambda}, W^\rho \right] = \eta^{\lambda\rho} W^\mu - \eta^{\mu\rho} W^\lambda. \tag{107}$$

Irreducible representations of the Poincaré algebra (and also the group, by exponentiation) are characterized by eigenvalues of its Casimir operators, the most important of which is the mass operator $M^2 = P_\mu P^{mu}$. In order to generalize the Casimir operator given by the square of four-momentum we must take into account similar contributions from all possible combination of Z_3 grades:

$$\mathcal{P}^2 = \overset{(0)}{\mathcal{P}}_\mu \overset{(0)}{\mathcal{P}}^\mu + \overset{(1)}{\mathcal{P}}_\mu \overset{(1)}{\mathcal{P}}^\mu + \overset{(2)}{\mathcal{P}}_\mu \overset{(2)}{\mathcal{P}}^\mu + \overset{(0)}{\mathcal{P}}_\mu \overset{(1)}{\mathcal{P}}^\mu + \overset{(1)}{\mathcal{P}}_\mu \overset{(2)}{\mathcal{P}}^\mu + \overset{(2)}{\mathcal{P}}_\mu \overset{(0)}{\mathcal{P}}^\mu, \quad (108)$$

This operator commutes with the full set of generators of the Lorentz-Poincaré algebra by virtue of (51, 52 and 53).

The Pauli-Lubanski 4-vector also possesses its Z_3-graded extensions. They are of the following form:

$$\overset{(0)}{\mathcal{W}}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}(\overset{(0)}{\mathcal{J}^{\nu\lambda}}\overset{(0)}{\mathcal{P}^\rho} + \overset{(1)}{\mathcal{J}^{\nu\lambda}}\overset{(2)}{\mathcal{P}^\rho} + \overset{(2)}{\mathcal{J}^{\nu\lambda}}\overset{(1)}{\mathcal{P}^\rho}),$$

$$\overset{(1)}{\mathcal{W}}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}(\overset{(2)}{\mathcal{J}^{\nu\lambda}}\overset{(2)}{\mathcal{P}^\rho} + \overset{(1)}{\mathcal{J}^{\nu\lambda}}\overset{(0)}{\mathcal{P}^\rho} + \overset{(0)}{\mathcal{J}^{\nu\lambda}}\overset{(1)}{\mathcal{P}^\rho}),$$

$$\overset{(2)}{\mathcal{W}}_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}(\overset{(1)}{\mathcal{J}^{\nu\lambda}}\overset{(1)}{\mathcal{P}^\rho} + \overset{(2)}{\mathcal{J}^{\nu\lambda}}\overset{(0)}{\mathcal{P}^\rho} + \overset{(0)}{\mathcal{J}^{\nu\lambda}}\overset{(2)}{\mathcal{P}^\rho}). \tag{109}$$

With these three graded Pauli-Lubanski vectors we can produce a Z_3-invariant extended Casimir operator of orbital spin:

$$\mathcal{W}^2 = \overset{(0)}{\mathcal{W}}_\mu \overset{(0)}{\mathcal{W}}{}^\mu + \overset{(1)}{\mathcal{W}}_\mu \overset{(1)}{\mathcal{W}}{}^\mu + \overset{(2)}{\mathcal{W}}_\mu \overset{(2)}{\mathcal{W}}{}^\mu + \overset{(0)}{\mathcal{W}}_\mu \overset{(1)}{\mathcal{W}}{}^\mu + \overset{(1)}{\mathcal{W}}_\mu \overset{(2)}{\mathcal{W}}{}^\mu + \overset{(2)}{\mathcal{W}}_\mu \overset{(0)}{\mathcal{W}}{}^\mu,$$