Lévy-Stable Distributions: Mathematical Properties and Explicit Representations

K.A. Penson

Sorbonne Université, Campus UPMC
75005 Paris

1. First concrete example: Lévy-Smirnov distribution moments ....
2. Relation with Laplace transform
3. For rational parameters $\alpha = l/k$, $k > l$ Laplace inversion
4. Implementation with Computer Algebra Systems

Joint work with K. Górska, A. Horzela (IFJ, PAN, Kraków), G. Dattoli

First concrete example
P. P. Lévy, Smirnov (1937)
— rather strange probability distribution —

$$g_{\frac{1}{2}}(x) = \frac{\exp\left(-\frac{1}{4x}\right)}{2\sqrt{\pi}\, x^{3/2}}, \quad 0 < x < \infty$$

Moments: $\rho(n) = \int_0^\infty x^n g_{\frac{1}{2}}(x)\,dx$

n	$\ldots -3$	-2	-1	0	1	2	$3 \ldots$
$\rho(n)$	120	12	2	1	∞	∞	∞

↑ normalized

Formula for moments:

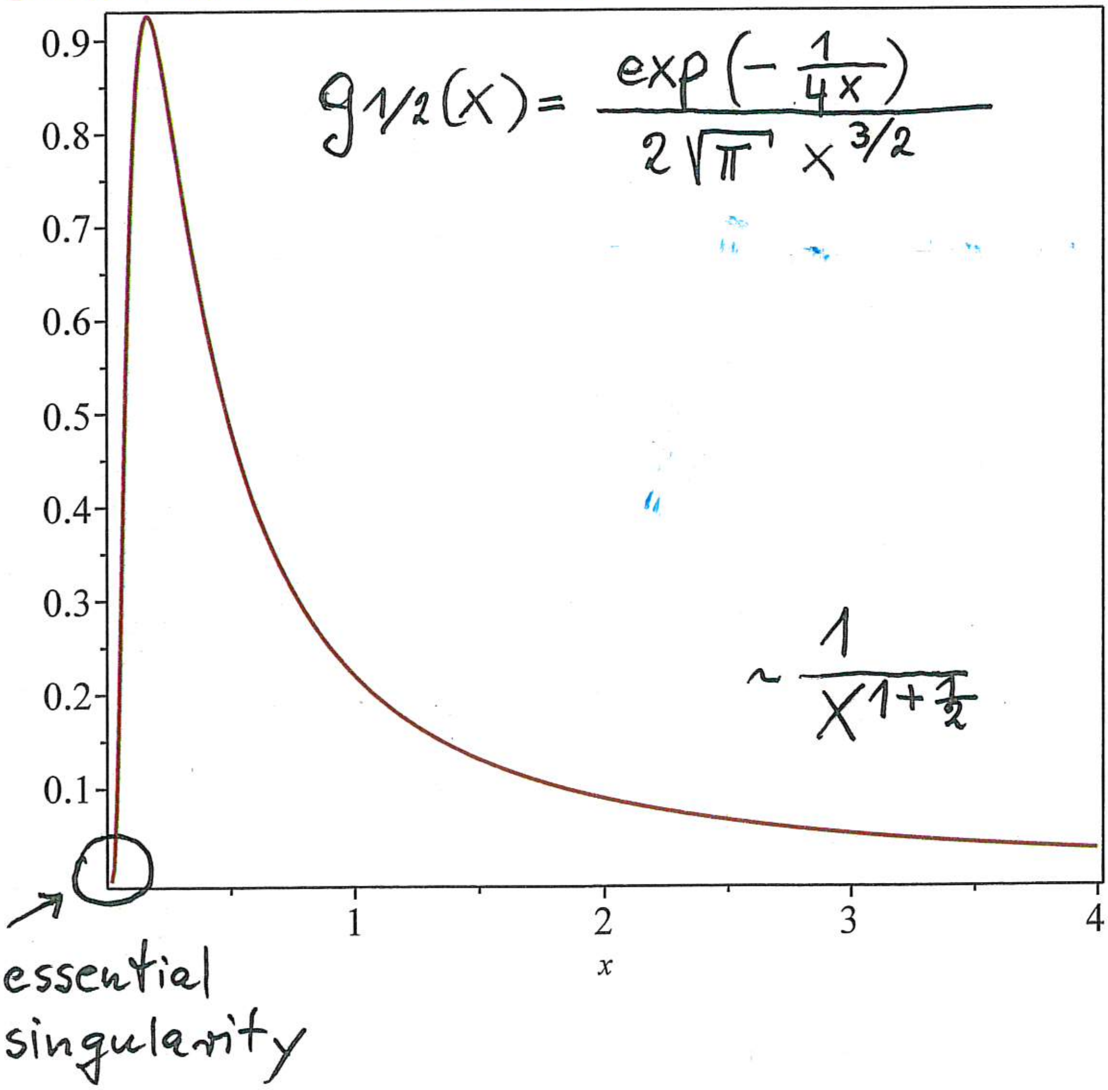
$$\rho(n) = \frac{\Gamma\left(\frac{1}{2} - n\right)}{2^{2n}\sqrt{\pi}}, \quad n < \frac{1}{2}$$

- All integer positive moments diverge (also fractional positive)
- All negative moments converge (also fractional negative)

```
> #  Lévy-Smirnov probability distribution on (0,infinity)
> g12:=proc(x)exp(-1/(4*x))/(2*sqrt(Pi)*x^(3/2));end;
```

$$g12 := \mathbf{proc}(x)\ 1/2 * \exp(-1/(4*x))/(\mathrm{sqrt}(\mathrm{Pi})*x\verb|^|(3/2))\ \mathbf{end\ proc} \tag{1}$$

```
> ?taylor
> plot(g12(x),x=0.02..4,axes=boxed);
```



Before continuing the calculations

- "stable" condition:

Laplace convolution $g(x) \circledast g(x)$

$$= \int_0^\infty g(y)\, g(x-y)\, dy \approx g(x)$$

Self-convolution preserves the form

Bochner, Feller

- the only form of Laplace transform is $\sim e^{-p^\alpha}$, $0<\alpha\, 1$

- the resulting Laplace inversion is always a positive function

For real α there seems to be no practicable solution.

For rational $\alpha = \frac{l}{k}$, $k > l$

$\mathcal{L}^{-1}[e^{-p^{(l/k)}}; x]$ EXISTS !!!

Where comes the index $\frac{1}{2}$ from?

Consider the Laplace transform of $g_{1/2}(x)$:

$$\mathcal{L}[g_{1/2}(x);p] = \int_0^\infty e^{-p\cdot x} g_{1/2}(x)\,dx$$

$$\left\{\begin{matrix} \text{per hand} \\ \text{or Tables} \end{matrix}\right\} = e^{-\sqrt{p}}\,.$$

$$\Downarrow$$

$$\mathcal{L}^{-1}[e^{-\sqrt{p}};x] = g_{1/2}(x)$$

How about more "general powers in the exponent"?

$$e^{-p^\alpha} \Longrightarrow g_\alpha(x)\,, \quad 0<\alpha<1$$

(related to Kohlrausch-Williams-Watts function)

Inverse Laplace of e^{-p^α}:

$$g_\alpha(x) = \mathcal{L}^{-1}[e^{-p^\alpha};x]$$

Further properties of all one-sided stable distributions

$0 < \alpha < 1$

A) $g_\alpha(x) \to 0, \quad x \to 0$

with essential singularity

$$\sim x^{P(\alpha)} \exp\left(-A(\alpha)\, x^{-\frac{\alpha}{1-\alpha}}\right),$$

$$P(\alpha) = \frac{\alpha - 2}{2(1-\alpha)}$$

(J. Mikusiński, 1959)

B) $g_\alpha(x) \to \sim x^{-(1+\alpha)}, \quad x \to \infty$

long tails

C) All the fractional moments

$$\int_0^\infty x^\mu g_\alpha(x)\, dx = \frac{\Gamma\left(-\frac{\mu}{\alpha}\right)}{\alpha\, \Gamma(-\mu)}$$

for $\boxed{-\infty < \mu < \alpha}$ are finite

D) all $g_\alpha(x)$ are unimodal

(have one maximum)

(W. Gawronski 1984)

Amazing circumstance: k, l integers

$\mathcal{L}^{-1}\left[e^{-p^{(l/k)}}; x\right]$ exists for $k > l$

$$\rightarrow g_{l/k}(x) = \frac{\sqrt{kl}}{(2\pi)^{(k-l)/2}} \frac{1}{x} G^{k,0}_{l,k}\left(\frac{l^l}{k^k x^l} \,\middle|\, \begin{matrix} \Delta(l,0) \\ \Delta(k,0) \end{matrix} \right)$$

Formula 2.2.1.19

of "Integrals and Series" vol. 5

(Gordon and Breach, Amsterdam, 1998)

$\Delta(k,a) = \frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+(k-1)}{k}$, a list.

$G^{\cdot\cdot}_{\cdot\cdot}\left(z \,\middle|\, \begin{matrix} \text{Parameters} \\ \text{Parameters} \end{matrix}\right)$ Meijer G function

defined exclusively via

Mellin Tr. $f^*(s) = \mathcal{M}[f(x); s] = \int_0^\infty x^{s-1} f(x)\, dx$

Inverse Mellin Tr. $f(x) = \mathcal{M}^{-1}[f^*(s); x]$

... Formula without proof

INTEGRALS AND SERIES

VOLUME 5: INVERSE LAPLACE TRANSFORMS

A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev

Gordon and Breach Science Publishers

Integrals and Series (Gordon and Breach, Amsterdam, 1998), Vols. 1–5.

[27] P. Humbert, Bull. Soc. Math. Fr. **69**, 121 (1945).

[28] B. D. Hughes, *Random Walks and Random Environments* (Clarendon Press, Oxford, 1995), Vol. 1.

[29] We have made extensive use of MAPLE® in this work.

[30] Here is the MAPLE® procedure LevyDist (k, l, x) used to calculate Eq. (2): LevyDist := proc(k, l, x) simplify(convert(sqrt($k * l$) * MeijerG([[], [seq($j1/l$, $j1 = 0..l-1$)]], [[seq($j2/k$, $j2 = 0..k-1$)], []], l^$l/(k$^$k * x$^l))/($x * (2 * Pi)$^(($k - l$)/2)), StandardFunctions)); end; Analogous syntax can be given for MATHEMATICA®.

2 lines Maple procedure given as Ref. [30] in

K.A. Penson & K. Górska PRL <u>105</u>, 210104 (2010)

for $g_{1/k}(x)$

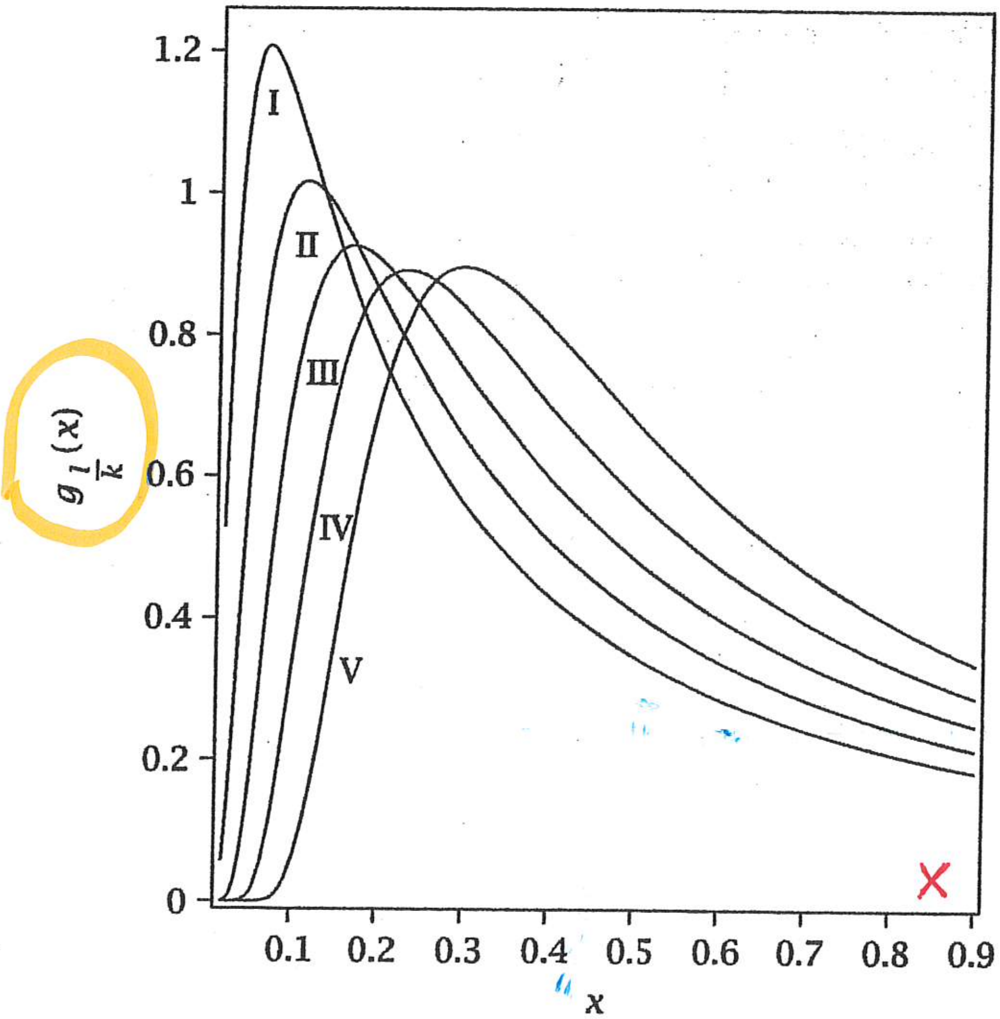


FIG. 2. Comparison of $g_{l/k}(x)$. Curves I, II, III, IV, and V correspond to $l/k = 2/5$, $9/20$, $1/2$, $11/20$, and $3/5$, respectively. Calculations were performed using Eqs. (3) and (4).

Ubiquitous objects in our calculation

Generalized hypergeometric functions:

Infinite series in the form:

argument

$$ {}_pF_q\left(\begin{matrix}\alpha_1, \alpha_2 \dots \alpha_p \\ \beta_1, \beta_2 \dots \beta_q\end{matrix}\middle| z\right) = $$

two lists of parameters

$$ = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(\alpha_1)_k \cdot (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_2)_k \cdots (\beta_q)_k} z^k $$

where $(\alpha)_k = \dfrac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \rightarrow$ **Pochhammer Symbol**

Gamma functions

Special List: $\Delta(k,a) = \dfrac{a}{k}, \dfrac{a+1}{k}, \dots \dfrac{a+k-1}{k}$

Previously Known cases :

$$\alpha = \frac{1}{2}, \underbrace{\frac{1}{3}, \frac{2}{3}}, \underbrace{\frac{3}{4}, \frac{1}{4}}$$

LS elementary

$K_\nu(\ldots)$ Modified Bessel

hypergeometric

Scher-Montroll conjecture (1975)
For rational α $g_\alpha(x)$ will be given by finite number of hypergeometric functions

$\Longrightarrow$ Confirmed by our closed form expressions

$\Longrightarrow$ Example : $\alpha = \frac{2}{9}$
requires 8 generalized hypergeometric functions of type ${}_3F_9$ $\Longrightarrow$ no problem !!!

Closed form of $\alpha = \frac{\ell}{k}$
$k > \ell$

$g_{1/k}(x) =$

numerical coefficients

$$= \sum_{j=1}^{k-1} \frac{b_j(k,\ell)}{x^{1+(j\ell)/k}} * {}_{\ell+1}F_k\left(\begin{matrix}1, \Delta(\ell, 1+\frac{j\ell}{k}) \\ \Delta(k, j+1)\end{matrix} \middle| \frac{(-1)^{k-\ell}\,\ell^\ell}{k^k x^\ell}\right)$$

gives asymptotics

- $k-1$ terms
- argument hypergeom $\sim \frac{1}{x^\ell}$

For Lévy-Smirnov $\alpha = \frac{1}{2}$; $\ell = 1$, $k = 2$

$$\frac{(-1)^{k-\ell}\,\ell^\ell}{k^k x^\ell} = -\frac{1}{4x}$$ O.K.

—*—

... entirely anti-intuitive

Example: $\alpha = \frac{1}{5}$

$$g_{1/5}(x) =$$

$$= \frac{\sqrt{5}\csc\left(\frac{\pi}{5}\right)\csc\left(\frac{2\pi}{5}\right)\mathrm{hypergeom}\left([\,],\left[\frac{2}{5},\frac{3}{5},\frac{4}{5}\right],\frac{1}{3125\,x}\right)}{20\,\Gamma\left(\frac{4}{5}\right)x^{6/5}} \qquad x^{6/5} = x^{1+\alpha} = x^{1+\frac{1}{5}}$$

$$- \frac{\sqrt{5}\csc\left(\frac{\pi}{5}\right)\csc\left(\frac{2\pi}{5}\right)\mathrm{hypergeom}\left([\,],\left[\frac{3}{5},\frac{4}{5},\frac{6}{5}\right],\frac{1}{3125\,x}\right)}{20\,\Gamma\left(\frac{3}{5}\right)x^{7/5}}$$

$$+ \frac{\sqrt{5}\,\Gamma\left(\frac{3}{5}\right)\csc\left(\frac{\pi}{5}\right)\mathrm{hypergeom}\left([\,],\left[\frac{4}{5},\frac{6}{5},\frac{7}{5}\right],\frac{1}{3125\,x}\right)}{40\,\pi\,x^{8/5}}$$

$$- \frac{\sqrt{5}\csc\left(\frac{2\pi}{5}\right)\Gamma\left(\frac{4}{5}\right)\mathrm{hypergeom}\left([\,],\left[\frac{6}{5},\frac{7}{5},\frac{8}{5}\right],\frac{1}{3125\,x}\right)}{120\,\pi\,x^{9/5}}$$

alternating sum

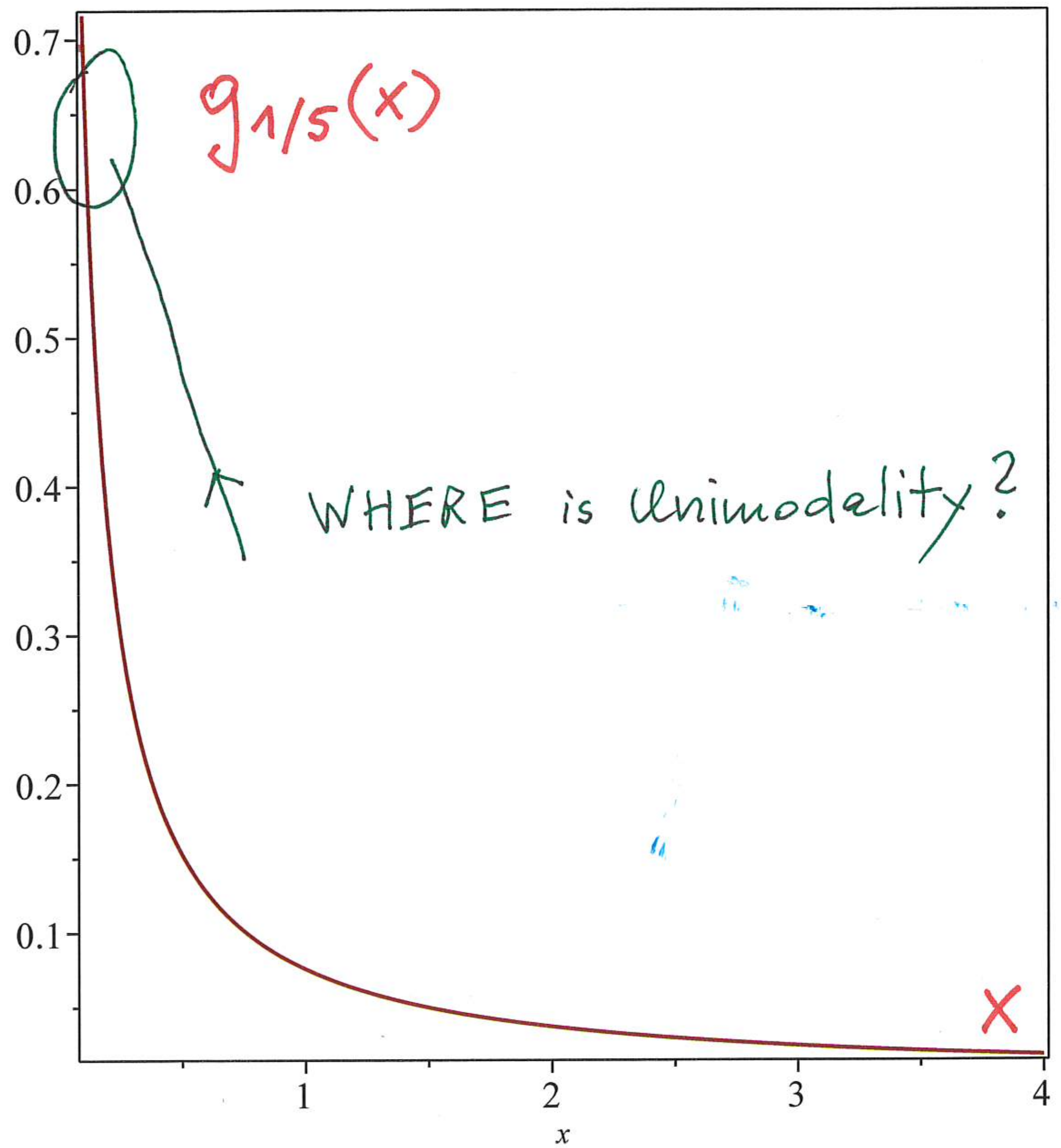
Notation Maple:

$$\mathrm{hypergeom}\left([\,],\left[\frac{6}{5},\frac{7}{5},\frac{8}{5}\right],\frac{1}{3125x}\right)$$

$$= {}_0F_3\left(\frac{6}{5},\frac{7}{5},\frac{8}{5};\frac{1}{3125x}\right)$$

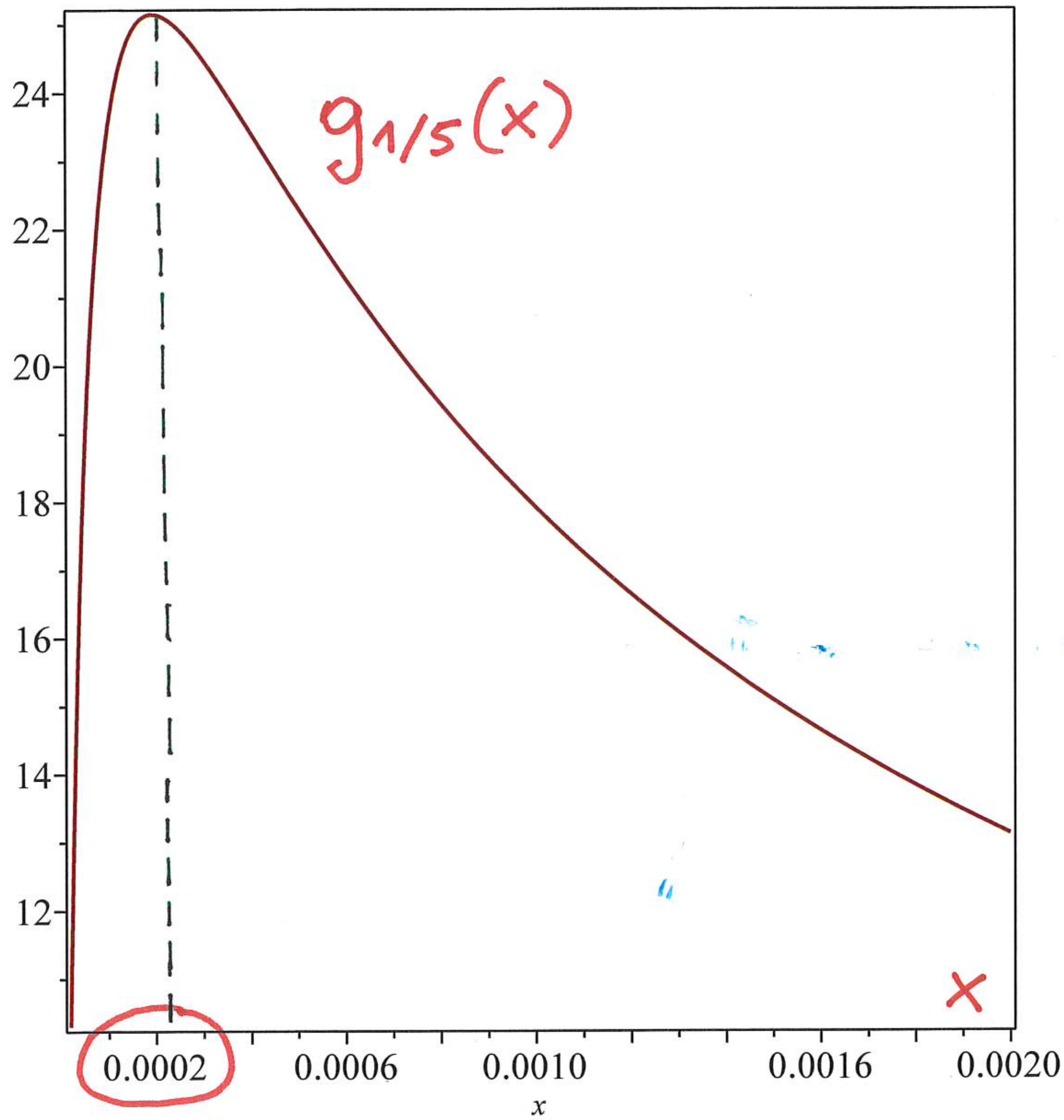
$g_{1/5}(x)$

```
> plot(expand(WW(5,1,x)),x=0.1..4,axes=boxed);
```



$g_{1/5}(x)$

```
> plot(expand(WW(5,1,x)),x=0.00002..0.002,axes=boxed);
```



$\sim 10^{-4} \to$ narrow range near zero!

Conversions encountered:

Meijer G-function → generalized hypergeometric

Generalized hypergeometric → standard special functions
(Bessel $K_\nu(x)$, MacDonald functions
Airy $Ai(x)$ etc.)

are automatically implemented by Computer Algebra Systems

Maple & Mathematica

Initial Programs of Extreme Simplicity
Results of Extreme Complexity

Application to Quasi-Relativistic Heat Equation
("Over"-simplified approach in 1D)

KAP, K. Górska, A. Horzela, G. Dattoli

Ann. Phys. (Berlin) 2017, 1700374 (2017)

NR = non-relativistic heat equation

$$\left.\begin{aligned} \partial_t \Psi_{NR}(x,t) &= \frac{1}{2}\partial_x^2 \Psi_{NR}(x,t) \\ \Psi_{NR}(x,0) &= f(x) \to \text{initial c.} \end{aligned}\right\} \text{Cauchy Problem}$$

- Solution: $\Psi_{NR}(x,t) = \exp\left(\frac{t}{2}\partial_x^2\right) f(x)$

$$= \int_{-\infty}^{\infty} \underbrace{\frac{\exp\left(-\frac{(x-u)^2}{2t}\right)}{\sqrt{2\pi t}}}_{\text{1D heat kernel}} f(u)\, du \qquad (*)$$

$\underbrace{\text{1D heat Kernel}}_{\text{Gauss-Weierstrass Transform of the initial condition } f(u)}$

Gauss-Weierstrass Transform of the initial condition $f(u)$

... can be calculated analytically for many $f(u)$

QR = quasi-relativistic heat equation

First: relativistic kinetic energy

$$E_K = \sqrt{c^2 \vec{p}^{\,2} + m^2 c^4}$$

1D $\rightarrow$ $p \sim i\frac{\partial}{\partial x}$ (dimensionless units)

$$\left.\begin{aligned} &\partial_t \Psi_{QR}(x,t) = \left(1 - \sqrt{1-\partial_x^2}\right)\Psi_{QR}(x,t) \\ &\Psi_{QR}(x,0) = f(x), \text{ initial c.} \end{aligned}\right\} \text{Cauchy problem}$$

- Solution: $\Psi_{QR}(x,t) = \exp(t)\exp\left(-t\sqrt{1-\partial_x^2}\right)f(x)$

"pseudodifferential" operator"

Rewrite the Laplace transform of $g_{1/2}(\xi)$:

$$e^{-t\sqrt{p}} = \int_0^\infty g_{1/2}(\xi)\, e^{-t^2 p \xi}\, d\xi \quad , \; p>0$$

for $p = \sqrt{1-\partial_x^2}$

$$\Psi_{QR}(x,t) = \left[e^{t} \int_{0}^{\infty} g_{1/2}(\xi)\, e^{-t^2 (1-\partial_x^2)\xi}\, d\xi \right] f(x)$$

$$= \left[e^{t} \int_{0}^{\infty} g_{1/2}(\xi)\, e^{-t^2 \xi}\, \underbrace{e^{t^2 \xi \partial_x^2} f(x)}\, d\xi \right]$$

from the proceeding equals to

$$e^{t^2 \xi \partial_x^2} f(x) = \Psi_{NR}(x, 2\xi t^2)$$

$$\Rightarrow \boxed{\Psi_{QR}(x,t) = e^{t} \int_{0}^{\infty} g_{1/2}(\xi)\, e^{-t^2 \xi}\, \Psi_{NR}(x, 2\xi t^2)\, d\xi}$$

Quasi-Relativistic Solution is obtained by integrating Non-Relativistic solution with the Lévy-Smirnov function.

(... the same initial condition)

Comparison of time evolution from initial Gaussian (t=0)

QR – quasi relativistic

NR – non-relativistic

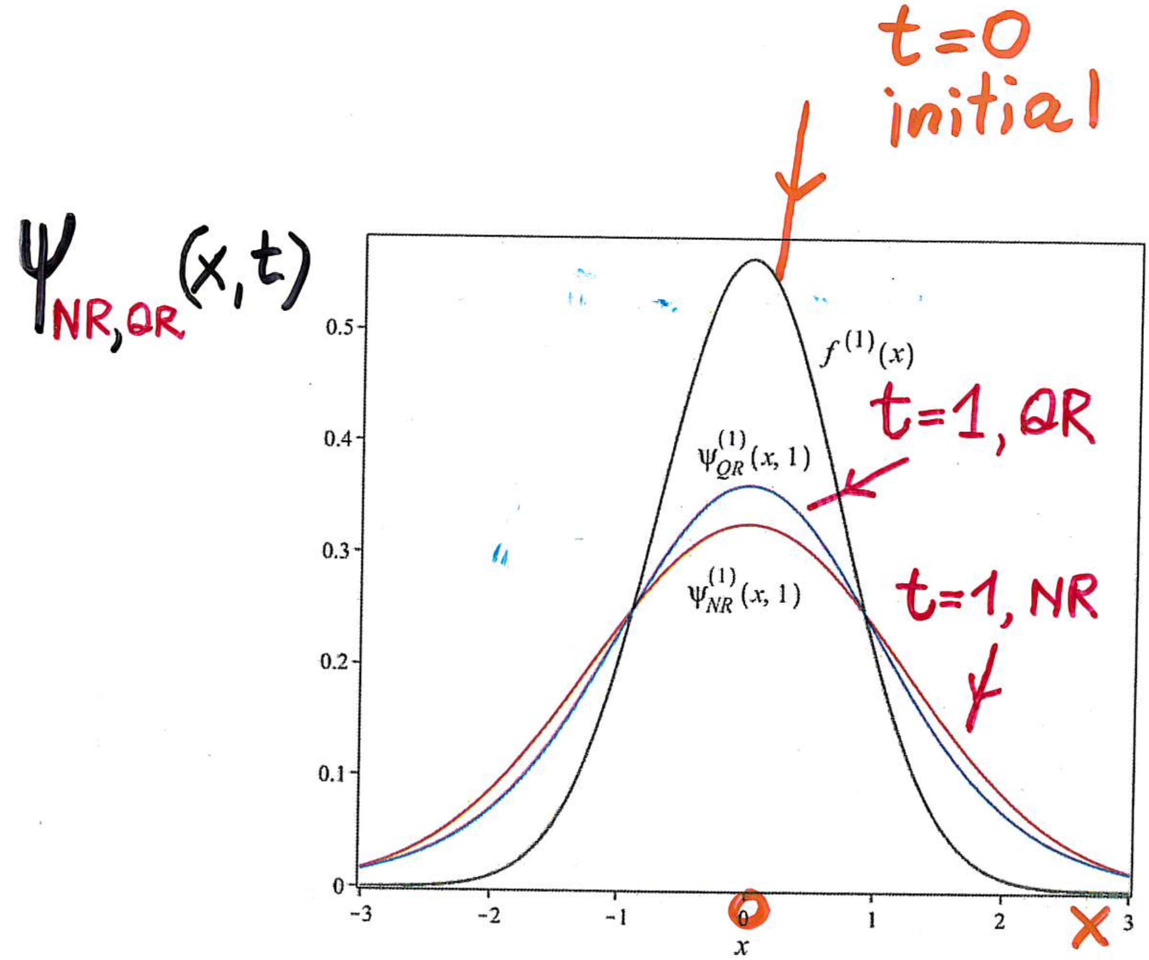


Figure 1. Comparison of QR and NR distributions with i.c. $f^{(1)}(x)$ for $t=1$: $\psi^{(1)}_{\mathrm{QR}}$ has less evolved than $\psi^{(1)}_{\mathrm{NR}}$. This fact holds for all t.

From the same initial condition the QR solution has less evolved than the NR solution

New properties of Laplace transform originating from Lévy laws

(K. A. Penson, K. Górska
J. Phys. A 49, 065201 (2016))

- define $\underbrace{\mathcal{L}[f(x);p]}_{\text{Laplace transform}} = F(p)$; $\underbrace{0 < \alpha < 1}$

A) $$\tilde{f}_\alpha(x) = \int_0^\infty \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) f(t)\, dt$$

$$\Longrightarrow \mathcal{L}[\tilde{f}_\alpha(x);p] = F(p^\alpha) \;;$$!

B) $$\bar{f}_\alpha(x) = \frac{x}{\alpha}\int_0^\infty \frac{1}{t}\,\frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) f(t)\, dt$$

$$\Longrightarrow \mathcal{L}[\bar{f}_\alpha(x);p] = p^{\alpha-1} F(p^\alpha) \;;$$!

Lévy functions $\Longrightarrow$ vehicle for calculations of inverse Laplace transforms

Transitivity of index α of Lévy laws

$(0 < \alpha, \beta < 1)$

We Know $g_\alpha(x)$ and $g_\beta(x)$.
Can we explicitly obtain $g_{\alpha\beta}(x)$ via a well defined operation?

$\boxed{\text{YES}}$:

$$\int_0^\infty \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) \frac{1}{y^{1/\beta}} g_\beta\left(\frac{t}{y^{1/\beta}}\right) dt =$$

$$= \frac{1}{y^{\frac{1}{\alpha\beta}}} g_{\alpha\beta}\left(\frac{x}{y^{\frac{1}{\alpha\beta}}}\right)$$

Verified on several α & β.