Open problems: $\gamma \in \ker(d=[\bullet\!\!-\!\!\bullet,\cdot]) \xrightarrow{\mathrm{Or}(\cdot)} Q(P)=\mathrm{Or}(\gamma)(P) \in \ker[\![P,\cdot]\!] = \operatorname{sym}\{\cdot,\cdot\}_P$.

Arthemy Kiselev (Groningen/IHÉS). (7/Nov/2019)

• Combinatorics & arithmetic for Physics •

Thanks: IHÉS, Nokia Fund.

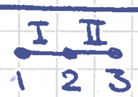
Refs: (M.K. [Ascona'96]) ... [Willwacher et al.] ... [1910.05844] & therein*

Lectures: IUM (2019, Moscow) = Ref. [18]* & IHES (2019: Nov/Dec).

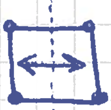
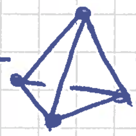
NB. $(\mathcal{M}^{r<\infty}_{affine/\mathbb{R}}, \{\cdot,\cdot\}_{Poisson}) \leftarrow$?: $\forall(\mathcal{M}^r, P)$, $Q(P) \overset{?}{\in} \operatorname{sym}\{\cdot,\cdot\}_P$

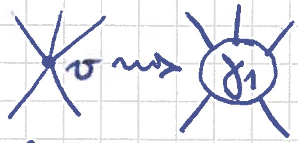
(!) May be $\nexists$ Darboux coords within affine atlas; nonlinear $P^{ij}(x)$.

§1 D.G.Lie: $(\mathrm{Vect}_{\mathbb{R}}(\gamma=\sum_a c_a\gamma_a) \leftarrow$ unoriented graph(s) $/ E(\gamma_a)=e_1\wedge\dots\wedge e_{\#Edge})$

Ex.  top  $\begin{Bmatrix} 1 & & 1' \\ 2 & & 2' \\ 3 & & 3' \end{Bmatrix}$: $I\wedge II \overset{def}{=} (-)^{I\rightleftarrows II}\, I'\wedge II' \Rightarrow \underbrace{(\gamma=-\gamma)}_{\gamma=\text{"}0\text{"}}$.

$E(\gamma)=I\wedge II$ $\quad E(\gamma')=I'\wedge II'$

Ex.  = "0". Ex.  $\overset{II}{=}$  = "0". Ex.  $\neq 0$.

$\gamma_1 \vec{\circ}\, \gamma_2 := \sum_{v\in \mathrm{Vert}(\gamma_2)}$ (insert γ_1 into vertex v:  in $\gamma_2\setminus\{v\}$, $E(\gamma_1)\wedge E(\gamma_2)$).

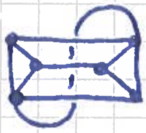
Ex. 

$\mathrm{Lie}^{Th.}$: $[\gamma_1,\gamma_2] = \gamma_1\vec{\circ}\,\gamma_2 - (-)^{\#E(\gamma_1)\cdot\#E(\gamma_2)}\, \gamma_2\vec{\circ}\,\gamma_1$. $\leftarrow$ [1811.10638]

$(d=[\bullet\!\!-\!\!\bullet,\cdot])^2 \equiv 0$: Blow-up each vertex:  $\leftarrow$ "edge".

Th. $[\gamma(\#V=n, \#E=2n-2) \in \ker d \Leftarrow \exists, \geq$ countably $\infty] \overset{Th.}{\cong} (\mathfrak{grt}=\mathrm{Lie}(GRT))$

[Willwacher (2010-15) Invent. Math.] $\qquad$ Drinfeld'1990.

Ex.  $=: \gamma_3 \in \ker d$; $\gamma_5 =$  $+\frac{5}{2}$ ; $\gamma_7 =$ (7-wheel) + 45 graphs

M.K. (1996) $\qquad$ M.K.; T. Willw. $\qquad$ [1710.00658]

? $\gamma_{2\ell+1} = ((2\ell+1)\text{-wheel}) + \langle ? \rangle \Leftarrow \mathfrak{grt}$. (Ex.) =?

NB. $[\cdots[\gamma_i,\gamma_j]\cdots\gamma_k]$ Free Lie

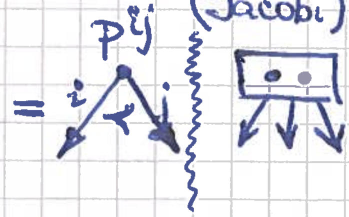
F. Brown: ? $\hookrightarrow \mathfrak{grt}$.

? $\ell \gg 1$ or $\ell\to\infty$: $\gamma_{2\ell+1}$ "graph" (mod $d(\gamma)$): diameter, valency spectra, ...).

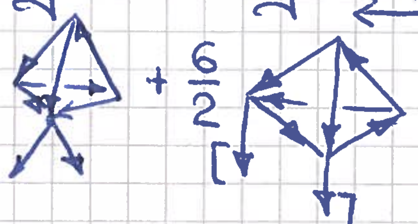
? $\dim H^{Gra}_{(n,2n-2)} = ?$

? $(\gamma := \bigsqcup_i (\gamma_i \in \ker d)) \in \ker d$: $\gamma_i = ?$ (off) $(\#V=n, \#E=2n-2)$.

§2 $\mathrm{Endo}\big(T^{\downarrow[1]}_{\text{polyvector}}(M^{r=\dim}_{\text{affine}})\big)$. ← Ex. $[\![\cdot,\cdot]\!]$ Schouten: $[\![P,P]\!]=0$. (Jacobi)

- Bi-vector $P = (P^{ij}(\underline{x}))$ "=" $\frac{1}{2}P^{ij}(\underline{x})\,\xi_i\xi_j$.
- Oriented graphs: $\overleftarrow{\partial}/\partial\xi_i^{(a)} \otimes \overrightarrow{\partial}/\partial x^i_{(b)} \rightsquigarrow a \xrightarrow{i} b$ } = 

- Idea $(T_{\text{poly}} \circlearrowleft \mathrm{Endo})$: { • Place multivector in each vertex of graph γ. • Orient every edge both ways, $\overrightarrow{\mathrm{Or}}$; sum up.

"Brick"  $= P$. Ex. $\overrightarrow{\mathrm{Or}}(\gamma_3)(P) =$  $\in \ker [\![P,\cdot]\!]$ [1608.01710] [1811.07878]

Ex. $\overrightarrow{\mathrm{Or}}(\gamma_5)(P)$; $\overrightarrow{\mathrm{Or}}(\gamma_7)(P)$.

? $[\![P, \overrightarrow{\mathrm{Or}}(\gamma\in\ker d)(P)]\!] = \underbrace{\Diamond}(P, [\![P,P]\!]) \doteq 0$ if P Poisson.

$\exists\, \Diamond_2 \neq \Diamond_1$, etc. } → 'Leibniz graphs' ⇐ $(\Diamond = \Diamond_{Ne_1}$ from $\overrightarrow{\mathrm{Or}})$ ✓

⇒ Topological identities $\Diamond_1 - \Diamond_2 \equiv 0$ in spaces of Leibniz graphs.

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§3 $\dot{P} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} P = \overrightarrow{\mathrm{Or}}(\gamma)(P) \in \mathrm{sym}\{\cdot,\cdot\}_P$. } ← $\forall(M^r_{\text{aff}}, P)$ & "$\dot{P}=P$" linear

? Nonlinear, proper $(\not\equiv 0|_{[\![P,P]\!]=0})$, & not from $\gamma\in\ker d$. ← ?

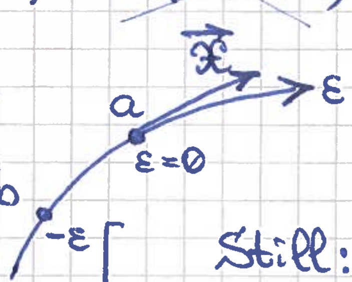
? $\dot{P} = Q(P)$ ← PDE: (?) integrability; (?) itself Hamiltonian ? ← (meta-principle)

? Poisson nontrivial ? — we know: $\overrightarrow{\mathrm{Or}}(\gamma_3) \Rightarrow \nexists\, \vec{\mathcal{X}}$ via graphs.

$Q(P) \overset{?}{=} \cancel{\nabla(P,[\![P,P]\!])} + [\![P, \vec{\mathcal{X}}(P)]\!]$

BUT

Empiric: $\forall P$, "$\exists$" $\vec{\mathcal{X}} = \vec{\mathcal{X}}(P)$, why?

Rem. 

$x'_{\text{new}}(a) := x_{\text{old}}(b)$. } ⇒ $\forall \vec{y}$, $\dot{\vec{y}} = [\vec{\mathcal{X}}, \vec{y}]$, bi-vector $\dot{P} = [\![\vec{\mathcal{X}}, P]\!]$, $\forall$-vector $\dot{\Omega} = [\![\Omega, \vec{\mathcal{X}}]\!]$

[Still: M^r_{affine}, $\vec{\mathcal{X}}(P)$ typically non-constant]

⇒ nonlinear $(\frac{1}{\infty}$; ε finite) diffeo on affine manifold, i.e. $(\mathfrak{grt} \Rightarrow \gamma\in\ker d)$ imitate a smooth structure on affine M^r.

? Beyond [M.K. (Ascona'96) = graphs]: ? other natural $\mathrm{sym}\{\cdot,\cdot\}_P$

? Quantisation(s); action $(\gamma\in\ker d)$: $\star \mapsto \star(\varepsilon)$; cyclic words (1992) [1702.00681] (1997) [1210.0726].