Schur positivity, Cluster monomials and Lattice polytopes

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Plan

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Free commutative monoid on $\mathbb{N}$

The free commutative monoid on **N** is formal finite sums of natural numbers. For example, an element of this monoid is

$$2 \oplus 3 \oplus 1 \oplus 10 \oplus 5.$$

Since these formal sums are commutative, we can order their summands in increasing order, or in decreasing order. In order to consider Young diagrams or partitions, we choose decreasing order like $10 \oplus 5 \oplus 3 \oplus 2 \oplus 1$. Such a sum can be drwn as a Young diagram: in this case, boxes with a left-justified shape of 10 boxes in the first row, 5 in the second row and so on.
Sum of two Young diagrams $Y_1 = n_1 \oplus \cdots \oplus n_k$, $n_i \geq n_{i+1}$, $i = 1, \ldots, k$, and $Y_2 = n'_1 \oplus \cdots \oplus n'_l$, $n'_i \geq n'_{i+1}$, $i = 1, \ldots, k'$, is defined as sum of elents of the monoid

$$Y_1 + Y_2 = n_1 \oplus \cdots \oplus n_k \oplus n'_1 \oplus \cdots \oplus n'_l.$$

We consider another operation, the lexicographical minimum of two Young diagram as diagram

$$\min(Y_1, Y_2) = Y_1, \text{ if } n_i = n'_i, i = 1, \dots, s, n_{s+1} < n'_{s+1}$$

and Y_2 otherwise.
These two operations endow this monoid with a tropical semiring structure, one has to think multiplication as the sum and sum as the minimum.
We also consider a partial subtraction, $Y_1 - Y_2 = Y$ if

$$Y_2 + Y = Y_1.$$

We have duality on the free commutative monoid. Namely, since we regard elements of the monoid as Young diagram, we can transpose Young diagrams. For example, dual to $10 \oplus 5 \oplus 3 \oplus 2 \oplus 1$ is

$$5 \oplus 4 \oplus 3 \oplus 2 \oplus 2 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1.$$

Duality allows us to endow Young diagrams with another structure of the semiring

$$Y_1 +^t Y_2 = (Y_1^t + Y_2^t)^t = (n_1 + n_1') \oplus (n_2 + n_2') \oplus \cdots.$$

and

$$\max(Y_1, Y_2) = (\min(Y_1^t, Y_2^t))^t.$$

For the dual monoid, we have a dual partial subtraction,

$$Y_1 -^t Y_2 = Y = (n_1 - n_1') \oplus \cdots,$$

if $Y +^t Y_2 = Y_1$. This means that $n_1 - n_1' \geq n_2 - n_2' \geq \ldots$.

Young diagrams and polynomial irreps of GL

Young diagrams with at most n rows (elements of the monoid with at most k summands) classify polynomial irreducible representations (irreps) of $GL(n, \mathbb{C})$. Namely character of $V((k_1) \oplus \cdot (k_n))$ is the Schur function $s_{(k_1,\ldots,k_n)}$.

For (k), the Schur function $s_{(k)}(x_1, \ldots) = \sum_{i_1 \leq \ldots \leq i_k} x_{i_1} \cdots x_{i_k}$ is called a full symmetric functions. These functions can be defined from

$$\sum_k h_k t^k = \prod_n (1 - x_n t)^{-1}.$$

The Jacobi-Trudy matrix is of the form

$$(h_{i-j})_{i,j \geq 1}.$$

The Jacobi-Trudy identities express Schur functions as minors of above matrices with columns set $(k_n, k_{n-1} + 1 \ldots, k_1 + n - 1)$, . Namely, we have

$$s_{(k_1,\ldots,k_n)} = \det(h_{k_i - i + j})_{i,j=1}^{n}.$$

We consider Schur functions as functions on the product space $\mathcal{Y} \times \{x_1, \ldots\}$, where $\mathcal{Y}$ is the set of the Young diagrams.

The Littlewood-Richardson rule

Schur polynomials form a vector space basis (over $\mathbb{Z}$) of the ring of symmetric polynomials in the variables $x_1, \dots x_N$. Since a product of symmetric polynomials is symmetric, we can expand the result in terms of Schur polynomials. In particular, define the Littlewood-Richardson coefficients $c^{\lambda}_{\mu,\nu}$ by

$$s_{\mu}(x_1, \dots x_N)s_{\nu}(x_1, \dots x_N) = \sum_{\lambda} c^{\lambda}_{\mu,\nu} s_{\lambda}(x_1, \dots x_N). \tag{1}$$

The LR-rule is a combinatorial description of the coefficients $c^{\lambda}_{\mu,\nu}$, namely it counts the number of semistandard Young tableaux of skew shape $\lambda \setminus \mu$ and of weight ν.
The rule was first stated by D.E.Littlewood and A.R. Richardson (1934, theorem III p.119), and the first rigorious proofs of the rule were given byM.-P. Sch́utzenberger (1976) (using some of ideas A. Lascoux and G.P. Thomas) and G.P.Thomas (1974).

For Young diagrams μ and ν, and N bigger than sum of part of μ and ν, there holds

$$c^{\mu\oplus\nu}_{\mu,\nu} = 1 \text{ and } c^{(\mu^t\oplus\nu^t)^t}_{\mu,\nu}.$$

Moreover the maximal Young diagram of (1) is $(\mu^t \oplus \nu^t)^t$ and the minimal one is $\mu \oplus \nu$.

Schur positivity

A symmetric function is Schur positive if its expansion on the basis of the Schur functions involves non-negative coefficients only.
For example, for a quadruple of partitions μ, ν, μ', ν', we have

$$s_\mu s_\nu - s_{\mu'} s_{\nu'} = \sum_\lambda (c^\lambda_{\mu,\nu} - c^\lambda_{\mu',\nu'}) s_\lambda(x_1, \dots x_N). \tag{2}$$

Then the LHS is Schur positive iff for any λ,

$$c^\lambda_{\mu,\nu} - c^\lambda_{\mu',\nu'} \geq 0.$$

For example, for a set X and a tuple $\{i_1, \dots, i_{2k+1}\}$ outside X, we have bilinear relations

$$s_{X\cup\{i_1,i_3,\dots,i_{2k+1}\}} s_{X\cup\{i_2,i_4,\dots,i_{2k}\}} - s_{X\cup\{i_2,i_3,\dots,i_{2k+1}\}} s_{X\cup\{i_1,i_4,\dots,i_{2k}\}}$$

and

$$s_{X\cup\{i_1,i_3,\dots,i_{2k+1}\}} s_{X\cup\{i_2,i_4,\dots,i_{2k}\}} - s_{X\cup\{i_2,i_3,\dots,i_{2k-1},i_{2k}\}} s_{X\cup\{i_1,i_4,\dots,i_{2k-2},i_{2k+1}\}}.$$

These polynomials are Schur positive.

Questions on Schur positivity of several types of expressions $s_\mu s_\nu - s_{\mu'} s_{\nu'}$ have been posed and studied in series of works, see [2, 7, 3, 6].
Lam, Postnikov, and Pyaljavskii [6] proved Schur positivity of

$$s_{\mu\vee\nu} s_{\mu\wedge\nu} - s_\mu s_\nu, \tag{3}$$

and affirmatively answer to open problems in [2, 7, 3, 6].

Base affine space

The coordinate ring of affine cone of the full flag variety, base affine space SL_n is a subalgebra of $\mathbb{C}[x_{ij}, i \leq j \leq n]$ span by flag minors Δ_I, $I \subset [n]$, of the upper-triangular matrices $X = (x_{ij}, i \leq j \leq n, 0, \text{otherwise})$. Moreover it is the invariant subalgebra of $\mathbb{C}[x_{ij}, i \leq j \leq n]$ under the action of the group of unitriangular matrices U,

$$\mathbb{C}[SL_n//N] = \mathbb{C}[X]^N.$$

We say that a function on the base affine space, that is a polynomial in flag minors, is Scgur positive, if its evaluation at the Jacobi-Trudy matrix is Schur positive.

The bi-linear expressions considered in LPP can be put in such a form of Schur positivity functions on the base affine space. This is one of reasons for us interested in such form of Schur positivity.

LPP results follows from Schur positivity of following functions: for a subsets $I = \{i_1, i_2, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$ and $I \vee J = \{\max(i_1, j_1), \ldots, \max(i_k, j_k)\}$, $I \wedge J = \{\min(i_1, j_1), \ldots, \min(i_k, j_k)\}$, then the polynomial are Schur positive

Cluster algebra

S.Fomin and A.Zelevinsky invented cluster algebras in 2001 in their study of positivity of dual canonical basis. Let us briefly recall the formalism of cluster algebras: For a positive integer r, an r-regular tree, denoted by $\mathbb{T}_r$, whose edges are labeled by $1, \ldots, r$, so that the r edges emanating from each vertex receive different labels. We denote by t_0 the root of $\mathbb{T}_r$. Then an edge of $\mathbb{T}_r$ is denoted by $t \rightarrow|_k t'$, indicating that vertices $t, t' \in \mathbb{T}_r$ form an edge (t, t') of $\mathbb{T}_r$ and $k \in [r]$ is the color of this edge.
For a case of geometric cluster algebras, a cluster seed is a pair: an ice quiver, $Q = (V, E)$, and a tuple of variables $\mathbf{x} = (x_j,\ j \in V)$, such that the collection $\{x_j, j \in V\}$ generates a field $\mathbf{C}[x_k,\ k \in V_f](x_j,\ j \in V_m)$.

Let us assign a cluster seed to a root t_0 of the tree T_r with $r = |V_m|$, and denote by $(\mathbf{x}_{t_0}, Q_{t_0})$ this seed. A seed pattern is an assignment of a cluster seed $(\mathbf{x}_t = (x_{j;t})_{j \in V(Q_t)}, Q_t)$ to every vertex $t \in \mathbb{T}_m$, such that the seeds assigned to the endpoints of any edge $t \to_k t'$ are obtained from each other by the seed mutation μ_k, $k \in V_m$. The mutation μ_k transforms the quiver and variables. Namely the mutation sends Q_t into a new quiver $Q_{t'} = \mu_k(Q_t)$ via a sequence of three steps. Firstly, for each oriented two-arrow path $u \to k \to w$, $u, w \in V(Q_t)$, add a new arrow $u \to w$. Secondly, reverse the direction of all arrows incident to the vertex v. Finally, repeatedly remove oriented 2-cycles until unable to do so.

The mutation μ_k assigns the variables to the vertices of $\mu_k(Q_t)$ by the following mutation rule: $\mu_k(x_{j;t}) = x_{j;t}$ if $j \neq k$, and

$$\mu_k(x_{k;t}) = \frac{\prod_{(i,k) \in E(Q_t)} x_{i;t} + \prod_{(k,j) \in E(Q_t)} x_{j;t}}{x_{k;t}}. \tag{4}$$

The ring $\mathbb{C}[SL_n/N]$ of regular functions on SL_n invariant under left multiplication by N is a $\mathcal{A}$-cluster algebra. We are interested in cluster variables of this algebra, special polynomials in flag minors. (One of our conjectures is that they are Schur positive.)
Custer variables are Laurent polynomials on variables of an initial seed. A seed is a pair of a quiver (a directed graph) and variables of a ring being assigned to the quiver vertices.

For a reduced decomposition $\mathbf{i} = i_1 \dots i_{l(w_0)}$ of the longest element w_0 of the Weyl group (the group of permutations of $[n]$ for SL_n), Berenstein, Fomin and Zelevinsky defined a seed $\Sigma(\mathbf{i})$ by the rule. The vertices of the quiver $\Gamma_{\mathbf{i}}$ are v_k, $k = 1, \dots, N$ and v_{-i}, $-i \in [-n] := \{-1, \dots, -n-1\}$.

The edges are defined as follows. For $k \in [-n]$ we set $i_k = -k$. For $k \in [l(w_0)]$ we denote by $k^+ = k^+_{\mathbf{i}}$ the smallest ℓ such that $k < \ell$ and $i_\ell = i_k$. If no such ℓ exists, we set $k^+ = l(w_0) + 1$. For $k \in [l(w_0)]$, we further let k^- be the largest index ℓ with that $\ell < k$ and $i_\ell = i_k$.

There is an edge connecting v_k and v_ℓ with $k < \ell$ if at least one of the two vertices is mutable and one of the following conditions is satisfied:

1. $\ell = k^+$,
2. $\ell < k^+ < \ell^+$, $c_{k,\ell} < 0$ and $k, \ell \in [N]$.

Edges of type (1) are called <u>horizontal</u> and are directed from k to ℓ.
Edges of type (2) are called <u>inclined</u> and are directed from ℓ to k.
The frozen vertices constitute the set v_{-i}, $-i \in [-n]$ union v_k such that $k^+ > N$.

The cluster variables of $\mathcal{S}(\mathbf{i})$ are the flag minors $\Delta_{\mathbf{i}|_{\leq k}\omega_{i_k},\omega_{i_k}}$, $k \in [N]$, where $\mathbf{i}|_{\leq k}$ denotes the subword $\mathbf{i}$ of the first k letters, and $\Delta_{\omega_i,\omega_i}$ attached to vertices v_k and v_{-i} respectively.
We can mutate at any unfrozen vertex v_k. This changes the quiver and the variables. The new variables is obtained by the $\mathcal{A}$-cluster mutation

We consider an initial seed corresponding to the reduced decomposition $121321 \cdots n-1n-2 \cdot 21$. The quiver Γ is the triangular graph embedded in the plane with the vertex set $\{i,j\}$, $1 \le i \le j \le n$, and the edges $((i,j),(i+1,j+1))$, $((i,j) \to (i-1,j)$, $((i,j) \to (i,j-1)$, $1 \le i \le j \le n$.
The variable attached to the vertex $v = (i,j)$ is the flag minor $\Delta_{\{i,i+1,\ldots,i+j-1\}}$. We call such minors as interval minors.
The cluster algebra with the such an initial seed contains all flag minors and the Laurent polynomials in the interval flag minors are polynomials in the set of all flag Δ_I, $I \subset [n]$. In the coordinate ring $\mathbb{C}[SL_n/N]$, these Laurent polynomials are polynomials in the flag minors indeed.

The free commutative monoid and subsets of $\mathbb{N}$

The map

$$(k_1) \oplus \ldots \oplus (k_n) \to \{k_n, k_{n-1} + 1, \ldots, k_1 + n - 1\},$$

a bijection between Young diagrams and subsets of $\mathbb{N}$.
The inverse image under this map of an interval $[i, i+1, \ldots, i+j-1]$ is $j-1$ sums of (i),

$$(i) \oplus (i) \oplus \ldots \oplus (i) =: (i)^j.$$

Thus the specification of variables of the initial seed to the Jacob-Trudy matrix yields the seed with variables being Schur functions of rectangular shapes $(i)^j$.

Theorem

Any cluster polynomial is Schur positive.

This theorem is due to joint work with D.Mironov and H.Oja. Here is an outline of the proof: we use results of D. Hernandes and B. Leckerc [4] on that the coordinate ring of the base affine space, $\mathbb{C}[SL_n/N]$, is isomorphic to the (complexified) Grothendieck ring of a monoidal subcategory of the finite-dimensional module category of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_{\mathfrak{n}})$. Specifically, this isomorphism sends the dual canonical basis to the class of simple modules. Recently, Kang-Kashiwara-Kim-Oh [**?**] proved that cluster monomials of $\mathbb{C}[N]$ are contained in the dual canonical basis via categorification by quiver Hecke algebra. Thus, using above isomorhism due to Hernandes-Leclerc, we get that cluster monomials correspond to the class of simple modules. Since $U_q(\mathfrak{sl}_{\mathfrak{n}})$ is a Hopf subalgebra of $U_q(\hat{\mathfrak{sl}}_{\mathfrak{n}})$, we get a decomposition of simple modules of $U_q(\hat{\mathfrak{sl}}_{\mathfrak{n}})$ into simple modules $U_q(\mathfrak{sl}_{\mathfrak{n}})$. We show that such a decomposition corresponds to the decomposition in Schur functions. This implies the desired positivity.

Some examples to this theorem:
Mutations at four valency vertices produce Schur functions and any Scur function is obtained by a sequence of mutations.
A simplest cluster variable, which is not a flag minor, is a quadratic polynomial in flag minors. Here are examples of such a variable

$$\Delta_{i-1,[i+1,i+j+1]}\Delta_{[i,i+j-1]} - \Delta_{i-1,[i+1,i+j-1]}\Delta_{[i,i+j+1]}.$$

The specialization of such a polynomial to the Schur functions gives us a quadraic polynomial in Schur functions

$$s_{(i^{j+1},i-1)}s_{i^j} - s_{i^{j+2}}s_{(i^{j-1},i-1)}. \tag{5}$$

Such functions are Schur positive. Some examples of calculations:

$$\frac{s_{(22)}s_{(4)}s_{(333)} + s_{(44)}s_{(3)}s_{(222)}}{s_{(33)}} = s_{(4322)} + s_{(5321)} + s_{(632)}$$

$$\frac{s_{(5)}s_{(444)}s_{(22)} + s_{(55)}\left(\frac{s_{(22)}s_{(4)}s_{(333)}+s_{(44)}s_{(3)}s_{(222)}}{s_{(33)}}\right)}{s_{(44)}} = s_{(5422)}+s_{(6421)}+s_{(742)}$$

$$\frac{\left(s_{(3333)}s_{(4322)} + s_{(5321)} + s_{(632)}\right) + s_{(2222)}s_{(444)}s_{(3)}}{s_{(333)}}$$
$$= s_{(43322)} + s_{(53321)} + s_{(6332)}$$

These cluster variables fit into the following case. Scandera established the sufficient condition of Schur positivity for

$$\Delta_I\Delta_J - \Delta_{I'}\Delta_{J'} \tag{6}$$

is Schur positive if, for any interval $K \subset \mathbb{N}$, there holds

$$\max(|I \cap K|, |J \cap K|) \leq \max(|I' \cap K|, |J' \cap K|). \tag{7}$$

All cluster variables of the form (6) satisfy (7), and thus are Schur positive. However, in contrast to Lam and al.[6] , this methods is not helpful for higher order mutations.

$$\frac{s_{(33)}s_{(5)}\big(s_{(443111)}+s_{(54311)}+s_{(5531)}\big)+\big(s_{(332111)}+s_{(43211)}+s_{(4421)}\big)s_{(4)}s}{s_{(44)}}$$

$$\begin{aligned}
&= s_{(5333111)}+s_{(5432111)}+s_{(543311)}+s_{(544211)}+s_{(54431)}+s_{(553211)}\\
&\quad+s_{(55421)}+s_{(6332111)}+2s_{(633311)}+s_{(6422111)}+s_{(6431111)}+3s_{(643211)}\\
&\quad+2s_{(64331)}+s_{(644111)}+2s_{(64421)}+s_{(6443)}+s_{(652211)}+s_{(653111)}\\
&\quad+2s_{(65321)}+s_{(65411)}+s_{(6542)}+s_{(7331111)}+2s_{(733211)}+s_{(73331)}\\
&\quad+s_{(7421111)}+2s_{(742211)}+3s_{(743111)}+3s_{(74321)}+s_{(7433)}+2s_{(74411)}\\
&\quad+s_{(7442)}+s_{(752111)}+2s_{(75221)}+2s_{(75311)}+s_{(7532)}+s_{(7541)}+2s_{(833111)}\\
&\quad+s_{(83321)}+2s_{(842111)}+s_{(84221)}+3s_{(84311)}+s_{(8432)}+s_{(8441)}\\
&\quad+2s_{(852,11)}+s_{(8522)}+s_{(8531)}+s_{(93311)}+s_{(94211)}+s_{(9431)}+s_{(9521)}
\end{aligned}$$

Theorem

Let Q^{in} be the initial seed of the cluster algebra $\mathbb{C}[SL_N^{w_0,e}]$.Let P be a cluster polynomial in flag minors. Then the expansion of the Schur specialization of P on the basis of Schur functions has terms labeled by lexmin and lexmax partitions with coefficients 1. Moreover these lexmin and lexmax partitions can be obtained by following the same sequence of mutations from Q^{in} as for a cluster variables corresponding to P but with respect to the two tropical semirings on the partitions.

For example,

$$\frac{\left(s_{(5,3,2,2,2)}+s_{(6,3,2,2,1)}+s_{(7,3,2,2)}\right)s_{(4,4,3)}s_{(3,3)}+s_{(3)}s_{(3,3,3,3)}s_{(2,2,2)}s_{(5,5)}}{\begin{array}{c}s_{(4,3,3,2,2,2)}+s_{(4,4,2,2,2,2)}+s_{(4,4,3,2,2,1)}+s_{(5,3,3,2,2,1)}+s_{(5,4,2,2,2,1)}+s_{(5,4,3,2,1,1)}\\+s_{(5,4,3,2,2)}+s_{(5,5,2,2,1,1)}+s_{(5,5,3,2,1)}+s_{(6,3,3,2,2)}+s_{(6,4,2,2,2)}\\+s_{(6,4,3,2,1)}+s_{(6,5,2,2,1)}+s_{(6,5,3,2)}+s_{(6,6,2,2)}\end{array}}$$
$$= s_{(5,4,3,3)}+s_{(6,4,3,2)}+s_{(7,4,3,1)}+s_{(8,4,3)}$$

the lexmin of concatenations of the numerator is $(5,4,4,3,3,3,3,2,2,2)$, and subtracting lexmin of denominator $(4,3,3,2,2,2)$ yields $(5,4,3,3)$ of RHS; lexmax sums, the nominator yields $(14,10,5,2)$, the denominator $(6,6,2,2)$, and we get $(8,4,3)$ of RHS. Note that the support here is the segment $[(5,4,3,3),(8,4,3)]$.

Conjecture

Any such a polynomial as a linear combination of Schur function has the full support. Namely, for a cluster polynomial, all integer points of the convex hull of vectors corresponding to partitions which support the Schur functions of the corresponding linear combination correspond to summands with positive coefficients.

Conjecture

If, for a cluster variable, the convex hull of the support of the Young diagrams is not a segment and a triangle, then this convex hull has no interior integer points.

Computation experiments which lead us to conjectures 1.2 and 1.3 were preformed using SageMath open source computer algebra system and polymake software for polyhedral geometry research. Note that identity check for cluster seeds actually can be done by checking that sets of arrays (footprint of the seed), associated with non-frozen subset of the seed, are identical(if this would occur not true, than the problem of comparing cluster seeds would involve NP-complete graph isomorphism problem). Each seed is encoded by it's exchange matrix (in the form of graph dictionary - mapping of pairs of vertices to number of edges connecting them), mapping of vertices to arrays, an mapping of vertices to Schur polynomials. We don't store or compute cluster variables.
To do mutation of cluster seed one needs to rearrange graph dictionary around mutating vertex and calculate new array using array multiplication and sum, that is done in [Koshevoy, 2014] by identification of cluster seeds with patterns of Young tableaux.

As identical seeds should have same associated Schur polynomials, our algorithm can start costly procedure of Schul polynomial computation when we get new seed. Moreover, we can identify Schur polynomials with arrays and compute new polynomial only when we encounter new array (as you can see, number of arrays grows much slower than number of seeds). For Schur polynomial computations we used SageMath. This computer algebra system already had efficient algorithm for Schur polynomial multiplication in infinite-dimensional algebra of symmetric functions over $\mathbb{Q}$.

On other hand division of symmetric functions in Schur polynomial basis is not presented in SageMath toolkit. One method of dividing Schur polynomials is to make set of liniar equations corresponding to equation $\sum_\lambda a_\lambda s_\lambda = (\sum_\mu b_\mu s_\mu)(\sum_\nu c_\nu x_\nu)$, which involves huge linear equations systems with coefficients computed from Littlewood-Richardson rule.

However, in our case we know that mutation of vertex should always yeild some sum of Schur polynomials, so division will be always successful. In this case we can change basis of algebra of symmetric functions to some algebraically independent basis (basis of elementary symmetric polynomials e_λ) and do multivariative polynomial division there with respect to any monomial order. Lastly one needs to convert result polynomial in elementary symmetric basis into Schur polynomial basis. This provides efficient way to compute mutation of Schur polynomials, yet it still takes significant time to complete, especially when some of components involved consist of thousand elementary Schur polynomials.

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