Towards a noncommutative Picard-Vessiot theory (with simple applications)

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Outline

INTRODUCTION

Picard-Vessiot theory of ordinary differential equation

$(\mathbf{k}, \partial)$ differential ring. $\mathrm{Const}(\mathbf{k}) = \{c \in \mathbf{k} | \partial c = 0\}$ is supposed to be a field.

$$(ODE) \quad (a_n\partial^n + a_{n-1}\partial^{n-1} + \ldots + a_0)y = 0, \quad a_0, \ldots, a_{n-1}, a_n \in \mathbf{k}.$$

a_n^{-1} is supposed to exist.

Definition

1. Let $y_1, \ldots, y_n$ be $\mathrm{Const}(\mathbf{k})$-linearly independent solutions of (ODE). Then $\{y_1, \ldots, y_n\}$ is called a fundamental set of solutions of (ODE) and it generates a $\mathrm{Const}(\mathbf{k})$-module of dimension at most n.
2. If[1] $M = \mathbf{k}\{y_1, \ldots, y_n\}$ and $\mathrm{Const}(M) = \mathrm{Const}(\mathbf{k})$ then M is called a Picard-Vessiot extension related to (ODE)
3. Let $\mathbf{k} \subset \mathbb{K}_1$ and $\mathbf{k} \subset \mathbb{K}_2$ be differential rings. An isomorphism of rings $\sigma : \mathbb{K}_1 \to \mathbb{K}_2$ is a differential $\mathbf{k}$-isomorphism if
$$\forall a \in \mathbb{K}_1, \quad \partial(\sigma(a)) = \sigma(\partial a) \text{ and, if } a \in \mathbf{k},\ \sigma(a) = a.$$
If $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$, the differential galois group of $\mathbb{K}$ over $\mathbf{k}$ is by
$$\mathrm{Gal}_{\mathbf{k}}(\mathbb{K}) = \{\sigma | \sigma \text{ is a differential } \mathbf{k}\text{-automorphism of } \mathbb{K}\}.$$

1. Let R_1, R_2 be differential rings s.t. $R_1 \subset R_2$. Let S be a subset of R_2. $R_1\{S\}$ denotes the smallest differential subring of R_2 containing R_1. $R_1\{S\}$ is the ring (over R_1) generated by S and their derivatives of all orders.

ALGEBRAIC COMBINATORIAL ASPECTS

Notations

- Let $(X^*, 1_{X^*})$ (resp. $(Y^*, 1_{Y^*})$) be the free monoid generated by $X := \{x_0, \ldots, x_m\}$ (resp. $Y := \{y_k\}_{k\geq 1}$). $\mathcal{X}$ will denote X or Y. Let $A\langle\mathcal{X}\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$) be the set of polynomials (resp. formal series) over $\mathcal{X}$ and with coefficients in the commutative ring A.
- For $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. Y^*), one defines on
  - $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X}) := (A\langle\mathcal{X}\rangle, \mathrm{conc}, 1_{\mathcal{X}^*}, \Delta_{\sqcup\!\sqcup}, \mathrm{e})$, $\Delta_{\sqcup\!\sqcup} x = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x$, or equivalently $u \sqcup\!\sqcup 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \sqcup\!\sqcup u = u$ and $xu \sqcup\!\sqcup yv = x(u \sqcup\!\sqcup yv) + y(xu \sqcup\!\sqcup v)$,
  - $\mathcal{H}_{\uplus}(Y) := (A\langle Y\rangle, \mathrm{conc}, 1_{Y^*}, \Delta_{\uplus}, \mathrm{e})$, $\Delta_{\uplus} y_i = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l$, or equivalently $u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u$ and $x_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v)$.
- Considering A as the differential ring of holomorphic functions on a simply connected domain Ω, denoted by $(\mathcal{H}(\Omega), \partial)$ and equipped 1_Ω as the neutral element, the differential ring $(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \mathbf{d})$ is defined as follows

$$\forall S \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle, \quad \mathbf{d}S = \sum_{w\in\mathcal{X}^*} (\partial\langle S|w\rangle)w \quad \in \mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle.$$

$\mathrm{Const}(\mathcal{H}(\Omega)) = \mathbb{C}.1_\Omega$ and $\mathrm{Const}(\mathcal{H}(\Omega)\langle\langle\mathcal{X}\rangle\rangle) = \mathbb{C}.1_\Omega\langle\langle\mathcal{X}\rangle\rangle$.

Representative series and Sweedler's dual

Theorem (rational series[2])

Let $S \in A\langle\langle\mathcal{X}\rangle\rangle$. The following assertions are equivalent

1. *The series S belongs to*[3] $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$.
2. *There exists a linear representation (ν, μ, η) (of rank n) for S with $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(A)$ s.t. $S = \sum_{w\in\mathcal{X}^*}(\nu\mu(w)\eta)w$.*
3. *The shifts*[4] *$\{S \triangleleft w\}_{w\in\mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w\in\mathcal{X}^*}$) lie within a finitely generated shift-invariant A-module.*

Moreover, if A is a field K, previous assertions are equivalent to

4. *There exists $(G_i, D_i)_{i\in F\,\mathit{finite}}$ s.t. $\Delta_{\mathtt{conc}}(S) = \sum_{i\in F\,\mathit{finite}} G_i \otimes D_i$.*

Hence,

$$\begin{array}{rcl} \mathcal{H}^{\circ}_{\shuffle}(\mathcal{X}) & = & (K^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \shuffle, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e}), \\ (\mathit{resp.}\ \mathcal{H}^{\circ}_{\stuffle}(Y) & = & (K^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \stuffle, 1_{\mathcal{X}^*}, \Delta_{\mathtt{conc}}, \mathrm{e})). \end{array}$$

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2. This form is a version over a ring of the form presented at CAP'2018.

3. $A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ is the (algebraic) closure by $\{\mathtt{conc}, +, *\}$ of $\widehat{A.\mathcal{X}}$ in $A\langle\langle\mathcal{X}\rangle\rangle$. It is closed under $\shuffle$. $A^{\mathrm{rat}}\langle\langle Y\rangle\rangle$ is also closed under $\stuffle$.

4. The *left* (resp. *right*) shift of S by P is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for $w \in \mathcal{X}^*$, $\langle P \triangleright S|w\rangle = \langle S|wP\rangle$ (resp. $\langle S \triangleleft P|w\rangle = \langle S|Pw\rangle$).

Kleene stars of the plane and conc-characters

Theorem (rational exchangeable series[5])

Let $A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ be the set of (syntactically) exchangeable[6] series and $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle$ the set of series admitting a linear representation with commuting matrices (hence, exchangeable). Then[7]

1. $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \subset A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle \cap A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$. *The equality holds when A is a field and, if $\mathcal{X}$ is finite then* $A_{\mathrm{exc}}^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle = \,\sqcup\!\sqcup\, \{A^{\mathrm{rat}}\langle\langle x\rangle\rangle\}_{x\in\mathcal{X}}$.
2. *If A is a $\mathbb{Q}$-algebra without zero divisors, $\{x^*\}_{x\in\mathcal{X}}$ (resp. $\{y^*\}_{y\in Y}$) are algebraically independent over $(A\langle\mathcal{X}\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ (resp. $(A\langle Y\rangle, \uplus, 1_{Y^*})$) within $(A^{\mathrm{rat}}\langle\langle\mathcal{X}\rangle\rangle, \sqcup\!\sqcup, 1_{\mathcal{X}^*})$ (resp. $(A^{\mathrm{rat}}\langle\langle Y\rangle\rangle, \uplus, 1_{Y^*})$). Moreover, x^* is a* conc-character.
3. *For any $x\in\mathcal{X}$, one has $A^{\mathrm{rat}}\langle\langle x\rangle\rangle = \{P(1-xQ)^{-1}\}_{P,Q\in A[x]}$ and if $A=K$ is an algebraically closed field then one also has* $K^{\mathrm{rat}}\langle\langle x\rangle\rangle = \mathrm{span}_K\{(ax)^* \sqcup\!\sqcup K\langle x\rangle | a\in K\}$.
4. $\forall S\in K\langle\langle\mathcal{X}\rangle\rangle$, *$K$ being a field,*
$$\Delta_{\mathrm{conc}}(S) = S\otimes S, \langle S|1_{\mathcal{X}^*}\rangle = 1 \iff S = \Big(\sum_{x\in\mathcal{X}} c_x x\Big)^* \text{ with } c_x\in K.$$

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5. This form is a version over a ring of the form presented at CAP'2018.
6. *i.e.* if $S\in A_{\mathrm{exc}}\langle\langle\mathcal{X}\rangle\rangle$ then $(\forall u,v\in\mathcal{X}^*)((\forall x\in\mathcal{X})(|u|_x = |v|_x) \Rightarrow \langle S|u\rangle = \langle S|v\rangle)$.
7. Let $S\in A\langle\langle\mathcal{X}\rangle\rangle$ s.t. $\langle S|1_{\mathcal{X}^*}\rangle = 0$. Then $S^* = \sum_{n\geq 0} S^n$, so called Kleene star of S.

Triangular sub bialgebras of $(A^{\mathrm{rat}}\langle\langle X\rangle\rangle, ⧢, 1_{X^*}, \Delta_{\mathtt{conc}}, \mathrm{e})$

Let (ν, μ, η) be a linear representation of $R \in A^{\mathrm{rat}}\langle\langle X\rangle\rangle$ and $\mathcal{L}$ be the Lie algebra generated by $\{\mu(x)\}_{x\in X}$.
Let $M(x) := \mu(x)x$, for $x \in X$. Then $R = \nu M(X^*)\eta$. If $\{\mu(x)\}_{x\in X}$ are triangular then let $D(X)$ (resp. $N(X)$) be the diagonal (resp. nilpotent) letter matrix s.t. $M(X) = D(X) + N(X)$ then $M(X^*) = ((D(X^*)T(X))^* D(X^*))$. Moreover, if $X = \{x_0, x_1\}$ then $M(X^*) = (M(x_1^*)M(x_0))^* M(x_1^*) = (M(x_0^*)M(x_1))^* M(x_0^*)$.

If A is an algagraically closed field, the modules generated by the following families are closed by $\mathtt{conc}$, ⧢ and coproducts :

(F_0) $E_1x_1 \ldots E_jx_1E_{j+1}$, where $E_k \in A^{\mathrm{rat}}\langle\langle x_0\rangle\rangle$,
(F_1) $E_1x_0 \ldots E_jx_0E_{j+1}$, where $E_k \in A^{\mathrm{rat}}\langle\langle x_1\rangle\rangle$,
(F_2) $E_1x_{i_1} \ldots E_jx_{i_j}E_{j+1}$, where $E_k \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle, x_{i_k} \in X$.

It follows then that

1. R is a linear combination of expressions in the form (F_0) (resp. (F_1)) iff $M(x_1^*)M(x_0)$ (resp. $M(x_0^*)M(x_1)$) is nilpotent,
2. R is a linear combination of expressions in the form (F_2) iff $\mathcal{L}$ is solvable. Thus, if $R \in A^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle ⧢ A\langle X\rangle$ then $\mathcal{L}$ is nilpotent.

NONCOMMUTATIVE PV THEORY AND INDEPENDENCE VIA WORDS

Iterated integrals and Chen series

Let $\mathcal{A} := \mathcal{H}(\Omega)$ and $\mathcal{C}_0$ be a differential subring of $\mathcal{A}$ ($\partial(\mathcal{C}_0) \subset \mathcal{C}_0$) which is an integral domain containing $\mathbb{C}$.
$\mathbb{C}\{\{(g_i)_{i\in I}\}\}$ denotes the differential subalgebra of $\mathcal{A}$ generated by $(g_i)_{i\in I}$, *i.e.* the $\mathbb{C}$-algebra generated by g_i's and their derivatives

$\{u_x\}_{x\in\mathcal{X}}$: elements in $\mathcal{C}_0 \cap \mathcal{A}^{-1}$ in correspondence with $\{\theta_x\}_{x\in\mathcal{X}}$ ($\theta_x = u_x^{-1}\partial$).

The iterated integral associated to $x_{i_1}\ldots x_{i_k} \in \mathcal{X}^*$, over the differential forms $\omega_i(z) = u_{x_i}(z)dz$, and along a path $z_0 \rightsquigarrow z$ on Ω, is defined by

$$\begin{aligned}
\alpha_{z_0}^z(1_{\mathcal{X}^*}) &= 1_\Omega,\\
\alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) &= \int_{z_0}^z \omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k).\\
\partial\alpha_{z_0}^z(x_{i_1}\ldots x_{i_k}) &= u_{x_{i_1}}(z)\int_{z_0}^z \omega_{i_2}(z_2)\ldots\int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k).
\end{aligned}$$

$$\begin{aligned}
\mathrm{span}_{\mathbb{C}}\{\partial^l\alpha_{z_0}^z(w)\}_{w\in X^*, l\geq 0} &\subset \mathrm{span}_{\mathbb{C}\{\{(u_x)_{x\in\mathcal{X}}\}\}}\{\alpha_{z_0}^z(w)\}_{w\in X^*}\\
&\subset \mathrm{span}_{\mathbb{C}\{\{(u_x^{\pm 1})_{x\in X}\}\}}\{\alpha_{z_0}^z(w)\}_{w\in X^*}\\
&\cong \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\} \otimes_{\mathbb{C}} \mathrm{span}_{\mathbb{C}}\{\alpha_{z_0}^z(w)\}_{w\in X^*}?
\end{aligned}$$

The Chen series, over $\{\omega_i\}_{i\in I}$ and along $z_0 \rightsquigarrow z$ on Ω, is defined by

$$C_{z_0\rightsquigarrow z} := 1_\Omega 1_{\mathcal{X}^*} + \sum_{w\in\mathcal{X}^*\mathcal{X}} \alpha_{z_0}^z(w)w.$$

Noncommutative differential equations

The Chen series, over $\{\omega_i\}_{i\in I}$ and along $z_0 \rightsquigarrow z$ on Ω, satisfies

$$(NCDE) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x\in\mathcal{X}} u_x x \quad (M \text{ is } ш\text{-primitive}).$$

More generally, $C_{z_0 \rightsquigarrow z}$ satisfies $\mathbf{d}^k S = Q_k S$, with $Q_k \in \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}\langle\mathcal{X}\rangle$ satisfying the recursion [8] $Q_0 = 1$ and $Q_k = Q_{k-1}M + \mathbf{d}Q_{k-1}$ $(k \geq 0)$.

1. The space of solutions of $(NCDE)$ is a right free $\mathbb{C}\langle\langle X\rangle\rangle$-module of rank 1.
2. By a theorem of Ree, $C_{z_0 \rightsquigarrow z}$ is a $ш$ $-$group-like solution of $(NCDE)$ and it can be obtained as the limit of a convergent Picard iteration, initialized at $\langle C_{z_0 \rightsquigarrow z} | 1_{\mathcal{X}^*}\rangle = 1_\Omega 1_{\mathcal{X}^*}$, for ultrametric distance.
3. If G and H are $ш$ $-$group-like solutions $(NCDE)$ there is a constant Lie series C such that $G = He^C$ (and conversely).

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8. More explicitly, Q_k can be computed as follows (summing over words $w = x_{i_1}\ldots x_{i_k}$ and derivation multiindices $\mathbf{r} = (r_1, \ldots, r_k)$ of degree $\deg \mathbf{r} = |w| = k$ and of weight $\mathtt{wgt}\ \mathbf{r} = k + r_1 + \ldots + r_k$)

$$Q_k = \sum_{\substack{\mathtt{wgt}\ \mathbf{r}=k \\ w\in\mathcal{X}^{\deg \mathbf{r}}}} \prod_{j=1}^{\deg \mathbf{r}} \binom{\sum_{j=1}^{j} r_j + j - 1}{r_k} \tau_{\mathbf{r}}(w), \quad \text{where}$$

$$\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1})\ldots\tau_{r_k}(x_{i_k}) = (\partial^{r_1} u_{x_{i_1}})x_{i_1}\ldots(\partial^{r_k} u_{x_{i_k}})x_{i_k} \in \mathbb{C}\{\{(u_x^{\pm 1})_{x\in\mathcal{X}}\}\}\langle\mathcal{X}\rangle.$$

First step of noncommutative PV theory

From this, it follows that

- the differential Galois group of $(NCDE)$ + $\sqcup\!\sqcup$ $-$group-like is the group[9] $\{e^C\}_{C\in\mathcal{L}ie_{\mathbb{C}.1_\Omega}\langle\langle\mathcal{X}\rangle\rangle}$.

Which leads us to the following definition

- the PV extension related to $(NCDE)$ is $\widehat{\mathcal{C}_0.\mathcal{X}}\{C_{z_0\rightsquigarrow z}\}$.

It, of course, is such that $\mathrm{Const}(\mathcal{C}_0\langle\langle\mathcal{X}\rangle\rangle) = \ker \mathbf{d} = \mathbb{C}.1_\Omega\langle\langle\mathcal{X}\rangle\rangle$.

On the other hand, the iterated integrals[10] satisfy

$$\forall u, v \in \mathcal{X}^*,\quad \alpha_{z_0}^z(u \sqcup\!\sqcup v) = \alpha_{z_0}^z(u)\alpha_{z_0}^z(v),$$

or equivalently, the Chen series satisfies

$$C_{z_0\rightsquigarrow z_0} = 1_\Omega \quad \text{and} \quad C_{z_0\rightsquigarrow z} = \sum_{w\in\mathcal{X}^*}\alpha_{z_0}^z(S_w)P_w = \prod_{l\in\mathcal{L}yn\mathcal{X}}^{\searrow} e^{\alpha_{z_0}^z(S_l)P_l},$$

where $\mathcal{L}yn\mathcal{X}$ denotes the set of Lyndon words related to $\mathcal{X}$, the linear basis $\{P_w\}_{w\in\mathcal{X}^*}$ (expanded after the basis $\{P_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ of $\mathcal{L}ie_{\mathbb{C}.1_\Omega}\langle\mathcal{X}\rangle$) and its graded dual basis $\{S_w\}_{w\in\mathcal{X}^*}$ (which contains the pure transcendence basis $\{S_l\}_{l\in\mathcal{L}yn\mathcal{X}}$ of the $\mathbb{C}-$ $\sqcup\!\sqcup$ algebra).

9. In fact, the Hausdorff group (group of characters) of $\mathcal{H}_{\sqcup\!\sqcup}(\mathcal{X})$.

10. Due to the fact that Ω is simply connected, the value of these iterated integrals only depend on the endpoints, (z_0, z), and not on the path.

Linear and algebraic independences over a differential field

Theorem (Basic triangular theorem over a differential field [11])

Let $\mathcal{C}_0$ a differential subfield of $\mathcal{A}$. Let $S \in \mathcal{A}\langle\langle X\rangle\rangle$ be a $\sqcup\!\sqcup$ −group-like solution of $\mathbf{d}S = MS$, with $M = \sum_{x\in\mathcal{X}} u_x x$ and $u_x \in \mathcal{C}_0 \subset \mathcal{A}$. The following assertions are equivalent

1. *the family $\{\langle S|w\rangle\}_{w\in\mathcal{X}^*}$ is $\mathcal{C}_0$-linearly independent,*
2. *the family $\{\langle S|l\rangle\}_{l\in\mathcal{L}yn\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,*
3. *the family $\{\langle S|x\rangle\}_{x\in\mathcal{X}}$ is $\mathcal{C}_0$-algebraically independent,*
4. *the family $\{\langle S|x\rangle\}_{x\in\mathcal{X}\cup\{1_{\mathcal{X}^*}\}}$ is $\mathcal{C}_0$-linearly independent,*
5. *the family $\{u_i\}_{i\in I}$ of $\mathcal{C}_0$ is s.t., for $f \in \mathcal{C}_0$ and $\{c_i\}_{i\in I}$ in $\mathbb{C}$, one has*
$$\sum_{i\in I} c_i u_i = \partial f \quad\Longrightarrow\quad (\forall i \in I)(c_i = 0),$$
6. $\partial\mathcal{C}_0 \cap \mathrm{span}_{\mathbb{C}}\{u_i\}_{i=0,\dots,m} = \{0\}$.

Remarque

In case $\mathcal{A} = \mathcal{H}(\Omega)$ with $\emptyset \neq \Omega$ connex, this theorem holds when $\mathcal{C}_0$ is only a differential ring.

11. This form is a group-like version of the abstract form of (Deneufchâtel, Duchamp, HNM & Solomon, 2011).

Linear & algebraic independences over a differential ring

Theorem (Basic triangular theorem over a differential ring[12])

Let $S \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ *be a group-like solution of*

$$\mathbf{d}S = MS, \quad \textit{with } M = \sum_{x \in \mathcal{X}} u_x x \quad \textit{and } u_x \in \mathcal{C}_0 \subset \mathcal{A},$$

where the commutative associative ring $\mathcal{A}$*, equipped with the differential operator* ∂*, is supposed to contain* $\mathbb{Q}$*. Then we have*

(1) *If* $H \in \mathcal{A}\langle\langle\mathcal{X}\rangle\rangle$ *is an other group-like solution then there exists* $C \in \mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{X}\rangle\rangle$ *such that* $S = He^C$ *(and conversely).*

(2) *If* $\mathcal{C}_0$ *is a differential* $\mathbb{C}$*-subalgebra of* $\mathcal{A}$*, the following assertions are equivalent*

- (a) $\{\langle S|w\rangle\}_{w \in \mathcal{X}^*}$ *is* $\mathcal{C}_0$*-linearly independent.*
- (b) $(\langle S|S_l\rangle)_{l \in \mathcal{L}yn\mathcal{X}}$ *is* $\mathcal{C}_0$*-algebraically independent.*
- (c) $(\langle S|x\rangle)_{x \in \mathcal{X} \cup \{1_{\mathcal{X}^*}\}}$ *is* $\mathcal{C}_0$*-algebraically independent.*
- (d) $\{\langle S|x\rangle\}_{x \in \mathcal{X} \cup \{1_{\mathcal{X}^*}\}}$ *is* $\mathcal{C}_0$*-linearly independent.*
- (e) *Let* $W(f_1, f_2) = d(f_1)f_2 - f_1 d(f_2)$ *(wronskian). For all* $(f_1, f_2) \in \mathcal{C}_0 \times \mathcal{C}_0^{\times}$ *and* $c = (c_x)_{x \in \mathcal{X}} \in \mathbb{C}^{(\mathcal{X})}$*, one has*

$$W(f_1, f_2) = f_2^2 \sum_{x \in \mathcal{X}} c_x u_x \quad \Longrightarrow \quad (\forall x \in \mathcal{X})(c_x = 0).$$

12. see also in the talk by G.H.E. Duchamp

Examples of positive cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = 1_\Omega, \mathcal{C}_0 = \mathbb{C}\{\{u_x^{\pm 1}\}\} = \mathbb{C}$.
   $\alpha_0^z(x^n) = \alpha_0^z(x^{\sqcup\!\sqcup\, n}/n!) = z^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = xS$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n)x^n = \sum_{n\geq 0} \frac{z^n}{n!}x^n = e^{zx}.$$
   Moreover, $\alpha_0^z(x) = z$ which is transcendent over $\mathcal{C}_0$ and the family $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f = 0$. Thus,
   if $\partial f = cu_x$ then $c = 0$.
2. $\Omega = \mathbb{C}\backslash]-\infty, 0], u_x(z) = z^{-1}, \mathcal{C}_0 = \mathbb{C}\{\{z^{\pm 1}\}\} = \mathbb{C}[z^{\pm 1}] \subset \mathbb{C}(z)$.
   $\alpha_1^z(x^n) = \alpha_1^z(x^{\sqcup\!\sqcup\, n}/n!) = \log^n(z)/n!$, for $n \geq 1$. Thus $\mathbf{d}S = z^{-1}xS$ and
   $$S = \sum_{n\geq 0} \alpha_1^z(x^n)x^n = \sum_{n\geq 0} \frac{\log^n(z)}{n!}x^n = z^x.$$
   Moreover, $\alpha_1^z(x) = \log(z)$ which is transcendent over $\mathbb{C}(z)$ then over $\mathbb{C}[z^{\pm 1}]$. The family the family $\{\alpha_1^z(x^n)\}_{n\geq 0}$ is $\mathbb{C}(z)$-free and then $\mathcal{C}_0$-free. Let $f \in \mathcal{C}_0$ then $\partial f \in \mathrm{span}_{\mathbb{C}}\{z^{\pm n}\}_{n\neq 1}$. Thus,
   if $\partial f = cu_x$ then $c = 0$.

Examples of negative cases over $\mathcal{X} = \{x\}, \mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = e^z, \mathcal{C}_0 = \mathbb{C}\{\{e^{\pm z}\}\} = \mathbb{C}[e^{\pm z}]$.
   $\alpha_0^z(x^n) = \alpha_0^z(x^{\,\sqcup\!\sqcup\, n}/n!) = (e^z - 1)^n/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = e^z x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n) x^n = \sum_{n \geq 0} \frac{(e^z - 1)^n}{n!} x^n = e^{(e^z - 1)x}.$$
   Moreover, $\alpha_0^z(x) = e^z - 1$ which is not transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = ce^z \in \mathcal{C}_0$ $(c \neq 0)$ then $W(f, 1_\Omega) = \partial f(z) = ce^z = cu_x(z)$.
2. $\Omega = \mathbb{C} \backslash]-\infty, 0], u_x(z) = z^a (a \notin \mathbb{Q})$,
   $\mathcal{C}_0 = \mathbb{C}\{\{z, z^{\pm a}\}\} = \mathrm{span}_{\mathbb{C}}\{z^{ka+l}\}_{k,l\in\mathbb{Z}}$.
   $\alpha_0^z(x^n) = \alpha_0^z(x^{\,\sqcup\!\sqcup\, n}/n!) = (a+1)^{-n} z^{n(a+1)}/n!$, for $n \geq 1$. Thus, $\mathbf{d}S = z^a x S$ and
   $$S = \sum_{n\geq 0} \alpha_0^z(x^n) x^n = \sum_{n \geq 0} \frac{z^{n(a+1)}}{(a+1)^n n!} x^n = e^{(a+1)^{-1} z^{(a+1)} x}.$$
   Moreover, $\alpha_0^z(x) = (a+1)^{-1} z^{a+1}$ which is not transcendent over $\mathcal{C}_0$ and $\{\alpha_0^z(x^n)\}_{n\geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = c(a+1)^{-1} z^{a+1} \in \mathcal{C}_0$ $(c \neq 0)$ then $W(f, 1_\Omega) = \partial f(z) = cz^a = cu_x(z)$.

EXTENDED REGULARIZATION OF DIVERGENT POLYZETAS BY NEWTON-GIRARD FORMULA

Families of eulerian functions

$\forall r \geq 1, \quad \Gamma_{y_r}(1+z) := e^{-f_r(z)} \quad$ and $\quad \mathrm{B}_{y_r}(a,b) := \Gamma_{y_r}(a)\Gamma_{y_r}(b)/\Gamma_{y_r}(a+b)$, where, for any $z \in \mathbb{C}$ such that $|z| < 1$,

$$f_1(z) := \gamma z - \sum_{k\geq 2} \zeta(k)(-z)^k/k \quad \text{and} \quad f_r(z) := -\sum_{k\geq 1} \zeta(kr)(-z^r)^k/k, r \geq 2.$$

For $r \geq 1$, let $\vartheta = e^{2\mathrm{i}\pi/r}$. We have, for $|z| < 1$,

$$f_r(z) = -\sum_{k\geq 1} \zeta(kr)(-z^r)^k/k = \sum_{j=0}^{r-1} f_1(\vartheta^j z) = -\sum_{j=0}^{r-1} \log(\Gamma(1+\vartheta^j z)).$$

Taking the exponential and using Weierstrass factorization, we also have

$$e^{f_r(z)} = \prod_{j=0}^{r-1} \frac{1}{\Gamma(1+\vartheta^j z)} = \prod_{j=0}^{r-1} e^{\gamma \vartheta^j z} \prod_{n\geq 1} \left(1 + \frac{\vartheta^j z}{n}\right) e^{-\frac{\vartheta^j z}{n}}.$$

Proposition

$\{f_r\}_{r\geq 1}$ and $\{e^{f_r}\}_{r\geq 1} \cup \{1_\Omega\}$ are[13] $\mathbb{C}$-linearly independent. Moreover, f_r is holomorphic[14] on the open unit disc and e^{f_r} (resp. e^{-f_r}) is entire (resp. meromorphic) admitting a countable set of isolated zeros (resp. poles) on the complex plan which is $\bigcup_{j=0}^{r-1} \vartheta^j \mathbb{N}_{\leq -1}$, for $r \geq 1$.

13. Since $(f_r)_{r\geq 1}$ is triangular then $(f_r)_{r\geq 1}$ is $\mathbb{C}$-linearly free. So is $(e^{f_r} - 1)_{r\geq 1}$, being triangular, then $(e^{f_r})_{r\geq 1}$ is $\mathbb{C}$-linearly free and free from 1.

14. $\forall r \geq 2, \zeta(2) \geq \zeta(r) \geq 1$: this proves that the radius of convergence of any the f_r is exactly one. In other words f_r is holomorphic on the open unit disc.

Independences by BTT (work in progress, 1/2)

$$M = \sum_{y_r \in Y} u_{y_r} y_r, \text{ with } \begin{Bmatrix} u_{y_r} &= e^{f_r} \partial f_r \\ \omega_r(z) &= u_{y_r}(z) dz \end{Bmatrix} \longleftrightarrow C_{0 \rightsquigarrow z} = \prod_{l \in \mathcal{L}ynY}^{\searrow} e^{\alpha_0^z(S_l) P_l}.$$

Let $F := \mathrm{span}_{\mathbb{C}}\{f_r\}_{r \geq 1}$, $E := \mathrm{span}_{\mathbb{C}}\{e^{f_r}\}_{r \geq 1}$ and let $\mathbb{C}[F], \mathbb{C}[E]$ denote their respective algebras. Let $\mathcal{F} := \mathbb{C}\{\{(f_r^{\pm 1})_{r \geq 1}\}\}, \mathcal{E} := \mathbb{C}\{\{(e^{\pm f_r})_{r \geq 1}\}\}$.

Since, for any $i, l, k \geq 1$, there exists $q_{i,l,k} \in \partial\mathcal{F} \setminus \mathbb{C}.1_\Omega$ such that $(\partial^i e^{\pm f_k})^l = q_{i,l,k} e^{\pm l f_k} \notin E$ then $\partial\mathcal{E} \subset \mathcal{C}_0$ where [15]

$$\begin{aligned} \mathcal{C}_0 &:= \mathrm{span}_{\mathbb{C}}\{q_{i_1,l_1,r_1} \ldots q_{i_k,l_k,r_k} e^{l_1 f_{r_1} + \ldots + l_k f_{r_k}}\}_{(i_1,l_1,r_1),\ldots,(l_k,r_k) \in \mathbb{N}_+ \times \mathbb{Z}_+ \times \mathbb{N}_+, k \geq 1} \\ &\subset \mathrm{span}_{\partial\mathcal{F}}\{e^{\phi_{r_1,\ldots,r_k}}\}_{r_1,\ldots,r_k \in \mathbb{N}_+, k \geq 1} \text{ with } \phi_{r_1,\ldots,r_k} := l_1 f_{r_1} + \ldots + l_k f_{r_k}. \end{aligned}$$

Let $0 \neq g \in \mathcal{C}_0 \subset \mathrm{Fr}(\mathcal{C}_0)$ and let $\{c_r\}_{r \geq 1}$ be a sequence, in $\mathbb{C}$, such that

$$\partial g = \sum_{r \geq 1} c_r u_r = \sum_{r \geq 1} c_r \partial e^{f_r} = \sum_{r \geq 1} c_r (\partial f_r) e^{f_r}.$$

$\partial g \neq 0$ is impossible because $\mathrm{Fr}(\mathcal{C}_0) \cap E = \{0\}$.
Hence, $\partial g = 0$ and then $\forall r \geq 1, c_r = 0$. ...

15. As linear combination of triangular holomorphic functions vanishing at zero, $\phi_{r_1,\ldots,r_k}$ is triangular and holomorphic satisfying $\phi_{r_1,\ldots,r_k}(0) = 0$ and $e^{\phi_{r_1,\ldots,r_k}}$ is then entire. They are $\mathbb{C}$-algebraically independent. Moreover, similarly to $\{f_r\}_{r \geq 1}$ and $\{e^{f_r}\}_{r \geq 1}$, the families $(\phi_{r_1,\ldots,r_k})_{k \geq 1}$ and $(e^{\phi_{r_1,\ldots,r_k}})_{k \geq 1}$ are $\mathbb{C}$-linearly independent.

Independences by BTT (work in progress, 2/2)

Theorem

$(e^{f_r})_{r\geq 1}$ (resp. $(f_r)_{r\geq 1}$) is algebraically independent over $\partial\mathcal{E}$ (resp. $\partial\mathcal{F}$). Hence, $\mathbb{C}[E]$ and $\partial\mathcal{E}$ are algebraically disjoint. So are $\mathbb{C}[F]$ and $\partial\mathcal{F}$.

Next, we have firstly the algebraic independence of $\{y^*\}_{y\in Y}$ over $(\mathbb{C}\langle Y\rangle, ⧢, 1_{Y^*})$ and then over $(\mathbb{C}.Y, ⧢, 1_{Y^*})$.
Secondly, with $\bar{u}_r = \partial f_r$, $r\geq 1$, taking iterated integrals, we get on the one hand, $\alpha_0^z(y_r^*) = e^{f_r(z)}$ and $\alpha_0^z(y_r^n) = \alpha_0^z(y_r^{⧢ n}/n!) = f_r^n(z)/n!$, $n\geq 0$.
On the other hand, since $(e^{f_r})_{r\geq 1}$ and $(f_r)_{r\geq 1}$ are algebraically free families, respectively, of $\mathbb{C}[E]$ and $\mathbb{C}[F]$ then, using the first fact and the injectivity of α_0^z (restricted in this case), it follows that

Corollary

1. *$(e^{f_r})_{r\geq 1}$ is algebraically independent over $\mathbb{C}[F]$.*
2. *$\mathbb{C}[F]$ and $\mathbb{C}[E]$ are algebraically disjoint.*
3. *$(f_r)_{r\geq 1}$ is algebraically independent over $\mathbb{C}[E]$.*
4. *$(\phi_{r_1,\ldots,r_k})_{k\geq 1}$ and $(e^{\phi_{r_1,\ldots,r_k}})_{k\geq 1}$ are algebraically independent, respectively, over $\mathbb{C}[F]$ and $\mathbb{C}[E]$.*

Back to polylogarithms : $u_0(z) = z^{-1}, u_1(z) = (1-z)^{-1}$

Here, $\Omega = \widetilde{\mathbb{C} \setminus \{0,1\}}, \mathcal{C} = \mathbb{C}\{\{u_0^{\pm 1}, u_1^{\pm}\}\} = \mathbb{C}[z, z^{-1}, (1-z)^{-1}] \subset \mathbb{C}(z)$ and

$$C_{z_0 \rightsquigarrow z} = \mathrm{L}(z)(\mathrm{L}(z_0))^{-1}, \quad \text{where}^{16} \quad \mathrm{L} = \sum_{w \in X^*} \mathrm{Li}_w\, w = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l} P_l},$$

$\mathrm{Li}_{x_0}(z) = \alpha_1^z(x_0) = \log(z)$ and, for $n_1, \ldots n_r \in \mathbb{N}_+$ and $z \in \mathbb{C}, |z| < 1$,

$$\mathrm{Li}_{x_0^{n_1-1}x_1 \ldots x_0^{n_r-1}x_1}(z) = \alpha_0^z(x_0^{n_1-1}x_1 \ldots x_0^{n_r-1}x_1) = \sum_{k_1 > \ldots > k_r > 0} \frac{z^{k_1}}{k_1^{n_1} \ldots k_1^{n_r}}.$$

The coefficients $\{\mathrm{H}_{y_{s_1} \ldots y_{s_r}}(n)\}_{n \geq 1}$ are defined by the following Taylor expansion

$$\frac{1}{1-z} \mathrm{Li}_{x_0^{n_1-1}x_1 \ldots x_0^{n_r-1}x_1}(z) = \sum_{n \geq 0} \mathrm{H}_{y_{s_1} \ldots y_{s_r}}(n) z^n.$$

The following morphisms of algebras are injective

$$\begin{aligned} \mathrm{Li}_\bullet &: (\mathbb{Q}\langle X \rangle, ш, 1_{X^*}) \longrightarrow (\mathbb{Q}\{\mathrm{Li}_w\}_{w \in X^*}, ., 1), & w &\longmapsto \mathrm{H}_w, \\ \mathrm{H}_\bullet &: (\mathbb{Q}\langle Y \rangle, \text{⧢}, 1_{Y^*}) \longrightarrow (\mathbb{Q}\{\mathrm{H}_w\}_{w \in Y^*}, ., 1), & w &\longmapsto \mathrm{Li}_w. \end{aligned}$$

Hence[17], $\{\mathrm{Li}_l\}_{l \in \mathcal{L}ynX}$ and $\{\mathrm{H}_l\}_{l \in \mathcal{L}ynY}$ are algebraically independent.

---

16. $\forall k \geq 1, \exists Q_k \in \mathcal{C}, \mathbf{d}^k C_{z_0 \rightsquigarrow z} = (\mathbf{d}^k \mathrm{L}(z))(\mathrm{L}(z_0))^{-1} = Q_k \mathrm{L}(z)(\mathrm{L}(z_0))^{-1}$.
Moreover, the PV extension related to $(NCDE)$ is $\mathcal{C}\langle\langle X \rangle\rangle\{C_{z_0 \rightsquigarrow z}\} = \mathcal{C}\langle\langle X \rangle\rangle\{\mathrm{L}\}$.
17. $\forall l \in \mathcal{L}ynX \setminus \{x_0\}$, then $l, S_l \in \mathbb{C}_+\langle X \rangle x_1$ and $\pi_Y(l) \in \mathcal{L}ynY$.
$\forall l \in \mathcal{L}ynY$ then $\pi_X(l) \in \mathcal{L}ynX \setminus \{x_0\}$.

Polyzetas and 3 characters of regularization

By a Abel's theorem, for $n_1 > 1$, one has

$$\zeta(n_1,\ldots,n_r) := \lim_{z\to 1} \mathrm{Li}_{x_0^{n_1-1}x_1\ldots x_0^{n_r-1}x_1}(z) = \lim_{n\to+\infty} \mathrm{H}_{y_{n_1}\ldots y_{n_r}}(n),$$

$$\mathcal{Z} := \mathrm{span}_{\mathbb{Q}}\{\mathrm{Li}_w(1)\}_{w\in x_0X^*x_1} = \mathrm{span}_{\mathbb{Q}}\{\mathrm{H}_w(+\infty)\}_{w\in Y^*\setminus y_1Y^*}.$$

We use then the one-to-one correspondences

$$(s_1,\ldots,s_r)\in\mathbb{N}_+^r \leftrightarrow y_{s_1}\ldots y_{s_r}\in Y^* \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1\in X^*x_1.$$

The following poly-morphism is surjective

$$\zeta : \begin{array}{c}(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}\langle X\rangle x_1, ш, 1_{X^*})\\ (\mathbb{Q}1_{Y^*}\oplus (Y-\{y_1\})\mathbb{Q}\langle Y\rangle, \text{⧢}, 1_{Y^*})\end{array} \longrightarrow\!\!\!\!\rightarrow (\mathcal{Z}, ., 1),$$

mapping, both, $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$ and $y_{s_1}\ldots y_{s_r}$ to $\zeta(s_1,\ldots,s_r)$.

It can be extended as characters as follows

$$\begin{array}{rcl}\zeta_{ш} : (\mathbb{R}\langle X\rangle, ш, 1_{X^*}) & \longrightarrow & (\mathbb{R}, ., 1),\\ \zeta_{\text{⧢}}, \gamma_\bullet : (\mathbb{R}\langle Y\rangle, \text{⧢}, 1_{Y^*}) & \longrightarrow & (\mathbb{R}, ., 1),\end{array}$$

s.t. $\zeta_{ш}(x_0) = \log(1) = 0$, $\zeta_{\text{⧢}}(l) = \zeta_{\text{⧢}}(\pi_Y l) = \gamma_{\pi_Y l} = \zeta(l)$, for $l\in\mathcal{L}ynX - X$, and

$$\begin{array}{rclll}\zeta_{ш}(x_1) = & 0 & = \mathrm{f.p.}_{z\to 1}\log(1-z), & \{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\ \zeta_{\text{⧢}}(y_1) = & 0 & = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\\ \gamma_{y_1} = & \gamma & = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_1(n), & \{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}.\end{array}$$

Extensions of $\mathrm{Li}_\bullet$ and of $\mathrm{H}_\bullet$ ($\mathcal{C}=\mathbb{C}\{z^a,(1-z)^b\}_{a,b\in\mathbb{C}}$)

Theorem (indexing by noncommutative rational series)

1. *If $R\in\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle$ with minimal representation of dimension n then*[18]
$$y(z_0,z)=\alpha_{z_0}^z(R)=:\langle R\|C_{z_0\rightsquigarrow z}\rangle=\langle R\|\mathrm{L}(z)(\mathrm{L}(z_0))^{-1}\rangle.$$
There exists $l=0,..,n-1$ s.t. $\{\partial^k y\}_{0\le k\le l}$ is $\mathcal{C}$-linearly independent and $a_l,\ldots,a_1,a_0\in\mathcal{C}$ s.t. $(a_l\partial^l+a_{l-1}\partial^{l-1}+\ldots+a_1\partial+a_0)y=0$.

2. *$\{\mathrm{Li}_w\}_{w\in X^*}$ is $\mathcal{C}$-linearly independent*[19]*. Moreover, the kernel of the following map is the $ш$ -ideal is generated by $x_0^*ш x_1^*-x_1^*+1$*
$$\mathrm{Li}_\bullet:(\mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\langle X\rangle\rangle ш\mathbb{C}\langle X\rangle,ш,1_{X^*})\twoheadrightarrow(\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*},.,1_\Omega),\quad R\mapsto\mathrm{Li}_R.$$

3. *The algebra $\mathcal{C}\{\mathrm{Li}_w\}_{w\in X^*}$ is closed under the differential operators $\theta_0=z\partial,\theta_1=(1-z)\partial$, and under their sections*[20] *ι_0,ι_1.*

Theorem (Kleene stars of the plane)

By Newton-Girard formula, the arithmetic function $\mathrm{H}_{(ty_r)^*}$ is given by[21]
$$\forall r\ge1,\forall t\in\mathbb{C},|t|<1,\quad \mathrm{H}_{(t^ry_r)^*}=\sum_{k\ge0}\mathrm{H}_{y_r^k}t^{kr}=\exp\left(-\sum_{k\ge1}\mathrm{H}_{y_{kr}}\frac{(-t^r)^k}{k}\right)$$
and $\mathrm{H}_{(\sum_{s\ge1}a_sy_s)^*}\mathrm{H}_{(\sum_{s\ge1}b_sy_s)^*}=\mathrm{H}_{(\sum_{s\ge1}(a_s+b_s)y_s+\sum_{r,s\ge1}a_sb_ry_{s+r})^*}$ $(|a_s|,|b_s|<1)$.

18. $\alpha_{z_0}^z:\mathbb{C}^{\mathrm{rat}}\langle\langle X\rangle\rangle\to\mathcal{H}(\Omega)$ is not injective : $\alpha_{z_0}^z(z_0x_0^*+(1-z_0)(-x_1)^*-1_{X^*})=0$.
19. The proof uses BTT (see also in the talk by G.H.E. Duchamp).
20. *i.e.* $\theta_0\iota_0=\theta_1\iota_1=\mathrm{Id}$. Note also that $[\theta_0,\theta_1]=\theta_0+\theta_1=\partial$.
21. $-\sum_{k\ge1}\mathrm{H}_{kr}(-t^r)^k/k$ is termwise dominated by $\|f_r\|_\infty$ and then $\mathrm{H}_{(t^ry_r)^*}$ is termwise dominated, in norm, by e^{f_r}.

Extended double regularization

Theorem (Regularization by Newton-Girard formula)

The characters $\zeta_{\sqcup\!\sqcup}$, $\gamma_\bullet$ can be extended algebraically as follows

$$\begin{array}{rcl} \zeta_{\sqcup\!\sqcup} : (\mathbb{C}\langle X\rangle \sqcup\!\sqcup \mathbb{C}^{\mathrm{rat}}_{\mathrm{exc}}\langle\!\langle X\rangle\!\rangle, \sqcup\!\sqcup, 1_{X^*}) & \longrightarrow & (\mathbb{C}, ., 1), \\ \forall t \in \mathbb{C}, |t| < 1, (tx_0)^*, (tx_1)^* & \longmapsto & 1_{\mathbb{C}}. \\ \gamma_\bullet : (\mathbb{C}\langle Y\rangle \uplus \{\mathbb{C}^{\mathrm{rat}}\langle\!\langle y_r\rangle\!\rangle\}_{r\geq 1}, \uplus, 1_{Y^*}) & \longrightarrow & (\mathbb{C}, ., 1), \\ \forall t \in \mathbb{C}, |t| < 1, \forall r \geq 1, (t^r y_r)^* & \longmapsto & \Gamma_{y_r}^{-1}(1+t). \end{array}$$

Moreover, the morphism $(\mathbb{C}[\{y_r^*\}_{r\geq 1}], \uplus, 1_{Y^*}) \longrightarrow (\mathbb{C}[\{e^{f_r}\}_{r\geq 1}], \times, 1)$, *maps* y_r^* *to* $\Gamma_{y_r}^{-1}$, *is injective*[22] *and* $\Gamma_{y_{2r}}(1-t) = \Gamma_{y_r}(1+t)\Gamma_{y_r}(1-t)$.

Corollary (comparison formula)

For any $z, a, b \in \mathbb{C}$ *such that* $|z| < 1$ *and* $\Re a > 0, \Re b > 0$, *one has*[23]

$$\mathrm{Li}_{x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1)^*]}(z) = \mathrm{Li}_{x_1[((a-1)x_0)^* \sqcup\!\sqcup (-bx_1)^*]}(z) = \mathrm{B}(z; a, b),$$

(partial Beta function) and $\mathrm{B}(1; a, b) = \mathrm{B}(a, b)$. *Hence,*

$$\mathrm{B}(a,b) = \frac{\gamma_{((a+b-1)y_1)^*}}{\gamma_{((a-1)y_1)^* \uplus ((b-1)y_1)^*}} = \zeta_{\sqcup\!\sqcup}(x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1)^*]) = \zeta_{\sqcup\!\sqcup}(x_1[((a-1)x_0)^* \sqcup\!\sqcup (-bx_1)^*]).$$

---

22. $\{y_r^*\}_{r\geq 1}$ and $\{e^{f_r}\}_{r\geq 1}$ are $\mathbb{C}$-algebraically independent.
23. $x_0[(ax_0)^* \sqcup\!\sqcup ((1-b)x_1)^*]$ and $x_1[((a-1)x_0)^* \sqcup\!\sqcup (-bx_1)^*]$ are of the form (F_2) which is closed by conc, $\sqcup\!\sqcup$ and co-products.

Polyzetas and extended eulerian functions

For $t_0, t_1 \in \mathbb{C}, |t_0| < 1, |t_1| < 1$, let $R := t_0^2 t_1 x_0[(t_0x_0)^* ⧢ (t_1x_1)^*]x_1$.
With $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1-z)^{-1}dz$, we get

$$\begin{aligned}\mathrm{Li}_R(1) &= t_0^2 t_1 \int_0^1 \frac{ds}{s} \int_0^s \left(\frac{s}{r}\right)^{t_0} \left(\frac{1-r}{1-s}\right)^{t_1} \frac{dr}{1-r} \\ &= t_0^2 t_1 \int_0^1 (1-s)^{t_0 t_1} s^{t_0-1} \int_0^s (1-r)^{t_0-1} r^{-t_0} ds dr.\end{aligned}$$

By changes of variables, $r = st$ and then $y = (1-s)/(1-st)$, we obtain

$$\begin{aligned}\zeta(R) &= t_0^2 t_1 \int_0^1 \int_0^1 (1-s)^{t_0 t_1} (1-st)^{t_0-1} t^{-t_0} dt ds \\ &= t_0^2 t_1 \int_0^1 \int_0^1 (1-ty)^{-1} t^{-t_0} y^{t_0 t_1} dt dy.\end{aligned}$$

By expending $(1-ty)^{-1}$ and then by integrating, we get on the one hand

$$\zeta(R) = \sum_{n \geq 1} \frac{t_0}{n - t_0} \frac{t_0 t_1}{n - t_0^2 t_1} = \sum_{k > l > 0} \zeta(k) t_0^k t_1^l.$$

Since $R = t_0 x_0 (t_0 x_0 + t_1 x_1)^* t_0 t_1 x_1$ then we get also on the other hand

$$\zeta(R) = \sum_{k>0} \sum_{l>0} \sum_{s_1+\ldots+s_l=k, s_1 \geq 2, s_2 \ldots, s_l \geq 1} \zeta(s_1, \ldots, s_l) t_0^k t_1^l.$$

Identifying the coeffients of $\langle \zeta(R) | t_0^k t_1^l \rangle$, we deduce the sum formula

$$\zeta(k) = \sum_{s_1+\ldots+s_l=k, s_1 \geq 2, s_2 \ldots, s_l \geq 1} \zeta(s_1, \ldots, s_l).$$

Riemann zeta function and eulerian functions

For $v = -u$ ($|u| < 1$), one gets

$$\frac{1}{\Gamma_{y_1}(1-u)\Gamma_{y_1}(1+u)} = \exp\left(-\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k}\right) = \frac{\sin(u\pi)}{u\pi}.$$

Taking the logarithms and then taking the Taylor expansions, one obtains

$$\begin{aligned}
-\sum_{k\geq 1}\zeta(2k)\frac{u^{2k}}{k} &= \log\left(1+\sum_{n\geq 1}\frac{(u\mathrm{i}\pi)^{2n}}{\Gamma_{y_1}(2n)}\right) \\
&= \sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{k\geq 1}(u\mathrm{i}\pi)^{2k}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma_{y_1}(2n_i)} \\
&= \sum_{k\geq 1}(u\mathrm{i}\pi)^{2k}\sum_{l\geq 1}\frac{(-1)^{l-1}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma_{y_1}(2n_i)}.
\end{aligned}$$

One can deduce then the following expression for $\zeta(2k)$:

$$\frac{\zeta(2k)}{\pi^{2k}} = k\sum_{l=1}^{k}\frac{(-1)^{k+l}}{l}\sum_{\substack{n_1,\ldots,n_l\geq 1\\ n_1+\ldots+n_l=k}}\prod_{i=1}^{l}\frac{1}{\Gamma_{y_1}(2n_i)} \in \mathbb{Q}.$$

Euler gave an other explicit formula using Bernoulli numbers $\{b_k\}_{k\in\mathbb{N}}$:

$$\frac{\zeta(2k)}{(2\mathrm{i}\pi)^{2k}} = -\frac{b_{2k}}{2(2k)!} \in \mathbb{Q}.$$

More about Riemann zeta function and eulerian functions

$$
\begin{array}{llllll}
 & \gamma_{(-t^2y_2)^*} & = & \gamma_{(ty_1)^*}\gamma_{(-ty_1)^*} & & \\
\Leftrightarrow & \Gamma_{y_2}^{-1}(1-t) & = & \Gamma_{y_1}^{-1}(1+t)\Gamma_{y_1}^{-1}(1-t) & & \\
\Leftrightarrow & e^{-\sum_{k\geq 2}\zeta(2k)t^{2k}/k} & = & \dfrac{\sin(t\pi)}{t\pi} & = & \displaystyle\sum_{k\geq 1}\frac{(ti\pi)^{2k}}{(2k)!}. \\
 & \gamma_{(-t^4y_4)^*} & = & \gamma_{(t^2y_2)^*}\gamma_{(-t^2y_2)^*} & & \\
\Leftrightarrow & \Gamma_{y_4}^{-1}(1-t) & = & \Gamma_{y_2}^{-1}(1+t)\Gamma_{y_2}^{-1}(1-t) & & \\
\Leftrightarrow & e^{-\sum_{k\geq 1}\zeta(4k)t^{4k}/k} & = & \dfrac{\sin(it\pi)}{it\pi}\dfrac{\sin(t\pi)}{t\pi} & = & \displaystyle\sum_{k\geq 1}\frac{2(-4t\pi)^{4k}}{(4k+2)!}.
\end{array}
$$

Since $\gamma_{(-t^4y_4)^*} = \zeta((-t^4y_4)^*), \gamma_{(-t^2y_2)^*} = \zeta((-t^2y_2)^*), \gamma_{(t^2y_2)^*} = \zeta((t^2y_2)^*)$ then, using the poly-morphism ζ, one deduces

$$
\begin{aligned}
\zeta((-t^4y_4)^*) = \quad & \zeta((-t^2y_2)^*)\zeta((t^2y_2)^*) & = \zeta((-t^2x_0x_1)^*)\zeta((t^2x_0x_1)^*)) \\
= \; & \zeta((-t^2x_0x_1)^* \sqcup\!\sqcup (t^2x_0x_1)^*) & = \zeta((-4t^4x_0^2x_1^2)^*).
\end{aligned}
$$

It follows then, by identification the coeffients of t^{2k} and t^{4k} :

$$
\zeta(\overbrace{2,\ldots,2}^{k\texttt{times}})/\pi^{2k} = 1/(2k+1)! \in \mathbb{Q},
$$

$$
\zeta(\overbrace{3,1,\ldots,3,1}^{k\texttt{times}})/\pi^{4k} = 4^k\zeta(\overbrace{4,\ldots,4}^{k\texttt{times}})/\pi^{4k} = 2/(4k+2)! \in \mathbb{Q}.
$$

THANK YOU FOR YOUR ATTENTION