q-cut-and-join operators and q-Capelli identity on Reflection Equation algebras

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Plan

The main actor of my talk is the so-called Reflection Equation algebra. It is defined by the following system

$$R\,L_1\,R\,L_1 - L_1\,R\,L_1\,R = 0,$$

where R is a Hecke type solution to the braid relation, $L = \|l_i^j\|_{1\le i,j\le N}$ is a matrix with entries l_i^j and $L_1 = L\otimes I$.

Observe that if R is a deformation of the usual flip P, the RE algebra is a deformation of the algebra $Sym(gl(N))$. Consequently there is the corresponding Poisson bracket. It is not the case of the QG $U_q(sl(N))$ which is not a deformation of the algebra $U(sl(N))$ as an algebra, only the coalgebraic structure is deformed.

Let $L = \|l_i^j\|_{1\le i,j\le N}$ be a matrix with commutative entries and $D = \|\partial_k^l\|_{1\le k,l\le N}$ be the matrix composed of the partial derivatives such that

$$\partial_k^l(l_i^j) = \delta_k^j\,\delta_i^l.$$

Consider the following matrix

$$\hat{L} = L\,D.$$

Its entries $\hat{l}_i^j$ meet the commutation relations

$$[\hat{l}_i^j, \hat{l}_k^l] = \delta_k^j\,\hat{l}_i^l - \delta_i^l\,\hat{l}_k^j$$

and consequently generate the enveloping algebra $U(gl(N))$.

It is well known that the elements $\mathrm{Tr}\,\hat{L}^k$ $k = 1, 2...$ belong to the center $Z(U(gl(N))$ of this algebra and generate it. They are called power sums.

Also, consider

$$\pi_\lambda : U(gl(N)) \to \mathrm{End}(V_\lambda)$$

irreducible representations of the algebra $U(gl(N))$. Classes of equivalent (up to conjugations) representations are labeled by the partitions

$$\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N \geq 0).$$

To such a partition λ we associate a Schur polynomial $s_\lambda(\alpha_1...\alpha_N)$. If the numbers α_i, $1 \geq i \geq N$ are eigenvalues of the matrix L, we'll also write $s_\lambda(L)$. This quantity can be expressed via the entries l_i^j of the matrix L as follows

$$s_\lambda(L) = Tr_{1..N} P_\lambda L_1\, L_2 ... L_N,$$

where P_λ is the idempotent corresponding to the partition λ. Note that the quantity $s_\lambda(L)$ does not depend on the standard tableau. Thus, the partial derivatives ∂_i^j can be applied to $s_\lambda(L)$, considered as an element of the commutative algebra $Sym(gl(N))$. Consequently, the differential operator

$$W = \mathrm{Tr}\, \hat{L} = \mathrm{Tr}\, L\, D$$

also can be applied to these elements.

The point is that the element $s_\lambda(L)$ is an eigenvector of the operator W:

$$W \triangleright s_\lambda(L) = \varphi_\lambda \, s_\lambda(L).$$

The higher counterparts of this operator

$$W^\Delta =: \mathrm{Tr}\, \hat{L}^{\Delta_1} ... \mathrm{Tr}\, \hat{L}^{\Delta_k} :$$

where $\Delta = (\Delta_1, \dots, \Delta_k)$ is a partition and $:\ :$ stands for the normal ordering, are called *the cut-and-join operators*.
In fact, these operators are close to the Casimir operators

$$p_k(\hat{L}) = \mathrm{Tr}\, \hat{L}^k \in U(gl(N)).$$

The elements $s_\lambda(L)$ are also eigenvectors of the Casimir operators. The corresponding eigenvalues were computed by Perelomov and Popov.

Observe that the term cut-and-join analysis was introduced by I.Goulden (partially with D.Jackson). This analysis arises from combinatorics related to the symmetric groups and is actively used in the Hurwitz theory by Morozov, Marshakov, Mironov, Orlov, Natanzon and others.

The restrictions of the cut-and-join operators on symmetric functions are habitually expressed via the power sums. In this form the simplest cut-and-join operator is

$$\Sigma_{i\geq 1, j\geq 1}\left(i\, j\, p_{i+j}\,\frac{\partial}{\partial p_i}\,\frac{\partial}{\partial p_j} + (i+j)\, p_i\, p_j\,\frac{\partial}{\partial p_{i+j}}\right).$$

Our aim is to get q-analogs of all this stuff. To this end we need the notion of Quantum Doubles.

Let A and B two associative unital algebras equipped with a map (called permutation map)

$$\sigma : A \otimes B \to B \otimes A, \ \ (a \otimes b) \mapsto \sigma(a \otimes b), \ \ a \in A, \ \ b \in B \qquad (1)$$

Definition

By a quantum double (QD) we mean the data (A, B, σ), where the map σ is defined by means of a braiding, different from a (super-)flip.

We will also speak about the permutation relations

$$a \otimes b = \sigma(a \otimes b).$$

Observe that if the algebras are introduced via relations on generators, these relations have to be compatible with the permutation relations.

Let us precise that by a braiding $R: V^{\otimes 2} \to V^{\otimes 2}$ we mean a solution of the braid relation

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R).$$

If a braiding R is subject to the relation

$$(q\, I - R)(q^{-1}\, I + R) = 0,\ \ q \in \mathbb{C} \setminus \{0, 1, -1\},$$

it is called Hecke symmetry.
The most known Hecke symmetries are those coming from the Quantum Groups (QG) $U_q(sl(N))$. However, there are many others which even are not deformations of (super-)flips.

With any Hecke symmetry R we can associate the following algebras

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \;\; \bigwedge_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle,$$

and two Quantum matrix algebras: the RTT one

$$R\,T_1\,T_2 - T_1\,T_2\,R = 0, \quad T = \|t_i^j\|_{1\le i,j\le N}$$

and the so-called Reflection Equation (RE) one (denoted $\mathcal{L}(R)$)

$$R\,L_1\,R\,L_1 - L_1\,R\,L_1\,R = 0, \quad L = \|l_i^j\|_{1\le i,j\le N}.$$

We say that a Hecke symmetry R is even if the Hilbert-Poincare series of the algebra $\bigwedge_R(V)$ is a polynomial. Its degree is called the rank of R.

Also, we will deal with modified RE algebras, defined by the system

$$R\,\hat{L}_1\,R\,\hat{L}_1 - \hat{L}_1\,R\,\hat{L}_1\,R = R\,\hat{L}_1 - \hat{L}_1\,R, \quad \hat{L} = \|\hat{l}_i^j\|.$$

Such an algebra will be denoted $\hat{\mathcal{L}}(R)$.
The generating matrix L of the algebra $\mathcal{L}(R)$ and that $\hat{L}$ of the algebra $\hat{\mathcal{L}}(R)$ are related by

$$L = I - (q - q^{-1})\hat{L}.$$

Thus, these algebras are isomorphic to each other. Recall that $q \neq \pm 1$.
Note that if a Hecke symmetry R tends to the usual flip P as $q \to 1$, the RE algebra tends to $Sym(gl(N))$, whereas its modified version tends to $U(gl(N))$. So, the above isomorphism fails as $q \to 1$.

Let (A, B, σ) be a QD and there exists a counit $\varepsilon : A \to \mathbb{C}$ coordinated with the algebraic structure of A in the following sense

$$\varepsilon(a\,b) = \varepsilon(a)\,\varepsilon(b), \;\; \varepsilon(1_A) = 1_{\mathbb{C}}.$$

Then it is possible to define an action of the algebra A on that B by setting

$$a \triangleright b := (Id \otimes \varepsilon_A)\sigma(a \otimes b), \; \forall\, a \in A, \; b \in B.$$

Typical examples of such QD are the so-called Heisenberg doubles composed of an RTT algebra and an RE one, associated with the same Hecke symmetry R. It is possible to define an action of the RE algebra (its elements are treated to be quantum vector fields) onto the corresponding RTT algebra (its elements are analogs of functions on $GL(N)$).

In 2011 we introduced another QD, where the RTT algebras were replaced by another copy of RE one. More precisely, we introduced a QD, such that $A = \hat{\mathcal{L}}(R)$, $B = \mathcal{L}(R)$, and the permutation relations

$$R\,\hat{L}_1\,R\,M_1 - M_1\,R\,\hat{L}_1\,R^{-1} = R\,M_1.$$

Here $\hat{L}$ (resp., M) is the generating matrix of A (resp., B). By introducing the matrix $D = M^{-1}\,\hat{L}$, we obtained a QD with the following defining relations

$$R\,M_1\,R\,M_1 = M_1\,R\,M_1\,R,$$

$$R^{-1}\,D_1\,R^{-1}\,D_1 = D_1\,R^{-1}\,D_1\,R^{-1},$$

$$D_1\,R\,M_1\,R = R\,M_1\,R^{-1}\,D_1 + R.$$

If we put $R = P$, the first system means that the entries m_i^j commute with each other, the second system means the same for the entries ∂_k^l and the third one leads to

$$\partial_k^l\, m_i^j = m_i^j\, \partial_k^l + \delta_k^j\, \delta_i^l, \text{ hence } \partial_k^l \triangleright m_i^j = \delta_k^j\, \delta_i^l,$$

i.e. the elements ∂_k^l are partial derivatives. Here we put $\varepsilon(\partial_i^j) = 0,\ \varepsilon(1_A) = 1_{\mathbb{C}}$.

The permutation relations in our approach play the role of the Leibniz rule. Thus, in order to apply an element ∂_i^j to a product $m_a^b\, m_c^d$ we have to permute the elements ∂_i^j and $m_a^b\, m_c^d$ and to apply the counit $\varepsilon(\partial_i^j) = 0, \varepsilon(1_A) = 1_{\mathbb{C}}$ to the factor from A.

The central relation between the generating matrices L, D and $\hat{L}$ is

$$\hat{L} = M\,D \Longleftrightarrow \hat{l}_i^j = m_i^k\,\partial_k^j,$$

which is similar to the classical one. The entries of the matrix D are called quantum partial derivatives. In order to get an analog of the operator W we have to apply the trace to the matrix $\hat{L} = M\,D$. The point is that this trace is not classical. It has to be coordinated with the initial Hecke symmetry R. We denote it Tr_R. It is defined by

$$\mathrm{Tr}_R\,A = \mathrm{Tr}\,C\,A,$$

where $C = C_R$ is a matrix determined by a given Hecke symmetry R.

Below, we impose two requirements on Hecke symmetries, we are dealing with: we assume R to be skew-invertible and even. Roughly speaking, a braiding R is called skew-invertible if for its the corresponding trace Tr_R can be defined.
Without going into detail, let us mention the main property of the corresponding R-trace Tr_R. In the space $\mathrm{End}(V)$ there exists a generalized Lie algebra with a bracket

$$[\,,\,]_R : \mathrm{End}(V)^{\otimes 2} \to \mathrm{End}(V),$$

such that the modified RE algebra has the meaning of its enveloping algebra. Then the following property is valid

$$\mathrm{Tr}_R\,[X,Y]_R = 0 \;\; \forall X, Y \in \mathrm{End}(V),$$

which is similar to the classical one.

Theorem

The elements $p_k(L) = \mathrm{Tr}_R\, L^k$ *(resp.,* $p_k(\hat{L}^k) = \mathrm{Tr}_R\, \hat{L}^k$*) are central in the algebra* $\mathcal{L}(R)$ *(resp.,* $\hat{\mathcal{L}}(R)$*).*

We call them *power sums*. They become Casimir operators while the algebra $\mathcal{L}(R)$ or $\hat{\mathcal{L}}(R)$ is represented somewhere.

So, our next aim is to construct a family of some representations of these algebras and perform the spectral analysis of the Casimir operators.
Afterwards, we'll define the normal ordering in our "braided setting" and define the corresponding analogs of the cut-and-join operators.

There exists a way to associate a central element of the algebra $\mathcal{L}(R)$ to any element of the Hecke algebra $H_k(q)$ (and even more, of the group algebra of the braid group). Recall that the Hecke algebra $H_k(q)$ is generated by elements 1, $\sigma_1 ... \sigma_{k-1}$ subject to the braid relations

$$\sigma_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\sigma_{i+1}$$

and the following ones

$$\sigma_i\,\sigma_j = \sigma_j\,\sigma_i \text{ if } |i-j| \geq 2,$$

$$(q\,1 - \sigma_i)(q^{-1}\,1 + \sigma_i) = 0.$$

Observe that for a generic q the representation theory of the Hecke algebra $H_k(q)$ is similar to that of the group algebra $\mathbb{C}[S_k]$ of the symmetric group.

In particular, for any partition $\lambda = (\lambda_1 \geq \lambda_1 \geq ... \geq \lambda_k)$ of the positive integer k and any standard tableau (denoted a) in the Hecke algebra $H_k(q)$ there exists a primitive idempotent $P_{(\lambda,a)} \in H_k(q)$.
Then the subspace $P_{(\lambda,a)} V^{\otimes k}$ defines just a representation of the RE algebra or the modified RE algebra, which is conjecturally irreducible.

Also, we define the following element of the algebra $\mathcal{L}(R)$ by

$$s_{(\lambda,a)}(M) = \mathrm{Tr}_{\,R(1\dots k)}\, P_{(\lambda,a)}(R)\, M_1 \dots M_{\overline{k}},$$

where

$$M_{\overline{1}} = M_1, \quad M_{\overline{k}} = R\, M_{\overline{k-1}}\, R^{-1}.$$

Observe that the elements $s_{(\lambda,a)}(M)$ does not depend on the tableau "a". So, we use for them the notation $s_\lambda(M)$. Also, note that this element is central in the algebra $\mathcal{L}(R)$.
These elements are called q-Schur (or simply Schur) polynomials. If λ is one-row (resp., one-column) diagram, we get *complete* (resp., *elementary*) *symmetric polynomials*. In this case the idempotents $P_{(\lambda,a)}$ are respectively R-symmetrizers and R-skew-symmetrizers.

Thus, the elementary symmetric polynomials are

$$e_k(M) = \mathrm{Tr}_{R(1\dots k)}(A^{(k)} M_1\, M_{\overline{2}} \dots M_{\overline{k}}), \quad k = 1, 2\dots$$

where $A^{(k)} : V^{\otimes k} \to V^{\otimes k}$ are skew-symmetrizers. They can be defined by recursion as follows

$$A^{(1)} = I, \;\; A^{(k)} = \frac{1}{k_q} A^{(k-1)} \left(q^{k-1} I - (k-1)_q\, R_{k-1} \right) A^{(k-1)},$$

$$k_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$

The power sums mentioned above can be also introduced in a similar way:

$$p_k(M) = \mathrm{Tr}_{R(12\dots k)}\, R_{k-1} \dots R_2\, R_1\, M_1\, M_{\overline{2}} \dots M_{\overline{k}} = \mathrm{Tr}_R\, M^k,$$

Definition

The operators, arising from all central elements of the algebra $\mathcal{L}(R)$ and that $\hat{\mathcal{L}}(R)$ are called q-Casimir (or simply, Casimir) ones.

Let us exhibit a few examples: $\mathrm{Tr}_R\, L$, $\mathrm{Tr}_R\, \hat{L}$, $\mathrm{Tr}_R\, L^2$, $\mathrm{Tr}_R\, \hat{L}^2$...

Our next aim is to perform the spectral analysis of some of the Casimir operators. In particular, we show that the q-Schur polynomials are eigenvectors of the Casimir operator $\mathrm{Tr}_R\, L$ and find the corresponding eigenvalue.

Proposition

The following holds

$$\mathrm{Tr}_R\, L \triangleright P_{(\lambda,a)}(R)\, M_1\, M_{\overline{2}} \ldots M_{\overline{k}} = \chi_\lambda(\mathrm{Tr}_R\, L) P_{(\lambda,a)}(R)\, M_1\, M_{\overline{2}} \ldots M_{\overline{k}},$$

where

$$\chi_\lambda(\mathrm{Tr}_R\, L) = \frac{m_q}{q^m} - \frac{\nu}{q^{2m}} \sum_{i=1}^{k} q^{-2\, c_i},$$

$\nu = q - q^{-1}$, *c_i is the content of the box containing the number i, and the sum is taken over all boxes of the tableau (λ, a).*

Recall that the content of a box is the difference of the numbers of the column and the row in which the box is located. Also, observe that in formula above the sum is taken over all boxes of the diagram λ and therefore it does not depend on the tableau.

This is the reason why instead of $\chi_{(\lambda,a)}$ we write χ_λ.

Corollary

The Schur polynomial $s_\lambda(M)$ is an eigenvector of the Casimir operator

$$\mathrm{Tr}_R\, L = e_1(L)$$

and the corresponding eigenvalue is $\chi_\lambda(\mathrm{Tr}_R\, L)$.

Definition

If the Schur polynomial s_λ is an eigenvector of a given Casimir operator $\mathcal{C}$

$$\mathcal{C}(s_\lambda(L)) = \chi_\lambda(\mathcal{C})\, s_\lambda(L),$$

then the corresponding eigenvalue $\chi_\lambda(\mathcal{C})$ is called λ- character of $\mathcal{C}$.

Proposition

λ-character of the Casimir element $e_1(L) = \mathrm{Tr}_R\, L$ can be cast under the following form

$$\chi_\lambda(e_1(L)) = \chi_\lambda(\mathrm{Tr}_R\, L) = q^{-1} \sum_{k=1}^{m} q^{-2(\lambda_k+(m-k))},$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$.

Now, by using the relation between the matrices L and $\hat{L}$, we compute λ-character of the first cut-and-join operator

$$W^1 = \mathrm{Tr}_R\, \hat{L}.$$

(For this cut-and-join operator we do not need to apply the normal ordering, so, it is equal to the first Casimir operator.)

Proposition

The following holds

$$W^1(s_\lambda(M)) = \chi_\lambda(\mathrm{Tr}_R\, \hat{L})\, s_\lambda(M),$$

where

$$\chi_\lambda(\mathrm{Tr}_R\, \hat{L}) = \frac{1}{q^{2m}} \sum_{i=1}^{m} \frac{1 - q^{-2\,(\lambda_k + (m-k))}}{q - q^{-1}}.$$

It is known that the generating matrix L of the algebra $\mathcal{L}(R)$ is subject to the following form of the Cayley-Hamilton (CH) identity

$$L^m - q\, e_1(L)\, L^{m-1} + q^2\, e_2(L)\, L^{m-2} \dots$$
$$+(-q)^{m-1}\, e_{m-1}(L)\, L + (-q)^m\, e_m(L)\, I = 0, \qquad (2)$$

where m is the rank of the Hecke symmetry R. The corresponding polynomial, called *characteristic* and meeting the relation $Q(L) = 0$, is

$$Q(t) = \sum_{k=0}^{m} t^{m-k}(-q)^k\, e_k(L).$$

Its roots $\{\mu_i\}_{1\le i\le m}$ are called the *eigenvalues* of the matrix L. They are elements of a central extension of the algebra $\mathcal{L}(R)$.

Thus, we have

$$q^k\, e_k(L) = \sum_{1\le i_1<\cdots<i_k\le m} \mu_{i_1}\ldots\mu_{i_k}. \tag{3}$$

In particular, $\mathrm{Tr}_R\, L = e_1(L) = q^{-1}\, \sum_{k=1}^m\, \mu_k$.
It is tempting to assign to the eigenvalues μ_i the values denoted $\chi_\lambda(\mu_i)$ and called λ-characters of the eigenvalues, coordinated with the relations (3).

Conjecture

The assignment $\chi_\lambda(\mu_i) = q^{-2(\lambda_i+(m-i))}$ is coordinated with the relations (3), i.e.

$$\chi_\lambda(e_k(L)) = q^{-k} \sum_{1\le i_1<\ldots<i_k\le m} \chi_\lambda(\mu_{i_1}).$$

Note that the matrix $\hat{L}$ is also subject to a version of the CH identity. Let $\hat{\mu}_i$ be the eigenvalues of the corresponding characteristic polynomial. They are related to the eigenvalues of the matrix L by the formula

$$\mu_i = 1 - \nu\hat{\mu}_i.$$

Thus, we get

$$\chi_\lambda(\hat{\mu}_k) = \frac{1 - q^{-2\,(\lambda_k + (m-k))}}{q - q^{-1}}, \; k = 1 \dots m.$$

By assuming that $R \to P$ as $q \to 1$ we pass to this limit and obtain λ-characters of the eigenvalues $\hat{\mu}_k$ of the matrix $\hat{L}$ generating $U(gl(N))$:

$$\chi_\lambda(\hat{\mu}_k) = \lambda_k + (m - k),\ k = 1 \dots N.$$

Note that this result is also valid for the generating matrix $\hat{L}$ of the modified RE algebra, associated with any involutive symmetry R, provided it is the $q = 1$ limit of the admissible Hecke symmetries.

Now, we pass to defining the higher cut-and-join operators W^{Δ}. To this end we proceed similarly to the classical case. However, we have to define normal ordering in our "braided context".
Thus, we have to push the factors D through the factors L to the right positions. To this end we use the following permutation relations

$$D_1\, R\, L_1\, R = R\, L_1\, R^{-1}\, D_1.$$

These permutation relations are obtained from those above by omitting the last (constant) term. This is a way of proceeding similar to the super-case.

Let us consider two examples. Namely, we compute the operators W^{Δ} for $\Delta = (2, 0 \dots 0)$ and $\Delta = (1, 1, 0 \dots 0)$.

Example

$$: \mathrm{Tr}_R \, \hat{L}^2 : = : \mathrm{Tr}_{\,R(12)}(L_1 \, \underline{D_1 \, R_1 \, L_1} \, D_1) :$$

$$= \mathrm{Tr}_{\,R(12)}(L_1 \, L_{\overline{2}} \, D_{\overline{2}} \, D_1 \, R_1^{-1}).$$

A straightforward computing leads to the following formula

$$: \mathrm{Tr}_R \, \hat{L}^2 : = \mathrm{Tr}_R \, \hat{L}^2 \, - \frac{m_q}{q^m} \, \mathrm{Tr}_R \, \hat{L}.$$

Example

$$:(\mathrm{Tr}_R\,\hat{L})^2:=:\mathrm{Tr}_{R(12)}(L_1\,\underline{D_1\,R_1\,L_1}\,D_1\,R_1^{-1}):=$$

$$\mathrm{Tr}_{R(12)}(L_1\,R_1\,L_1\,R_1^{-1}\,D_1\,R_1^{-1}\,D_1\,R_1^{-1})=\mathrm{Tr}_{R(12)}(L_1\,L_{\overline{2}}\,D_{\overline{2}}\,D_1\,R_1^{-2}).$$

Also, we get the following relation

$$:(\mathrm{Tr}_R\,\hat{L})^2:=\;(\mathrm{Tr}_R\,\hat{L})^2\;-q^{-2m}\,\mathrm{Tr}_R\,\hat{L}.$$

However, as we said above, the cut-and join operators (more precisely, their restrictions on the symmetric polynomials) are usually expressed via the power sums. Being presented in this form, these operators are differential ones. (The simplest one is exhibited above.)

In our setting it is also possible to compute the action of the q-Casimir and q-cut-and-join operators on the power sums $p_k(M)$. However, this action cannot be represented under a form of differential operators. Their explicit form is an open problem.

In order to compute the action of any operator under consideration onto the power sums $p_k(M)$ it is necessary to know the relation between symmetric polynomials $s_\lambda(M)$ and $p_k(M)$ in our setting, i.e. under assumption that M is the generating matrix of a RE algebra. First, observe that for our Schur polynomials the Littlewood-Richardson formula

$$s_\lambda(M)s_\mu(M) = \sum c^\nu_{\lambda,\mu} s_\nu(M)$$

with the classical coefficients is valid.

However, their relations with the power sums are deformed. q-analogs of the Wronski relations are

$$(-1)^k k_q e_k(M) + \sum_{r=0}^{k-1} (-q)^r e_r(M)\, p_{k-r}(M) = 0,$$

$$k_q h_k(M) - \sum_{r=0}^{k-1} q^{-r} h_r(M)\, p_{k-r}(M) = 0.$$

Nevertheless, the classical Newton identities are still valid

$$\sum_{r=0}^{k} (-1)^r e_r(M) h_{k-r}(M) = 0.$$

Now, we pass to the second topic, namely, to the Capelli identity. The classical version of this identity is as follows

$$rDet(\hat{L}+K) = detL\, detD,$$

where K is the diagonal matrix $diag(0, 1, ..., n-1)$ and $rDet$ is the so-called row-determinant.
Observe that the term $rDet(\hat{L}+K)$ in the l.h.s. can be cast under the following form

$$Tr_{1..N}A^{(N)}\hat{L}_1\,(\hat{L}+I)_2\,(\hat{L}+2I)_3...(\hat{L}+(N-1)I)_N.$$

Our next aim is to exhibit a q-analog of the Capelli identity. Conjecturally, it has the form

$$\mathrm{Tr}_{R(1\dots m)} A^{(m)} L_1 (L_{\overline{2}} + q\, I)(L_{\overline{3}} + q^2\, 2_q\, I)\dots(L_{\overline{m}} + q^{m-1}\,(m-1)_q\, I) =$$

$$q^{m(m-1)}\, det_R\, L\, det_{R^{-1}}\, D.$$

Here m is the rank of R. (Note that in the classical case $m = N$.) Whereas the determinants in the r.h.s. are defined by the formulae

$$det_R\, L = \mathrm{Tr}_{R(1\dots m)} A^{(m)} L_1\, L_{\overline{2}} \dots L_{\overline{m}},$$

$$det_{R^{-1}}\, D = \mathrm{Tr}_{R(1\dots m)} A^{(m)}\, D_{\overline{m}}\, D_{\overline{m-1}} \dots D_1.$$

In this connection I want to put the following question: when is it possible to write down the determinants entering the Capelli identity (or its q-analog) under the form of row- or column-determinant?

The answer is the following: in general it is not possible.

However, for any even skew-invertible Hecke symmetry R, the quantum determinant for the generating matrix of the corresponding (non-modified) RE algebra can be defined as explained above.

Moreover, if R is a deformation of the usual flip P (for instance, it comes from $U_q(sl(N))$ or it is a Crammer-Gervais R-matrix), the quantum determinant can be cast under the form of either row- or column- determinant.

Also, the answer is positive if $\hat{L}$ is the generating matrix of $U(gl(N))$.

By concluding the talk, I want to observe the following.

The first attempt of introducing a q-analog of the Capelli identity was undertaken by Noumi, Umeda, Wakayama in 1994. Their construction is related to the RTT algebra. Though R is the standard Hecke symmetry, i.e. it comes from the QG $U_q(sl(N))$, their Capelli identity is not a deformation of the classical one. Whereas ours tends to the classical one while $R \to P$.

In 1996 A.Okounkov introduced the notion of quantum immanants. Our technique enables us to introduce q-analogs of these objects.

Many thanks