# Coefficientwise Hankel-total positivity

Alan Sokal
University College London / New York University

Combinatorics and Arithmetic for Physics

IHES, Bures-sur-Yvette, 7 November 2019

A big project in collaboration with
Mathias Pétréolle, Bao-Xuan Zhu, Jiang Zeng,
Andrew Elvey Price, Alex Dyachenko, . . .

**Key references:**

1. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. **32**, 125–161 (1980).
2. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux* (UQAM, 1983).
3. Pétréolle–Sokal–Zhu, Lattice paths and branched continued fractions, arXiv:1807.03271
4. Pétréolle–Sokal, LP&BCF II, arXiv:1907.02645

## Total positivity

A (finite or infinite) matrix of real numbers is called *totally positive* if all its minors are nonnegative.

**Applications:**

- Mechanics of oscillatory systems
- Zeros of polynomials and entire functions
- Numerical linear algebra
- Approximation theory
- Stochastic processes
- Statistics
- Lie theory and cluster algebras
- Representation theory of the infinite symmetric group
- Theory of immanants
- Planar discrete potential theory and the planar Ising model
- **Stieltjes moment problem**
- **Enumerative combinatorics**

⋮

# Hankel-total positivity

Given a sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$, we define its *Hankel matrix*

$$H_\infty(\boldsymbol{a}) \;=\; (a_{i+j})_{i,j\geq 0} \;=\; \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- We say that the sequence $\boldsymbol{a}$ is *Hankel-totally positive* if its Hankel matrix $H_\infty(\boldsymbol{a})$ is totally positive.
- This implies that the sequence is *log-convex*, but is much stronger.

**Fundamental Characterization** (Stieltjes 1894, Gantmakher–Krein 1937):

For a sequence $\boldsymbol{a} = (a_n)_{n\geq 0}$ of real numbers, the following are equivalent:

(a) $\boldsymbol{a}$ is Hankel-totally positive.

(b) There exists a positive measure $\mu$ on $[0, \infty)$ such that $a_n = \int x^n \, d\mu(x)$ for all $n \geq 0$.
[That is, $(a_n)_{n\geq 0}$ is a **Stieltjes moment sequence**.]

(c) There exist numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that

$$\sum_{n=0}^{\infty} a_n t^n \;=\; \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

in the sense of formal power series.

[**Stieltjes-type continued fraction** with nonnegative coefficients]

# From numbers to polynomials

## [or, From counting to counting-with-weights]

**Some simple examples:**

1. Counting subsets of $[n]$: $a_n = 2^n$

   Counting subsets of $[n]$ by cardinality: $P_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$

2. Counting permutations of $[n]$: $a_n = n!$

   Counting permutations of $[n]$ by number of cycles:
   $P_n(x) = \sum_{k=0}^{n} \left[{n \atop k}\right] x^k$ (Stirling cycle polynomial)

   Counting permutations of $[n]$ by number of descents:
   $P_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$ (Eulerian polynomial)

3. Counting partitions of $[n]$: $a_n = B_n$ (Bell number)

   Counting partitions of $[n]$ by number of blocks:
   $P_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$ (Bell polynomial)

4. Counting non-crossing partitions of $[n]$: $a_n = C_n$ (Catalan number)

   Counting non-crossing partitions of $[n]$ by number of blocks:
   $P_n(x) = \sum_{k=0}^{n} N(n,k)\, x^k$ (Narayana polynomial)

These polynomials can also be **multivariate**!
(count with many simultaneous statistics)

An industry in combinatorics: $q$-Narayana polynomials, $p, q$-Bell polynomials, ...

## Coefficientwise total positivity

- Consider sequences and matrices whose entries are *polynomials* with real coefficients in one or more indeterminates $\mathbf{x}$.
- A matrix is *coefficientwise totally positive* if every minor is a polynomial with nonnegative coefficients.
- A sequence is *coefficientwise Hankel-totally positive* if its Hankel matrix is coefficientwise totally positive.
- More generally, can consider sequences and matrices with entries in a *partially ordered commutative ring*.

But now there is no analogue of the Fundamental Characterization!

Coefficientwise Hankel-TP is **combinatorial**, not analytic.

Coefficientwise Hankel-TP *implies* that $(P_n(\mathbf{x}))_{n \geq 0}$ is a Stieltjes moment sequence for all $\mathbf{x} \geq 0$, but it is *stronger*.

# Coefficientwise Hankel-TP in combinatorics

Many interesting sequences of combinatorial polynomials $(P_n(x))_{n\ge 0}$ have been proven in recent years to be *coefficientwise log-convex*:

- Bell polynomials $B_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$
  (Liu–Wang 2007, Chen–Wang–Yang 2011)
- Narayana polynomials $N_n(x) = \sum_{k=0}^{n} N(n,k)\, x^k$
  (Chen–Wang–Yang 2010)
- Narayana polynomials of type B: $W_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k$
  (Chen–Tang–Wang–Yang 2010)
- Eulerian polynomials $A_n(x) = \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k$
  (Liu–Wang 2007, Zhu 2013)

Might these sequences actually be *coefficientwise Hankel-totally positive*?

- In many cases I can prove that the answer is **yes**, by using the Flajolet–Viennot method of *continued fractions*.
- In several other cases I have strong **empirical evidence** that the answer is **yes**, but no proof.
- The continued-fraction approach gives a *sufficient but not necessary* condition for coefficientwise Hankel-total positivity.

## **Combinatorics of continued fractions** (Flajolet 1980)

We consider two types of continued fractions:

- Stieltjes type (S-fractions):

$$f(t) = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cfrac{\alpha_3 t}{1 - \cdots}}}}$$

- Jacobi type (J-fractions):

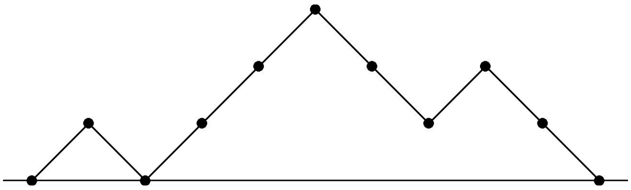
$$f(t) = \cfrac{1}{1 - \gamma_0 t - \cfrac{\beta_1 t^2}{1 - \gamma_1 t - \cfrac{\beta_2 t^2}{1 - \gamma_2 t - \cfrac{\beta_3 t^2}{1 - \gamma_3 t - \cdots}}}}$$

For lack of time I will show here only Stieltjes.

Jacobi can also be handled, but Hankel-TP is more subtle.
(needs method of production matrices)

## Combinatorics of Stieltjes-type continued fractions

A *Dyck path* of length $2n$ is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $(2n,0)$ using steps $(1,1)$ ["rise"] and $(1,-1)$ ["fall"]:



**Theorem** (Flajolet 1980): As an identity in $\mathbb{Z}[\boldsymbol{\alpha}][[t]]$, we have

$$\cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}} \;=\; \sum_{n=0}^{\infty} S_n(\alpha_1, \ldots, \alpha_n)\, t^n$$

where $S_n(\alpha_1, \ldots, \alpha_n)$ is the generating polynomial for Dyck paths of length $2n$ in which each fall starting at height $i$ gets weight $\alpha_i$.

$S_n(\boldsymbol{\alpha})$ is called the *Stieltjes–Rogers polynomial* of order $n$.

**Theorem** (A.S. 2014, based on Viennot 1983): The sequence $(S_n(\boldsymbol{\alpha}))_{n \ge 0}$ of Stieltjes–Rogers polynomials is coefficientwise Hankel-totally positive in the polynomial ring $\mathbb{Z}[\boldsymbol{\alpha}]$.

Proof uses the Karlin–McGregor–Lindström–Gessel–Viennot lemma on families of nonintersecting paths.

Can now specialize $\boldsymbol{\alpha}$ to *nonnegative* elements in any partially ordered commutative ring, and get Hankel-TP.

## Example 1: Narayana polynomials

- Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$
- Count numerous objects of combinatorial interest:
  - Dyck paths of length $2n$ with $k$ peaks
  - Non-crossing partitions of $[n]$ with $k$ blocks
  - Non-nesting partitions of $[n]$ with $k$ blocks
- Narayana polynomials $N_n(x) = \sum_{k=0}^{n} N(n,k)\, x^k$
- Ordinary generating function $\mathcal{N}(t,x) = \sum_{n=0}^{\infty} N_n(x)\, t^n$
- Elementary "renewal" argument on Dyck paths implies

$$\mathcal{N} \;=\; \frac{1}{1 - tx - t(\mathcal{N} - 1)}$$

  which can be rewritten as

$$\mathcal{N} \;=\; \frac{1}{1 - \dfrac{xt}{1 - t\mathcal{N}}}$$

- Leads immediately to S-type continued fraction

$$\sum_{n=0}^{\infty} N_n(x)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{t}{1 - \cfrac{xt}{1 - \cfrac{t}{1 - \cdots}}}}}$$

**Conclusion:** The sequence $(N_n(x))_{n\geq 0}$ of Narayana polynomials is coefficientwise Hankel-totally positive.

# Example 2: Bell polynomials

- Stirling number $\left\{ {n \atop k} \right\}$ = # of partitions of $[n]$ with $k$ blocks
- Bell polynomials $B_n(x) = \sum\limits_{k=0}^{n} \left\{ {n \atop k} \right\} x^k$
- Ordinary generating function $\mathcal{B}(t,x) = \sum\limits_{n=0}^{\infty} B_n(x)\, t^n$
- Flajolet (1980) expressed $\mathcal{B}(t,x)$ as a J-type continued fraction
- Can be transformed to an S-type continued fraction

$$\sum_{n=0}^{\infty} B_n(x)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{1t}{1 - \cfrac{xt}{1 - \cfrac{2t}{1 - \cdots}}}}}$$

**Conclusion:** The sequence $(B_n(x))_{n \ge 0}$ of Bell polynomials is coefficientwise Hankel-totally positive.

- Can extend to polynomial $B_n(x,p,q)$ that enumerates set partitions w.r.t. blocks ($x$), crossings ($p$) and nestings ($q$):

$$\sum_{n=0}^{\infty} B_n(x,p,q)\, t^n \;=\; \cfrac{1}{1 - \cfrac{xt}{1 - \cfrac{[1]_{p,q} t}{1 - \cfrac{xt}{1 - \cfrac{[2]_{p,q} t}{1 - \cdots}}}}}$$

where $[n]_{p,q} = \dfrac{p^n - q^n}{p - q} = \sum\limits_{j=0}^{n-1} p^j q^{n-1-j}$

- Implies coefficientwise Hankel-TP jointly in $x, p, q$

## Example 3: Narayana polynomials of type B

The polynomials

$$W_n(x) \;=\; \sum_{k=0}^{n} \binom{n}{k}^2 x^k$$

- Grand Dyck paths with weight $x$ for each peak
- Coordinator polynomial of the classical root lattice $A_n$
- Rank generating function of the lattice of noncrossing partitions of type B on $[n]$

- There is no S-type continued fraction *in the ring of polynomials*: we have

$$\alpha_1, \alpha_2, \ldots \;=\; 1+x,\, \frac{2x}{1+x},\, \frac{1+x^2}{1+x},\, \frac{x+x^2}{1+x^2},\, \frac{1+x^3}{1+x^2},\, \frac{x+x^3}{1+x^3},\, \ldots$$

- However, there *is* a nice *J-type* continued fraction:

$$\sum_{n=0}^{\infty} W_n(x)\, t^n \;=\; \cfrac{1}{1-(1+x)t-\cfrac{2xt^2}{1-(1+x)t-\cfrac{xt^2}{1-(1+x)t-\cfrac{xt^2}{1-\cdots}}}}$$

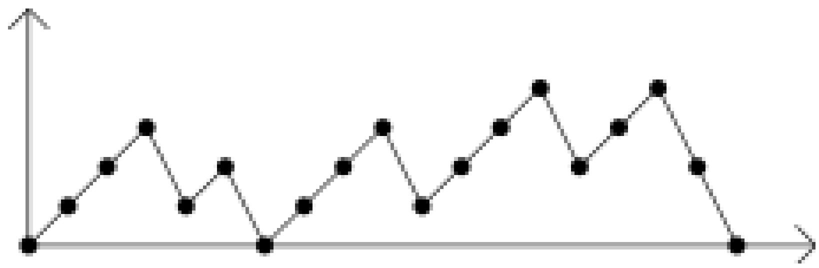
- By J-fraction theory (not explained here) one can show:

  **Theorem** (A.S. unpublished 2014, Wang–Zhu 2016):
  The sequence $(W_n(x))_{n\ge 0}$ is coefficientwise Hankel-TP.

# A new tool: Branched continued fractions

Generalize Dyck paths: Fix an integer $m \geq 1$.

An *m-Dyck path* of length $(m+1)n$ is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $((m+1)n, 0)$ using steps $(1,1)$ ["rise"] and $(1,-m)$ ["$m$-fall"]:



A 2-Dyck path of length 18.

Let $S_n^{(m)}(\boldsymbol{\alpha})$ be the generating polynomial for $m$-Dyck paths of length $(m+1)n$ in which each $m$-fall starting at height $i$ gets weight $\alpha_i$.

We call $S_n^{(m)}(\boldsymbol{\alpha})$ the *m-Stieltjes–Rogers polynomial* of order $n$.

**Theorem** (Pétréolle–A.S.–Zhu 2018): The sequence $(S_n^{(m)}(\boldsymbol{\alpha}))_{n \geq 0}$ of $m$-Stieltjes–Rogers polynomials is coefficientwise Hankel-TP in the polynomial ring $\mathbb{Z}[\boldsymbol{\alpha}]$.

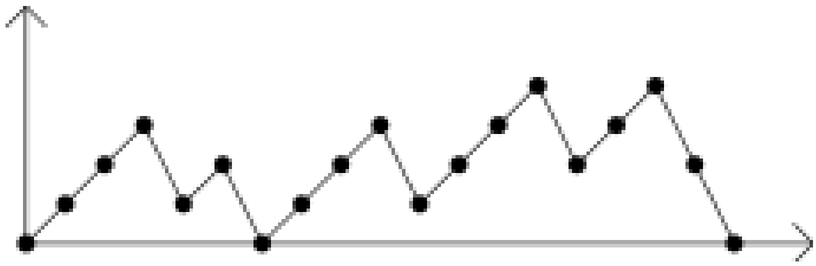
Proof is essentially identical to the one for $m = 1$!

**Remark:** $S_n^{(m)}(\boldsymbol{\alpha})$ are the Taylor coefficients of (extremely ugly) *branched continued fractions*.

# A new tool: Branched continued fractions

Generalize Dyck paths: Fix an integer $m \geq 1$.

An *m-Dyck path* of length $(m+1)n$ is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $((m+1)n, 0)$ using steps $(1,1)$ ["rise"] and $(1,-m)$ ["$m$-fall"]:



A 2-Dyck path of length 18.

Let $S_n^{(m)}(\boldsymbol{\alpha})$ be the generating polynomial for $m$-Dyck paths of length $(m+1)n$ in which each $m$-fall starting at height $i$ gets weight $\alpha_i$.

We call $S_n^{(m)}(\boldsymbol{\alpha})$ the *m-Stieltjes–Rogers polynomial* of order $n$.

**Theorem** (Pétréolle–A.S.–Zhu 2018): The sequence $(S_n^{(m)}(\boldsymbol{\alpha}))_{n\geq 0}$ of $m$-Stieltjes–Rogers polynomials is coefficientwise Hankel-TP in the polynomial ring $\mathbb{Z}[\boldsymbol{\alpha}]$.

Proof is essentially identical to the one for $m = 1$!

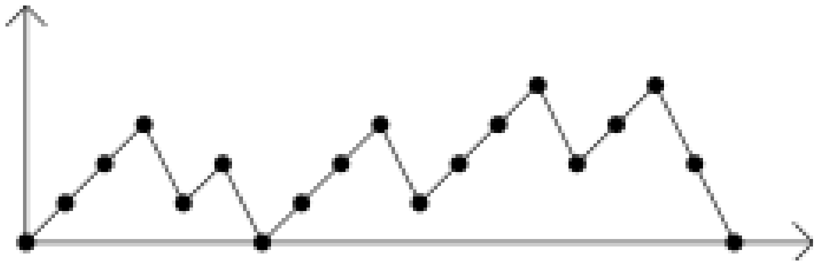
**Remark:** $S_n^{(m)}(\boldsymbol{\alpha})$ are the Taylor coefficients of (extremely ugly) *branched continued fractions:*

$$f(t) = \cfrac{1}{1 - \cfrac{\alpha_m t}{\left(1 - \cfrac{\alpha_{m+1} t}{\left(1 - \frac{\alpha_{m+2} t}{(\cdots)\cdots(\cdots)}\right)\cdots\left(1 - \frac{\alpha_{2m+1} t}{(\cdots)\cdots(\cdots)}\right)}\right)\cdots\left(1 - \cfrac{\alpha_{2m} t}{\left(1 - \frac{\alpha_{2m+1} t}{(\cdots)\cdots(\cdots)}\right)\cdots\left(1 - \frac{\alpha_{3m} t}{(\cdots)\cdots(\cdots)}\right)}\right)}}$$

# A new tool: Branched continued fractions

Generalize Dyck paths: Fix an integer $m \geq 1$.

An *m-Dyck path* of length $(m+1)n$ is a path in the right quadrant $\mathbb{N} \times \mathbb{N}$ from $(0,0)$ to $((m+1)n, 0)$ using steps $(1,1)$ ["rise"] and $(1, -m)$ ["$m$-fall"]:



A 2-Dyck path of length 18.

Let $S_n^{(m)}(\boldsymbol{\alpha})$ be the generating polynomial for $m$-Dyck paths of length $(m+1)n$ in which each $m$-fall starting at height $i$ gets weight $\alpha_i$.

We call $S_n^{(m)}(\boldsymbol{\alpha})$ the *m-Stieltjes–Rogers polynomial* of order $n$.

**Theorem** (Pétréolle–A.S.–Zhu 2018): The sequence $(S_n^{(m)}(\boldsymbol{\alpha}))_{n \geq 0}$ of $m$-Stieltjes–Rogers polynomials is coefficientwise Hankel-TP in the polynomial ring $\mathbb{Z}[\boldsymbol{\alpha}]$.

Proof is essentially identical to the one for $m = 1$!

**Remark:** $S_n^{(m)}(\boldsymbol{\alpha})$ are the Taylor coefficients of (extremely ugly) *branched continued fractions*.

**Non-obvious fact:** The $S_n^{(m)}(\boldsymbol{\alpha})$ get more general as $m$ grows.

## Branched continued fractions: An example

- $n! = \int_0^\infty x^n e^{-x}\, dx$ is a Stieltjes moment sequence.
- Euler showed in 1746 that

$$\sum_{n=0}^{\infty} n!\, t^n \;=\; \cfrac{1}{1-\cfrac{1t}{1-\cfrac{1t}{1-\cfrac{2t}{1-\cfrac{2t}{1-\cdots}}}}}$$

- The entrywise product of Stieltjes moment sequences is also one.
- So $(n!)^2$ is a Stieltjes moment sequence.
- Straightforward computation gives for $(n!)^2$

$$\alpha_1, \alpha_2, \ldots \;=\; 1,\ 3,\ \tfrac{20}{3},\ \tfrac{164}{15},\ \tfrac{3537}{205},\ \tfrac{127845}{5371},\ \tfrac{4065232}{124057},\ \tfrac{244181904}{5868559},\ \tfrac{38418582575}{721944303},\ \ldots$$

- The $\boldsymbol{\alpha}$ are indeed positive, but what the hell are they???

- $(n!)^2$ has a nice $m$-branched continued fraction with $m = 2$:

$$\boldsymbol{\alpha} \;=\; 1, 1, 2, 4, 4, 6, 9, 9, 12, \ldots$$

- Similar results hold for $(n!)^m$, $(2n-1)!!^m$, $(mn)!$ and much more general things.
- But these are special cases of something vastly more general ...

## Branched continued fractions for ratios of contiguous hypergeometric functions

- Euler also showed in 1746 that

$$\sum_{n=0}^{\infty} a(a+1)\cdots(a+n-1)\, t^n \;=\; \cfrac{1}{1-\cfrac{at}{1-\cfrac{1t}{1-\cfrac{(a+1)t}{1-\cfrac{2t}{1-\cdots}}}}}$$

- And this is the $b=1$ special case of

$$\frac{{}_2F_0\left(\left.\begin{matrix} a, b \\ - \end{matrix}\right| t\right)}{{}_2F_0\left(\left.\begin{matrix} a, b-1 \\ - \end{matrix}\right| t\right)} = \cfrac{1}{1-\cfrac{at}{1-\cfrac{bt}{1-\cfrac{(a+1)t}{1-\cfrac{(b+1)t}{1-\cdots}}}}}$$

  (${}_2F_0$ limiting case of Gauss continued fraction for ${}_2F_1$)

- We generalize this to ratios of contiguous ${}_{m+1}F_0$: the result is an $m$-branched continued fraction ...

# Branched continued fractions for ratios of contiguous hypergeometric functions (bis)

**Theorem** (Pétréolle–A.S.–Zhu 2018): For each $m \geq 1$,

$$\frac{{}_{m+1}F_0\left(\begin{array}{c} a_1, \ldots, a_{m+1} \\ — \end{array} \middle| \, t\right)}{{}_{m+1}F_0\left(\begin{array}{c} a_1, \ldots, a_m, a_{m+1}-1 \\ — \end{array} \middle| \, t\right)} \;=\; \sum_{n=0}^{\infty} S_n^{(m)}(\boldsymbol{\alpha})\, t^n$$

where the $\boldsymbol{\alpha}$ are very simple polynomials in $a_1, \ldots, a_{m+1}$:

$$\boldsymbol{\alpha} \;=\; a_1 \cdots a_m,\; a_2 \cdots a_{m+1},\; a_3 \cdots a_{m+1}(a_1+1),\; a_4 \cdots a_{m+1}(a_1+1)(a_2+1),\; \ldots$$

**Corollary:** The polynomials $P_n^{(m)}(a_1, \ldots, a_m; a_{m+1}) \;=\; S_n^{(m)}(\boldsymbol{\alpha})$ arising as the Taylor coefficients of this ratio are coefficientwise Hankel-TP jointly in $a_1, \ldots, a_{m+1}$.

Can obtain many examples by specialization of $a_1, \ldots, a_{m+1}$.

**Even more generally:** For every $r, s \geq 0$ we find an $m$-branched continued fraction with $m = \max(r-1, s)$ for ratios of contiguous ${}_rF_s$.

- Generalizes Gauss continued fraction for ${}_2F_1$.
- Can further generalize to $q$-hypergeometric functions ${}_r\phi_s$.
- But corollaries for Hankel-TP are more subtle than for $s = 0$. (Already this was the case for ${}_2F_1$ compared to ${}_2F_0$.)

## Coefficientwise Hankel-TP seems to be very common . . . but not so easy to prove

There are *many* cases where:

- I find **empirically** that a sequence $(P_n(x))_{n\geq 0}$ is coefficientwise Hankel-TP . . .
- But I am **unable to prove it** because there is neither an S-type nor a J-type continued fraction in the ring of polynomials (and maybe no branched continued fraction, either?).

- Domb polynomials
- Apéry polynomials
- Boros–Moll polynomials
- Ramanujan polynomials
- Inversion enumerators for trees
- Reduced binomial discriminant polynomials

$\vdots$

The last two examples are closely related to the problem of *counting connected graphs* . . .

and hence to the $q \to 0$ limit of the $q$-state Potts model.

## Generating polynomials of connected graphs

- Let $c_{n,m} = \#$ of connected simple graphs on vertex set $[n]$ having $m$ edges

- Define the *generating polynomial of connected graphs*

$$
\begin{aligned}
C_n(v) &= \sum_{m=n-1}^{\binom{n}{2}} c_{n,m}\, v^m \\
&= n^{n-2} v^{n-1} + \ldots + v^{\binom{n}{2}}
\end{aligned}
$$

- No useful explicit formula for the polynomials $C_n(v)$ or their coefficients is known.

- But they have the well-known exponential generating function

$$
\sum_{n=1}^{\infty} \frac{x^n}{n!} C_n(v) \;=\; \log \sum_{n=0}^{\infty} \frac{x^n}{n!} (1+v)^{n(n-1)/2}
$$

- In particular we have

$$
C_n(-1) \;=\; (-1)^{n-1}(n-1)!
$$

- Of course we also have

$$
C_n(0) \;=\; 0 \qquad \text{for } n \ge 2
$$

  since $C_n(v)$ has an $(n-1)$-fold zero at $v = 0$.

- Make change of variables $y = 1+v$ and define $\overline{C}_n(y) = C_n(y-1)$:

$$
\sum_{n=1}^{\infty} \frac{x^n}{n!} \overline{C}_n(y) \;=\; \log \sum_{n=0}^{\infty} \frac{x^n}{n!} y^{n(n-1)/2}
$$

- These formulae can be considered either as identities for formal power series or as analytic statements valid when $|1+v| \le 1$ (resp. $|y| \le 1$).

# Inversion enumerator for trees

- Let $T$ be a tree with vertex set $[n]$, rooted at the vertex 1.
- An *inversion* of $T$ is an ordered pair $(j,k)$ of vertices such that $j > k > 1$ and the path from 1 to $k$ passes through $j$.
- Let $i_{n,\ell}$ denote the number of trees on $[n]$ having $\ell$ inversions.
- Define the *inversion enumerator for trees*

$$\begin{aligned} I_n(y) &= \sum_{\ell=0}^{\binom{n-1}{2}} i_{n,\ell}\, y^\ell \\ &= (n-1)! + \ldots + y^{\binom{n-1}{2}} \end{aligned}$$

- The polynomial $I_n(y)$ turns out to be related to $C_n(v)$ by the beautiful formula

$$C_n(v) = v^{n-1} I_n(1+v)$$

 or equivalently

$$\overline{C}_n(y) = (y-1)^{n-1} I_n(y)$$

- This shows in particular that $I_n(0) = (n-1)!$ and $I_n(1) = n^{n-2}$.
- It is useful to define the normalized polynomials

$$I_n^\star(y) = \frac{I_n(y)}{(n-1)!}$$

 which have nonnegative rational coefficients and constant term 1.

## Inversion enumerator for trees: Stieltjes moment property

**Theorem** (Laguerre 1883):
For $0 < y < 1$, the deformed exponential function

$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}\, y^{n(n-1)/2}$$

is an entire function in the Laguerre–Pólya class $LP^+$:

$$F(x,y) \;=\; \prod_{k=0}^{\infty} \biggl(1 \,+\, \frac{x}{\xi_k(y)}\biggr)$$

with $0 < \xi_0(y) < \xi_1(y) < \xi_2(y) < \ldots$.

**Consequences for $\overline{C}_n(y)$ and $I_n(y)$:**

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}\, \overline{C}_n(y) \;=\; \log F(x,y) \;=\; \sum_{k=0}^{\infty} \log\biggl(1 \,+\, \frac{x}{\xi_k(y)}\biggr)$$

$$\Longrightarrow \quad \overline{C}_n(y) \;=\; (-1)^{n-1}\,(n-1)! \sum_{k=0}^{\infty} \xi_k(y)^{-n}$$

$$\Longrightarrow \quad I^\star_{n+1}(y) \;=\; (1-y)^{-n} \sum_{k=0}^{\infty} \xi_k(y)^{-(n+1)}$$

$$\Longrightarrow \quad (I^\star_{n+1}(y))_{n\ge 0} \text{ is a Stieltjes moment sequence for } 0 \le y \le 1$$

And the same is true for $I_{n+1}(y) = n!\, I^\star_{n+1}(y)$.
(product of Stieltjes moment sequences is a Stieltjes moment sequence)

I *conjecture* that these things hold also for $y > 1$.

## Inversion enumerator for trees: Coefficientwise Hankel-TP?

Might these results also hold *coefficientwise* in $y$?

**Fact 1.** $I_n(y)$ has strictly positive coefficients.

- Nonnegativity is obvious; strict positivity takes a bit of work.

**Fact 2.** $I_n(y)$ has log-concave coefficients.

- Special case of a deep result of Huh, arXiv:1201.2915, on the log-concavity of the $h$-vector of the independent-set complex for matroids representable over a field of characteristic 0: apply it to $M^*(K_n)$.
- **Open problem:** Find an elementary direct proof.

Now form the sequence $\boldsymbol{I} = (I_{n+1}(y))_{n \ge 0}$.

**Conjecture 1.** The sequence $\boldsymbol{I}$ is coefficientwise Hankel-totally positive.

- I have checked this through the $10 \times 10$ Hankel matrix.
- Even the coefficientwise log-convexity $I_{n-1}I_{n+1} \succeq I_n^2$ seems to be an **open problem**!

**Conjecture 2.** The $2 \times 2$ minors $I_{m-1}I_{n+1} - I_m I_n \quad (1 \le m \le n)$ have coefficients that are log-concave.

- I have checked this through $n = 165$.
- It is false for minors of size $3 \times 3$ and higher.

# Inversion enumerator for trees: Coefficientwise Hankel-TP? (continued)

Now look at the normalized polynomials $\boldsymbol{I}^\star = (I^\star_{n+1}(y))_{n\geq 0}$.

**Conjecture 3.** The sequence $\boldsymbol{I}^\star$ is coefficientwise Hankel-totally positive.

- I have checked this through the $10 \times 10$ Hankel matrix.
- **Note:** At *fixed real* $y$, the result for $\boldsymbol{I}^\star$ implies the one for $\boldsymbol{I}$. But this argument does *not* work in $\mathbb{R}[y]$!

**Conjecture 4.** *All* the Hankel minors of $\boldsymbol{I}^\star$ have coefficients that are log-concave.

- I have checked this through the $10 \times 10$ Hankel matrix.
- For the $2 \times 2$ minors, I have checked it for $1 \leq m \leq n \leq 165$.

# (Tentative) Conclusion

- Many interesting sequences $(P_n(\mathbf{x}))_{n\geq 0}$ of combinatorial polynomials are (or appear to be) coefficientwise Hankel-totally positive.
- In some cases this can be proven by the Flajolet–Viennot method of continued fractions.
  - When S-fractions exist, they give the simplest proofs.
  - Sometimes S-fractions don't exist, but J-fractions can work.
  - Sometimes neither S-fractions nor J-fractions exist, but branched S-fractions do.
  - Branched S-fractions are a powerful (but not universal) tool.
- Alas, in many cases *none* of these methods work!
- **New methods of proof will be needed:**
  - Differential operators?
  - Direct study of Hankel minors?
  - . . . ???
- Coefficientwise Hankel-TP is a big phenomenon that we understand, at present, only very incompletely.

*Dedicated to the memory of Philippe Flajolet (1948–2011)*