The Calabi-Yau geometry of the sunset Feynman graphs

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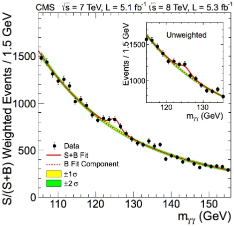
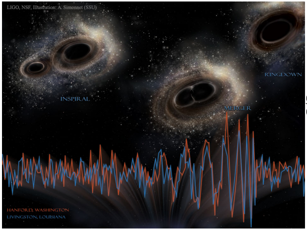
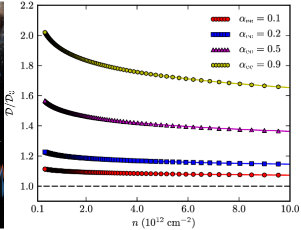
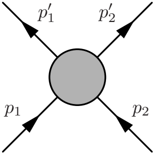
based on work to appear with

Charles Doran, Andrey Novoseltsev









Scattering amplitudes are the fundamental tools for making contact between quantum field theory description of nature and experiments

- Comparing particule physics model against datas from accelators
- Post-Minkowskian expansion for Gravitational wave physics
- Various condensed matter and statistical physics

Physics arguments indicate that any quantum field theory amplitude can be expanded on a **finite** basis of integral functions

$$A^{\mathrm{L-loop}}_{n-\mathrm{part.}} = \sum_{i\in\mathcal{B}(L)} \mathrm{coeff}_i\, \mathrm{Integral}_i + \mathrm{Rational\ function}$$

- What is the dimension of the basis $\mathcal{B}(L)$?
- What are the functions in the basis?
  - Feynman integrals are highly transcendental functions with a lot singularities
- How can we achieve such decomposition?

Feynman Graph Motive

Feynman Integrals: parametric representation

The integral functions in the basis are Feynman integrals with L-loop and n internal edges

$$I_\Gamma(\underline{s}, \underline{m}) = \Gamma\left(n - \frac{LD}{2}\right) \int_{\Delta_n} \Omega_\Gamma; \qquad \Omega_\Gamma := \frac{\mathcal{U}_\Gamma(\underline{x})^{n-\frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{n-\frac{LD}{2}}} \prod_{i=1}^{n-1} dx_i$$

The domain of integration is the positive quadrant

$$\Delta_n := \{x_1 \geqslant 0, \ldots, x_n \geqslant 0 | [x_1, \ldots, x_n] \in \mathbb{P}^{n-1}\}$$

The integral is an analytic function of the space-time dimension D with the Laurent expansion near $D_c \in \mathbb{N}^*$

$$I_\Gamma(\underline{s}, \underline{m}) = \sum_{r \geqslant -m} (D - D_c)^r I_\Gamma^{(r)}(\underline{s}, \underline{m}) \qquad D = D_c - 2\epsilon; \qquad 0 \leqslant \epsilon \ll 1$$

Feynman Integrals: parametric representation

The graph polynomial is homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times \left(\sum_{i=1}^{n} m_i^2 x_i\right) - \mathcal{V}_\Gamma(\underline{s}, \underline{x})$$

$$\mathcal{U}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L\\ 0\leqslant a_i\leqslant 1}} u_{a_1,\dots,a_n} \prod_{i=1}^{n} x_i^{a_i}, \qquad \mathcal{V}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L+1\\ 0\leqslant a_i\leqslant 1}} S_{a_i,\cdots,a_n} \prod_{i=1}^{n} x_i^{a_i}$$

$u_{a_1,\dots,a_n} \in \{0,1\}$ and $S_{a_i,\cdots,a_n}$ are linear combination of the kinematic variables

From $\mathcal{F}_\Gamma$ we can reconstruct the associated Feynman graph Γ

- the number of edges is n
- the loop order is $L = \deg(\mathcal{F}_\Gamma) - 1$
- Number of vertices $v = 1 + n - L$ from Euler characteristic of the planar graph

Feynman Integrals: parametric representation

$$I_\Gamma(\underline{s},\underline{m}) = \Gamma\left(n - \frac{LD}{2}\right) \int_{\Delta_n} \Omega_\Gamma; \quad \Omega_\Gamma := \mathrm{Res}_{X_\Gamma} \left(\frac{\mathcal{U}_\Gamma(\underline{x})^{n-\frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{n-\frac{LD}{2}}} \prod_{i=1}^{n-1} dx_i \right)$$

Algebraic differential form $\Omega_\Gamma \in H^{n-1}(\mathbb{P}^{n-1} \backslash X_\Gamma)$ on the complement of the graph hypersurface

$$X_\Gamma := \{\mathcal{U}_\Gamma(\underline{x}) = 0 \& \mathcal{F}_\Gamma(\underline{x}) = 0, \underline{x} \in \mathbb{P}^{n-1}\}$$

- All the singularities of the Feynman integrals are located on the graph hypersurface
- Generically the graph hypersurface has non-isolated singularities

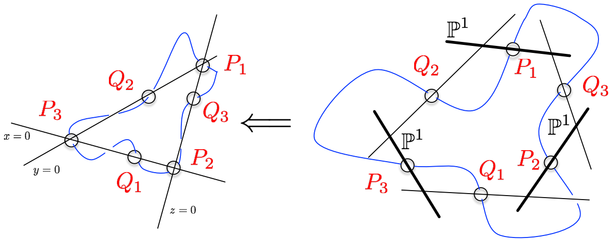
Feynman integral and periods

$\Delta_n \notin H^{n-1}(\mathbb{P}^{n-1}\backslash X_\Gamma)$ because

$$\partial\Delta_n \cap X_\Gamma = \{(1,0,\dots,0),(0,1,0,\dots,0),\dots,(0,\dots,0,1)\}$$

we have to look at the relative cohomology $H^\bullet(\mathbb{P}^{n-1}\backslash X_\Gamma; Д_n\backslash Д_n \cap X_\Gamma)$

The normal crossings divisor $Д_n := \{x_1\cdots x_n = 0\}$ and X_Γ are separated by performing a series of iterated blowups of the complement of the graph hypersurface [Bloch, Esnault, Kreimer]



Differential equation

The Feynman integral *are* periods of the relative cohomology after performing the appropriate blow-ups

$$\mathfrak{M}(\underline{s}, \underline{m}^2) := H^\bullet(\widetilde{\mathbb{P}^{n-1}} \backslash \widetilde{X_F}; \widetilde{\mathit{\Pi}_n} \backslash \widetilde{\mathit{\Pi}_n} \cap \widetilde{X_\Gamma})$$

Since Ω_Γ varies with the kinematic variables $\underline{s}$ and internal mass $\underline{m}$ one needs to study a variation of (mixed) Hodge structure

The Feynman integrals inhomogenous differential equation

$$L_{PF}\, I_\Gamma = S_\Gamma$$

Generically there is an inhomogeneous term $S_\Gamma \neq 0$ due to the boundary components $\partial\Delta_n$

Deriving this differential equation is difficult in general and requires a lot of computer resources and is still a major question in QFT

Gel'fand-Zelevinsky-Kapranov approach

Unitarity in QFT motivates considering the following period integral

$$\pi_\Gamma = \frac{1}{(2i\pi)^n} \int_{|x_1|=\cdots=|x_n|=\epsilon} \Omega_\Gamma \qquad 0 < \epsilon \ll 1$$

Consider the toric polynomial

$$\mathcal{F}^{\text{toric}}_{\Delta(\Gamma)}(\underline{x}) = \sum_{\nu \in \Delta(\Gamma) \cap \mathbb{Z}^{n+1}} f_\nu \, x^\nu$$

- $\mathcal{F}_\Gamma(\underline{x})$ is a specialisation of the toric deformation parameters to the physical locus $f_\nu \mapsto (\underline{s}, \underline{m})$. The map is *linear*
- π_Γ is a period integral for the Calabi-Yau hypersurface X_{Δ° and is a GZK $\mathcal{A}$-hypergeometric series

The Feynman graph hypersurface is highly non generic

- The system often resonant and reducible
- Obtaining the minimal order Picard-Fuchs operator this way is not an easy task as one must restrict the D-module

Geometry for Feynman graph motives

What is the geometry governing the Feynman graph motive?

1. One-loop graph hypersurface degree 2 in $\mathbb{P}^{n-1}$
   - The motive structure is the one of dilogarithms
2. Two-loop graph hypersurface degree 3 in $\mathbb{P}^{n-1}$
   - for $n = 3$ Brown-Levin elliptic multiple polylogarithms [Bloch, Vanhove; Adams, Bogner, Weinzierl]
   - For $n \geqslant 4$ motivic elliptic curve for the mixed Hodge structure [Bloch, Doran, Kerr, Vanhove (work in progress)]
3. family of sunset graphs : $n-1$-loop graph hypersurface degree n in $\mathbb{P}^{n-1}$ define Calabi-Yau $n-2$-fold [Doran, Novoseltsev, Vanhove]

Creative Telescoping

We want to derive the differential equation

$$L_{PF}\int_\Gamma \Omega_\Gamma = S_\Gamma$$

The differential form Ω_Γ is functions of the kinematics parameters $\underline{s} = \{p_i \cdot p_j\}$ and the internal masses $\underline{m} = \{m_1, \dots, m_n\}$ which are all non vanishing.

For a given subset of kinematic parameters $\underline{z} := (z_1, \dots, z_r) \subset \underline{s} \cup \underline{m}$ we want to construct a differential operator $T_{\underline{z}}$ such that

$$T_{\underline{z}}\Omega_\Gamma = 0$$

such that

$$T_{\underline{z}} = L_{PF}(\underline{s}, \underline{m}, \underline{\partial_z}) + \sum_{i=1}^{n} \partial_{x_i} Q_i(\underline{s}, \underline{m}, \underline{\partial_z}; \underline{x}, \underline{\partial_x})$$

where the finite order differential operator

$$L_{PF}(\underline{s}, \underline{\partial_z}) = \sum_{\substack{0\leqslant a_i\leqslant o_i\\ 1\leqslant i\leqslant r}} p_{a_1,\dots,a_r}(\underline{s}, \underline{m}) \prod_{i=1}^{r} \left(\frac{d}{dz_i}\right)^{a_i}$$

$$Q_i(\underline{s}, \underline{m}^2, \underline{\partial_z}) = \sum_{\substack{0\leqslant a_i\leqslant o'_i\\ 1\leqslant i\leqslant r}} \sum_{\substack{0\leqslant b_i\leqslant \tilde{o}_i\\ 1\leqslant i\leqslant n}} q^{(i)}_{a_1,\dots,a_r}(\underline{s}, \underline{m}, \underline{x}) \prod_{i=1}^{r} \left(\frac{d}{dz_i}\right)^{a_i} \prod_{i=1}^{n} \left(\frac{d}{dx_i}\right)^{b_i}$$

- The orders o_i, o'_i, $\tilde{o}_i$ are positive integers
- $p_{a_1,\dots,a_r}(\underline{s}, \underline{m})$ polynomials in the kinematic variables
- $q^{(i)}_{a_1,\dots,a_r}(\underline{s}, \underline{m}, \underline{x})$ rational functions in the kinematic variable and the projective variables $\underline{x}$.

Integrating over a cycle γ gives

$$0 = \oint_\gamma T_{\underline{z}} \Omega_\Gamma = L_{PF}(\underline{s}, \underline{m}, \partial_{\underline{z}}) \oint_\gamma \Omega_\Gamma + \oint_\gamma d\beta_\Gamma$$

For a cycle $\oint_\gamma d\beta_\Gamma = 0$ (e.g. maximal cut) we get

$$L_{PF}(\underline{s}, \underline{m}, \partial_{\underline{z}}) \oint_\gamma \Omega_\Gamma = 0$$

For the Feynman integral I_Γ we have

$$0 = \int_{\Delta_n} T_{\underline{z}} \Omega_\Gamma = L_{PF}(\underline{s}, \underline{m}, \partial_{\underline{z}}) I_\Gamma + \int_{\Delta_n} d\beta_\Gamma$$

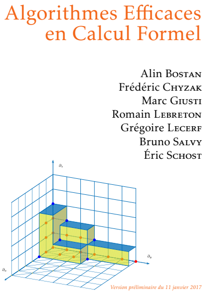
since $\partial\Delta_n \neq \emptyset$

$$L_{PF}(\underline{s}, \underline{m}, \partial_{\underline{z}}) I_\Gamma = S_\Gamma$$

This can done using the creative telescoping method introduced by Doron Zeilberger (1990) and the algorithm by F. Chyzak

This works in all case even when the graph hypersurface does not have isolated singularities (which is the generic case)

This algorithm gives immediately the minimal order differential operator no need for reducing the system



A Fast Approach to Creative Telescoping

Christoph Koutschan

Abstract. In this note we reinvestigate the task of computing creative telescoping relations in differential-difference operator algebras. Our approach is based on an ansatz that explicitly includes the denominators of the delta parts. We contribute several ideas of how to make an implementation of this approach reasonably fast and provide such an implementation. A selection of examples shows that it can be superior to existing methods by a large factor.

Mathematics Subject Classification (2010). Primary 68W30; Secondary 33F10.

Keywords. holonomic functions, special functions, symbolic integration, symbolic summation, creative telescoping, Ore algebra, WZ theory.

1. Introduction

The method of creative telescoping nowadays is one of the central tools in computer algebra for attacking definite integration and summation problems. Zeilberger with his celebrated holonomic systems approach [17] was the first to recognize its potential for making these tasks algorithmic for a large class of functions. In the realm of holonomic functions, several algorithms for computing creative telescoping relations have been developed in the past. The methodology described here is not an algorithm in the strict sense because it involves some heuristics. But since it works pretty well on nontrivial examples we found it worth to be written down. Additionally we believe that it is the method of choice for really big examples. Our implementation is contained in the Mathematica package `HolonomicFunctions` as the command `FindCreativeTelescoping`. The package can be downloaded from the RISC combinatorics software webpage:

http://www.risc.uni-linz.ac.at/research/combinat/software/

Throughout this paper we will work in the following setting. We assume that a function f to be integrated or summed satisfies some linear difference-differential relations which we represent in a suitable operator algebra (Ore algebra). We use the symbol D_x to denote the derivation operator w.r.t. x and S_n for the shift operator w.r.t. n. Such an algebra can be viewed as a polynomial ring in the respective operators, with coefficients being rational functions in the corresponding variables, subject to the commutation rules $D_x x = x D_x + 1$ and $S_n n = n S_n + S_n$. Ideally, all the relations for f generate a ∂-finite left ideal, i.e., a zero-dimensional left ideal in the operator algebra. If additionally f is holonomic (a notion that can be made formal by D-module theory), then the existence of creative telescoping relations is guaranteed by theory (i.e., by the elimination property of holonomic modules). Chyzak, Kauers, and Salvy [6] have shown that creative telescoping is also possible for higher-dimensional ideals under certain conditions. We tacitly assume that any input to a creative

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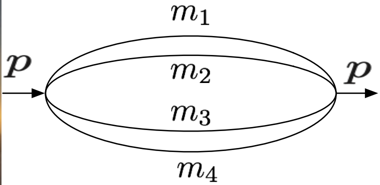
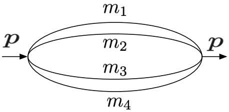
The method provides as well the space of annihilator (the GKZ system of equations) with respect to all the parameters

The sunset graphs family



The sunset family of graph



The graph polynomial for the $n-1$-loop sunset

$$\mathcal{F}_n^{\ominus}(\underline{x}) = x_1 \cdots x_n \left(\phi_n^{\ominus}(\underline{x}) - p^2\right) ;\ \phi_n^{\ominus}(\underline{x}) = \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)\left(m_1^2 x_1 + \cdots + m_n^2 x_n\right)$$

The Feynman integral in $D = 2$

$$I_n^{\ominus}(p^2, \underline{m}^2) = \int_{x_1 \geqslant 0, \ldots, x_n \geqslant 0} \frac{1}{p^2 - \phi_n^{\ominus}(\underline{x})} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

and the classical period

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \int_{|x_1| = \cdots = |x_n| = 1} \frac{1}{p^2 - \phi_n^{\ominus}(\underline{x})} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

The sunset family and generalized Apéry numbers

The classical period

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{p^2 - \phi_n^{\ominus}(\underline{x})} \prod_{i=1}^{n-1} \frac{dx_i}{x_i}$$

has the series expansion

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \sum_{k\geqslant 0} (p^2)^{-k-1} A_n(k, \underline{m}^2)$$

with

$$A_n(k, \underline{m}^2) = \sum_{\substack{r_1+\cdots+r_n=k \\ r_i\geqslant 0}} \left(\frac{(r_1+\cdots+r_n)!}{r_1!\cdots r_n!}\right)^2 \prod_{i=1}^{n} (m_i^2)^{r_i}$$

The sunset integrals and L-function values

For the special value $p^2 = m_1^2 = \cdots = m_n^2 = 1$ the sunset Feynman integral becomes a pure period integral [Bloch, Kerr, Vanhove]

$$I_n^{\ominus}(1,\ldots,1) = \int_{x_i \geqslant 0} \frac{\prod_{i=1}^{n-1} d\log x_i}{1 - \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)(x_1 + \cdots + x_n)}$$

Using impressive numeric experimentations [Broadhust] found that $I_n^{\ominus}(1,\ldots,1)$ is given by L-function values in the critical band. For large n the L-function are from moments Kloosterman sums over finite fields

The sunset integrals and L-function values

For the special value $p^2 = m_1^2 = \cdots = m_n^2 = 1$ the sunset Feynman integral becomes a pure period integral [Bloch, Kerr, Vanhove]

$$I_n^{\ominus}(1,\ldots,1) = \int_{x_i \geqslant 0} \frac{\prod_{i=1}^{n-1} d\log x_i}{1 - \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)(x_1 + \cdots + x_n)}$$

These special values realise explicitly Deligne's conjecture relating period integrals to L-values in the critical band

$n = 3$: elliptic curve case : $I_3^{\ominus}(1,\ldots,1) = \frac{1}{2}\zeta(2)$
$n = 4$: $K3$ Picard rank 19 : $I_4^{\ominus}(1,\ldots,1) = \frac{12\pi}{\sqrt{15}} L(f_{K3}, 2)$ [Bloch, Kerr, Vanhove]

- $L(f_{K_3}, s)$ is the L-function of $H^2(K3, \mathbb{Q}_\ell)$
- Functional equation $L(f_{K3}, s) \propto L(f_{K3}, 3 - s)$
- $f_{K3} = \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) \sum_{m,n} q^{m^2+4n^2+mn}$

The sunset integrals and exponential motives

The Feynman integral for $0 \leqslant p^2 \leqslant (m_1 + \cdots + m_n)^2$

$$I_n^{\ominus}(p^2, \underline{m}^2) = 2^{n-1} \int_0^\infty u I_0(\sqrt{p^2} u) \prod_{i=1}^n K_0(m_i u) du$$

The classical period for $p^2 \geqslant (m_1 + \cdots + m_n)^2$

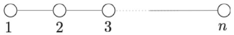
$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \frac{1}{2} \int_0^\infty u K_0(\sqrt{p^2} u) \prod_{i=1}^n I_0(m_i u) du$$

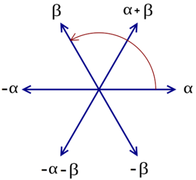
where we have the modified Bessel function of the first kind

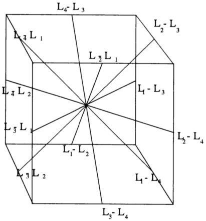
$$I_0(z) = \frac{1}{2i\pi} \int_{|t|=1} e^{-\frac{z}{2}(t+\frac{1}{t})} d\log t; \qquad K_0(z) = \int_0^{+\infty} e^{-\frac{z}{2}(t+\frac{1}{t})} d\log t$$

There are exponential period integrals in the sense of the non classical exponential motives (cf. [Fresàn,Jossen] for recent work on this)

Sunset Calabi-Yau







Sunset graphs toric variety $X_{p^2}(A_n)$ [Verrill]

The sunset graph polynomial

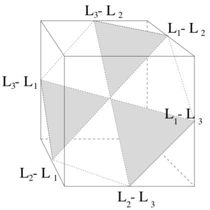
$$\mathcal{F}_n^{\ominus} = x_1 \cdots x_n \left(\left(\sum_{i=1}^{n} m_i^2 x_i \right) \left(\sum_{i=1}^{n} \frac{1}{x_i} \right) - p^2 \right)$$

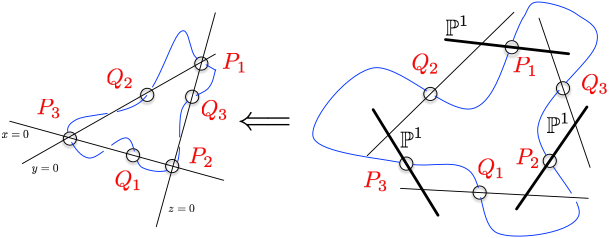
is a character of the adjoint representation of A_{n-1} with support on the polytope generated by the A_{n-1} root lattice

- The Newton polytope Δ_n for $\mathcal{F}_n^{\ominus}$ is reflexive with only the origin as interior point
- The toric variety $X(A_{n-1})$ is the graph of the Cremona transformations $X_i \to 1/X_i$ of $\mathbb{P}^{n-1}$
  $X(A_{n-1})$ is obtained by blowing up the strict transform of the points, lines, planes etc. spanned by the subset of points $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$ in $\mathbb{P}^{n-1}$

Two-loop Sunset toric variety $X(A_2)$

$$(m_1^2x_1 + m_2^2x_2 + m_3^2x_3)(x_1x_2 + x_1x_3 + x_2x_3) = p^2x_1x_2x_3$$





- The toric variety is $X(A_2) = Bl_3(\mathbb{P}^2) = dP_6$ blown up at 3 points
- The subfamily of anticanonical hyperspace is non generic The combinatorial structure of the **NEF** partition describes precisely the mass deformations
- True for all n

Sunset graphs pencils of variety $\mathcal{X}_{p^2}(A_n)$ [Verrill]

For $p^2 \in \mathbb{P}^1$ we define the pencil in the ambient toric variety $X(A_{n-1})$

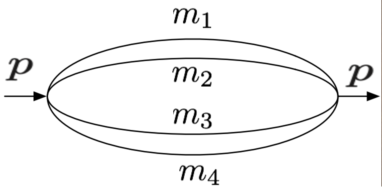
$$\mathcal{X}_{p^2}(A_{n-1}) = \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_{n-1}) | x_1 \cdots x_n \left(\sum_{i=1}^{n} m_i^2 x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) - p^2 x_1 \cdots x_n = 0\}$$

The fiber at $p^2 = \infty$ is $\varPi_n = \{x_1 \cdots x_n = 0\}$

Since $\varPi_n$ is linearly equivalent to the anti-canonical divisor of $X(A_{n-1})$ the family has trivial canonical divisor: We have a family of (singular) Calabi-Yau $n-2$-fold

This is specific to this family of associated with root lattice of A_n

The Iterative fibration



The Iterative fibration

The sunset family $\left(\sum_{i=1}^{n} m_i^2 x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) - p^2 = 0$ is birational to a complete intersection variety in $\mathbb{P}^n$

$$\frac{1}{x_0} + \sum_{i=1}^{n} \frac{1}{x_i} = 0; \qquad p^2 x_0 + \sum_{i=1}^{n} m_i^2 x_i = 0$$

Obviously $X(A_{n-1})$ is obtained from $X(A_{n-2})$ with the substitutions

$$\frac{1}{x_{n-1}} \to \frac{1}{x_{n-1}} + \frac{1}{x_n}; \qquad m_{n-1}^2 x_{n-1} \to m_{n-1}^2 x_{n-1} + m_n^2 x_n$$

$X(A_{n-1})$ is fibrered over $X(A_1) = \mathbb{P}^1$ with generic fibers $X(A_{n-2})$

$$X(A_{n-2}) \to X(A_{n-1}) \to X(A_1) = \mathbb{P}^1$$

The Iterative fibration

The geometric phenomenon at work that the n-loop sunset corresponds to a family of Calabi-Yau $(n-1)$-folds each of which is a double cover of the (rational) total space of a family of $(n-1)$-loop sunset Calabi-Yau $(n-2)$-folds.

At the level of the integrals this

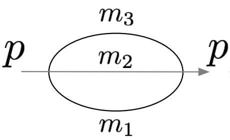
$$I_n^{\ominus}(p^2, \underline{m}^2) = \int_0^{+\infty} I_{n-1}^{\ominus}\left(p^2, \underline{m}^2, (m_{n-1}^2 + t^{-1} m_n^2)(1+t)\right) d\log t$$

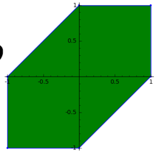
and for the classical period

$$\pi_n^{\ominus}(p^2, \underline{m}^2) = \frac{1}{2i\pi} \int_{|t|=1} \pi_{n-1}^{\ominus}\left(p^2, \underline{m}^2, (m_{n-1}^2 + t^{-1} m_n^2)(1+t)\right) d\log t$$

This construction allows to understand the geometry and build the PF operator for all loop orders [Doran, Novoseltsev, Vanhove]

The two-loop sunset graph [Bloch, Kerr, Vanhove]





The pencil of sunset elliptic curve

$$\mathcal{X}_{p^2}(A_2) = \{(p^2, \underline{x}) \in \mathbb{P}^2 \times X(A_2) | (m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3)(x_1 x_2 + x_1 x_3 + x_2 x_3) = p^2 x_1 x_2 x_3\}$$

The fibers types are

- Generic case $m_1 \neq m_2 \neq m_3$

$$I_2(0) + I_6(\infty) + I_1(\mu_1) + \cdots + I_1(\mu_4); \qquad \mu_i = (\pm m_1 \pm m_2 \pm m_3)^2$$

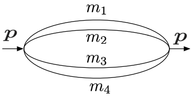
- single mass $m_1 = m_2 = m_3 \neq 0$: modular curve $X_1(6)$

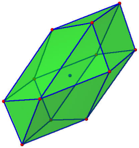
$$I_2(0) + I_6(\infty) + I_3(m^2) + I_1(9m^2)$$

The Feynman integral is an elliptic dilogarithm [Bloch, Kerr,Vanhove]

$$H^2(\mathbb{P}^2 \backslash \{x_1 x_2 x_3 = 0\}, X_\ominus, \mathbb{Q}(2))$$

The 3-loop case : pencil of $K3$





$$\mathcal{X}_{p^2}(A_3) := \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_3) | \left(m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4\right)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_4}\right) = p^2\}$$

Generic anticanonical $K3$ hypersurface in the toric threefold X_{Δ° has Picard rank 11

The physical locus for the sunset has at least Picard rank 16

masses	fibers	Mordell-Weil	Picard rank
(m_4, m_1, m_2, m_3)	$8I_1 + 2I_2 + 2I_6$	2	16
$(m_4 = m_1, m_2, m_3)$	$8I_1 + I_4 + 2I_6$	2	17
$(m_4, m_1, m_2 = m_3)$	$4I_1 + 4I_2 + 2I_6$	1	17
$(m_4 = m_1, m_2 = m_3)$	$4I_1 + 2I_2 + I_4 + 2I_6$	1	18
$(m_4 = m_1 = m_2, m_3)$	$8I_1 + I_4 + 2I_6$	3	18
$(m_4, m_1 = m_2 = m_3)$	$4I_1 + 4I_2 + 2I_6$	2	18
$(m_4 = m_1 = m_2 = m_3)$	$4I_1 + 2I_2 + I_4 + 2I_6$	2	19

$|Pic| = 19$ motive of an elliptic 3-log $H^3(\mathbb{P}^3 \backslash \mathcal{I}_4, X_4, \mathbb{Q}(3))$ [Bloch, Kerr, Vanhove]

The Picard-Fuchs operator: three loop sunset

$$\begin{tikzcd}
& L_3 & \\
L_4 \arrow[ur, "m_4=m_1"] & & L_4 \arrow[ul, "m_4=m_1"'] \\
& L_5 \arrow[ul, "m_3=m_1"] \arrow[ur, "m_3=m_4"'] & \\
& L_6 \arrow[u, "m_2=m_1"] &
\end{tikzcd}$$

$$L_r = (\alpha \frac{d}{dp^2} + \beta) \circ L_{r-1}$$

The Picard-Fuchs operators for the Feynman integral for general parameters $m_4 \neq m_1 \neq m_2 \neq m_3$

$$L_6 = \sum_{r=0}^{6} q_r(s) \left(\frac{d}{dp^2}\right)^r$$

is order 6 and degree 25

$$q_6(p^2) = \tilde{q}_6(p^2) \times \prod_{\epsilon_i = \pm 1} (p^2 - (\epsilon_1 m_1 + \epsilon_2 m_2 + \epsilon_3 m_3 + \epsilon_4 m_4)^2)$$

with $\tilde{q}_6(p^2)$ degree 17 contains the apparent singularities

The 4-loop case : pencil of CY 3-fold

$$\mathcal{X}_{p^2}(A_4) := \{(p^2, \underline{x}) \in \mathbb{P}^1 \times X(A_4) | \left(m_1^2 x_1 + \cdots + m_5^2 x_5\right) \left(\frac{1}{x_1} + \cdots + \frac{1}{x_5}\right) = p^2\}$$

This gives a pencil of nodal Calabi-Yau 3-fold

For a (small or big) resolution $\hat{W}$ is

- $h^{12}(\hat{W}) = 5$ for the 5 masses case : 30 nodes
- $h^{12}(\hat{W}) = 1$ for the 1 mass case $m_1 = \cdots = m_5$: 35 nodes
- $h^{12}(\hat{W}) = 0$ for $p^2 = m_1 = \cdots = m_5 = 1$: rigid case birational to the Barth-Nieto quintic
  - $I_5^{\ominus}(1, \ldots, 1) = 48\zeta(2)L(f, 2)$ [Broadhurst]
  - f weight 4 and level 6 modular form $f = (\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau))^2$
  - This L-series is precisely the one for $H^3(X(A_4), \mathbb{Q}_\ell)$ [Verrill]
  - Functional equation $L(f, s) \propto L(f, 4 - s)$
  - Again we have a manifestation of Deligne's conjecture

The Picard-Fuchs operator : 4 loop sunset

$$\begin{tikzcd}
& L_4 & \\
L_6 \arrow[ur] & & L_6 \arrow[ul] \\
L_8 \arrow[u] \arrow[urr] & & L_8 \arrow[u] \arrow[ull] \\
& L_{10} \arrow[ul] \arrow[ur] & \\
& L_{12} \arrow[u] &
\end{tikzcd}$$

The Picard-Fuchs operators for the Feynman integral for general parameters $m_1 \neq m_2 \neq m_3 \neq m_4 \neq m_5$

$$L_{12} = \sum_{r=0}^{12} q_r(s) \left(\frac{d}{dp^2}\right)^r$$

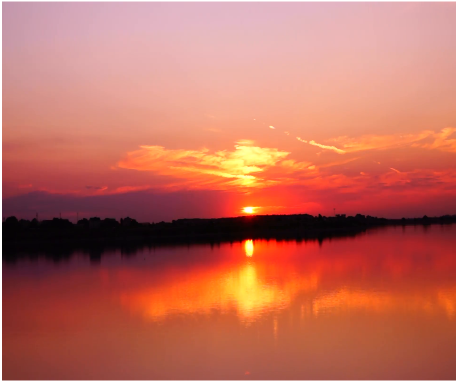
is order 12 and degree 121

$$q_{12}(p^2) = \tilde{q}_{12}(p^2) \times (p^2)^{12} \prod_{\epsilon_i=\pm 1} (p^2 - (\epsilon_1 m_1 + \cdots + \epsilon_5 m_5)^2)$$

with $\tilde{q}_{12}(p^2)$ degree 98 contains the apparent singularities

$$L_r = \left(\alpha \left(\frac{d}{dp^2}\right)^2 + \beta \frac{d}{dp^2} + \gamma\right) \circ L_{r-2}$$

Sunset Mirror Symmetry



Sunset local Gromov-Witten invariants

It was shown that the sunset Feynman integral takes the expression

[Bloch, Kerr, Vanhove]

$$I_3^{\ominus}(p^2) = \pi_3^{\ominus}(p^2)\left(3R_0^2 + \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell>0\\(\ell_1,\ell_2,\ell_3)\in\mathbb{N}^3\setminus(0,0,0)}} \ell(1-\ell R_0)\, N^{\text{loc.}}_{\ell_1,\ell_2,\ell_3} \prod_{i=1}^{3} Q_i^{2\ell_i}\right).$$

- $N^{\text{loc.}}_{\ell_1,\ell_2,\ell_3}$ are rational genus 0 Gromov-Witten numbers
- With $\pi_3^{\ominus}(p^2) = \frac{d}{dp^2} R_0$

$$R_0 := \int_{|x_1|=|x_2|=|x_3|=1} \log(p^2 - \phi_3^{\ominus}) \prod_{i=1}^{3} \frac{d\log x_i}{2\pi i}$$

Classical and regularised periods I

The classical period associated to a Laurent polynomial $f : (\mathbb{C}^\times)^n \to \mathbb{C}$ is

$$\pi_f(t) = \left(\frac{1}{2\pi i}\right)^n \int_{|x_1|=\cdots=|x_n|=1} \frac{1}{1 - tf(x_1, \ldots, x_n)} \prod_{i=1}^n \frac{dx_i}{x_i}$$

- This period satisfies an ordinary differential equation $L\pi_f(t) = 0$ with $L = \sum_{i=0}^r p_r(t) \left(\frac{d}{dt}\right)^i$ of minimal order
- In our case the Laurent polynomial is given by the second Symanzik graph polynomial

Classical and regularised periods II

The (regularised) quantum period $G_V(t)$ of a Fano manifold V

$$G_V(t) = 1 + \sum_{\beta\in H_2(V,\mathbb{Z})} |-K_V\cdot\beta|!\langle[pt]\psi^{-K_V\cdot\beta-2}\rangle^V_{0,1,\beta}\, t^{K_V\cdot\beta}$$

where $\langle[pt]\psi^{-K_V\cdot\beta-2}\rangle^V_{0,1,\beta}$ is a 1-pointed genus 0 Gromov–Witten invariant with descendants for anticanonical degree $K_V\cdot\beta$ curves on V

- This (regularised) quantum period is annihilated by a quantum differential operator $\hat L^{\hat G} = \sum_{i=0}^s \hat q_r(t)\left(\frac{d}{dt}\right)^i$

A complex projective manifold V of complex dimension n is called Fano if the anticanonical line bundle $-K_V = \wedge^n T_X$ is ample.

Fano/ LG mirror symmetry I

Mirror symmetry predicts that the mirror of a Fano $n-1$-fold V is a pair (Y, w) called a Landau-Ginzburg model where Y is an $n-1$-fold and the superpotential $w \in \Gamma(Y, \mathcal{O}_Y)$ is a regular function

The Gromov-Witten theory of V should be related to the Hodge theory of the fibers of $w : Y \to \mathbb{A}^1$ as follows : the regularised quantum period $\hat{G}_V$ of V coincides with the classical period π_w defined by

$$\pi_w(t) = \int_\Gamma \frac{dx_1 \cdots dx_n}{1 - tw(x_1, \dots, x_n)}$$

A Laurent polynomial $f \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$ is mirror to a smooth Fano variety V of dimension $n-1$ if the classical period π_f coincide with the regularised quantum period $\hat{G}_V$.

Sunset Landau-Ginzburg mirror symmetry

The LG superpotential is the sunset graph polynomial

$$w = \mathcal{F}_n^{\ominus}(\underline{x}) = x_1 \cdots x_n \left(p^2 - \phi_n^{\ominus}(\underline{x})\right)$$

is homogeneous of degree n in $\mathbb{P}^{n-1}$ therefore the central charge is $c = 3(n-2)$ in agreement with the statement that $\mathcal{X}_{p^2}(A_{n-1})$ is a Calabi-Yau $n-2$-fold

We can know use the mirror symmetry between Landau-Ginzburg model and Fano varieties

The mirror sunset theorem

Theorem

The pencils of sunset Calabi-Yau $(n-1)$-folds form Landau-Ginzburg models mirror to weak Fano n-folds. Specifically, the "all equal masses" case is known to be mirror to the toric Fano variety whose N-lattice polytope is the Newton polytope of the n-loop sunset Feynman graph hypersurfaces. This is just the type $(1, 1, \dots, 1)$ hypersurface in $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ ($n+1$ factors).

[Doran, Novoseltsev, Vanhove (to appear)]

A motivic conjecture

For graph with more edge the graph polynomial does not define a Calabi-Yau but based on in depth-analysis of the graph polynomial geometry we can make the following conjecture

Conjecture (Motivic Mirror Conjecture (short version))

- *Feynman integrals satisfy irreducible Fuchsian systems over momentum space*
- *ODE are are inhomogeneous differential equations whose homogeneous part is the Picard-Fuchs equation of a pencil of Calabi-Yau varieties*
- *These pencils can be interpreted as Landau-Ginzburg models, for which the internal mass parameters are complex structure deformations, mirror to weak Fano varieties, for which the internal mass parameters are deformations in the Kähler cone.*

[Doran, Novoseltsev, Vanhove (to appear)]

Conclusion

- We have put forward the new relation between Feynman integrals and mirror symmetry between Fano / LG model
- It is a new result that all the sunset Feynman integrals compute the genus 0 relative Gromov-Witten invariants

Generic Feynman graphs is more intricate

- For Feynman graph with $\deg(\mathcal{F})_\Gamma = L$ in $\mathbb{P}^n$ with $n > L + 1$ we do not have a Calabi-Yau
  - Multiple potential LG models?
  - Relations with the Doran-Harder-Thompson construction for Tyurin degenerations cf. [Doran (strings math 2015)]
- For higher-point two-loop integrals (ie with graph polynomial of degree 3 in $\mathbb{P}^n$ with $n \geqslant 3$) there seems to be an extension of the mirror symmetry to other del Pezzo surfaces