GEOMETRY, MATTER AND PHYSICS

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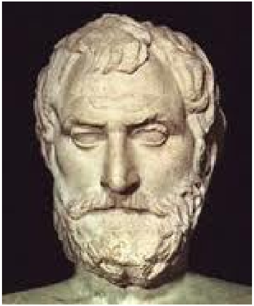
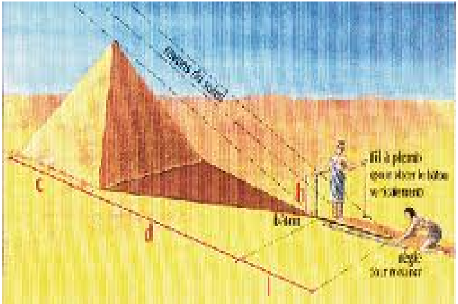
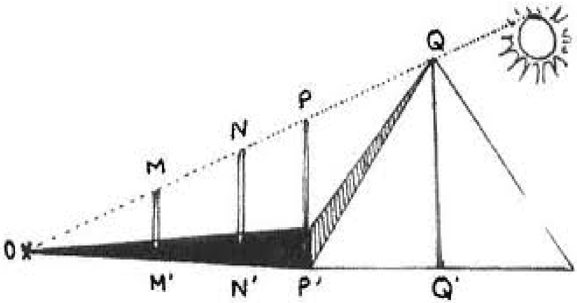
Geometry, Matter and Light





Thales of Miletus ($-625 \sim -547$) became famous for his prediction of solar eclipse of -585, and of his ability to evaluate dimensions of objects at a distance, by comparing their shadows.

Geometry, Matter and Light



MM' : OM = QQ' : OQ

Geometry and Reality

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- It has become such a commonplace, that the physical aspects of this fundamental experiment are rarely considered in a more detailed manner. In fact, Thales has performed an important physical experiment relating different definitions of geometry
- Let us analyze the premices and hypotheses that enabled Thales to draw his conclusions and to state the theorem of parallel lines cutting an angle formed by two non-parallel straight lines.

Geometry and Reality

- **The first two assumptions are that the line OQ' on the ground is indeed a straight line, and that the two segments, the height of the Pyramid QQ' and the stick MM' are also straight, and form the same angle with the line of the ground $OM'Q'$ (in this case, the straight angle of 90^o).**

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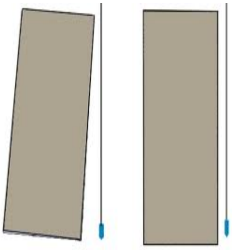
- **This is a physical statement, and the fact that the two objects are straight and vertical was checked using of the well known instruments based on the exploitation of gravity.**

Geometry and Material World



Two instruments used for checking whether a straight line or a plane is vertical or horizontal. Both are based on the use of terrestrial gravity defining local vertical direction and horizontal planes (equipotential surfaces).

Geometry, Matter and Light

- **The fact that the two segments are vertical and straight is based on the assumption that the string sustaining a heavy object in the gravitational field on the surface of Earth may serve as a definition of *vertical straight line*. Checking the horizontality of the ground is performed using the same principle.**

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- **The fact that the stick remains straight and stiff is due to the assumption that it is made of a material whose cohesion is sufficient to keep its shape unchanged (a solid body).**
- **As seen from our present perspective, this hypothesis is based on the assumption that atoms can form stable structures able to keep unchanged under reasonable conditions (e.g. the ambient temperature not exceeding certain values).**

Incidentally, the ability of atoms and molecules to form stable periodic structures makes possible an alternative definition of straight lines and right (and not only right) angles. Crystals represented in Fig. (1) show remarkable linear structure as well as apparently perfect angles, 90^o in the case of cubic lattice of $NaCl$, and 60^o and 120^o in the case of quartz (SiO_2).

Figure: **Crystals of ordinary salt $NaCl$, of quartz SiO_2, and an example of crystalline lattice (SiO_2 - wurtzite). The interatomic forces impose the shapes and the geometry of solid bodies.**

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- The geometry based on solid bodies' shapes is independent of gravitational field that determines parallel vertical lines and the horizontal plane in Thales' experiment. From the present point of view, the existence of stable configurations of atoms, as well as that of atoms themselves, can be understood only using the principles of quantum mechanics, until now seemingly independent of gravitational phenomena.

- But this is not the end of the story. A third type of straight line is involved in the experiment, the light ray along the line $QPNMO$. The character of this line is due to the properties of electromagnetic waves' propagation in vacuo (as far as the influence of air can be neglected), which *a priori* is independent of gravitational phenomena as well as of the forces predominant on the atomic level.

- **But this is not the end of the story. A third type of straight line is involved in the experiment, the light ray along the line** $QPNMO$**. The character of this line is due to the properties of electromagnetic waves' propagation in vacuo (as far as the influence of air can be neglected), which** *a priori* **is independent of gravitational phenomena as well as of the forces predominant on the atomic level.**
- **Light wavefronts and rays set forth an alternative notion of straight lines and angles, resulting in** *conformal geometry*, **which preserves the notions of straight lines and angles, but ignores the notions of length and distance.**

Geometry, Matter and Light

- **The results of Thales' experiment can be interpreted in two ways. In fact, he established the coincidence of three completely different definitions of a straight line. The first came from the natural shape a string with a heavy body attached at its end takes in the gravitational field.**

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- The second definition comes from the material shape of the stick. The third is just a light ray.
- With the height of the pyramid considered as a known quantity, as well as the height of the stick, the result of Thales can be interpreted as a proof that the light rays follow straight lines, too.

Of matter, time and space

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  Since the advent of modern physics, the description of the world surrounding us is based on three essential realms, already present in the Thales" experiment, which are
- **Space and time**
- **Material bodies**
- **Forces acting between them**

Fundamental relationship

- **Newton's third law of dynamics**

$$\mathbf{a} = \frac{1}{m}\,\mathbf{F}. \tag{1}$$

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- **shows the relation between three different realms which are dominant in our perception and description of physical world: massive bodies (m), force fields responsible for interactions between the bodies ("F") and space-time relations defining the acceleration ("a").**

Three realms

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- we deliberately wrote

$$\mathbf{a} = \frac{1}{m}\,\mathbf{F}. \tag{2}$$

in order to separate the directly observable entity (a) from the product of two entities whose definition is much less direct and clear.

Causal relationship

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- Also, by putting the acceleration alone on the left-hand side, we underline the causal relationship between the phenomena: the force is the cause of acceleration, and not vice versa.
- In modern language, the notion of force is generally replaced by that of a field.
- The fact that the three ingredients are related by the equation (1) may suggest that perhaps only two of them are fundamentally independent, the third one being the consequence of the remaining two.

Three aspects

The three aspects of theories of fundamental interactions can be symbolized by three orthogonal axes, as shown in following figure, which displays also three choices of pairs of independent properties from which we are supposed to be able to derive the third one.

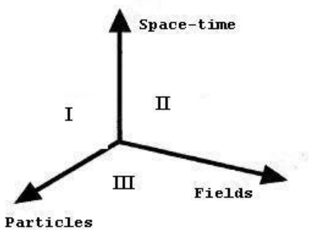

Figure: The three realms of Physics

Three types of theories

- **The attempts to understand physics with only two realms out of three represented in (17) have a very long history. They may be divided in three categories, labeled *I*, *II* and *III* in the Figure.**

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- The attempts to understand physics with only two realms out of three represented in (17) have a very long history. They may be divided in three categories, labeled *I*, *II* and *III* in the Figure.
- In the category *I* one can easily recognize Newtonian physics, presenting physical world as collection of material bodies (particles) evolving in absolute space and time, interacting at a distance. Newton considered light being made of tiny particles, too; the notion of fields was totally absent.

Three types of theories

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- Theories belonging to the category *II* assume that physical world can be described uniquely as a collection of fields evolving in space-time manifold. This approach was advocated by lord Kelvin, Einstein, and later on by Wheeler.
- As a follower of Maxwell and Faraday, Einstein believed in the primary role of fields and tried to derive the equations of motion as characteristic behavior of singularities of fields evolving in space-time.

Three types of theories

- The category *III* represents an alternative point of view supposing that the existence of matter is primary with respect to that of the space-time, which becomes an "emergent" realm - an euphemism for "illusion". Such an approach was advocated recently by N. Seiberg and E. Verlinde.

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- The category *III* represents an alternative point of view supposing that the existence of matter is primary with respect to that of the space-time, which becomes an "emergent" realm - an euphemism for "illusion". Such an approach was advocated recently by N. Seiberg and E. Verlinde.
- It is true that space-time coordinates cannot be treated on the same footing as conserved quantities such as energy and momentum; we often forget that they exist rather as bookkeeping devices, and treating them as real objects is a "bad habit", as pointed out by D. Mermin.

Three types of theories

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- Many of those properties do not require any mention of space and time on the quantum mechanical level, as was demonstrated by Born and Heisenberg in their version of matrix mechanics, or by von Neumann's formulation of quantum theory in terms of the C^* algebras.
- The non-commutative geometry is another example of formulation of space-time relationships in purely algebraic terms.

Three types of theories

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- **In other words, the Lorentz symmetry observed on the macroscopic level, acting on what we perceive as space-time variables, is an averaged version of the symmetry group acting in the Hilbert space of quantum states of fundamental particle systems.**

The Lorentz covariance

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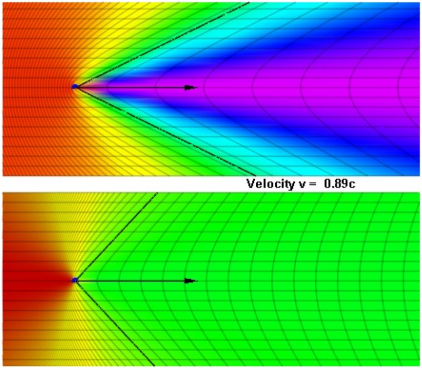
- Consequently, with high enough precision one can infer that the Doppler effect is relativistic, i.e. the frequency ω and the wave vector $\mathbf{k}$ form an entity that is seen differently by different inertial observers, and passing from $\frac{\omega}{c}, \mathbf{k}$ to $\frac{\omega'}{c}, \mathbf{k}'$ is the Lorentz transformation.

The Lorentz covariance



Relativistic versus Galilean Doppler effect.

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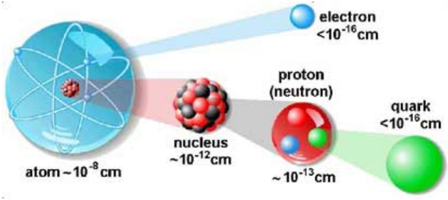
- All acts of observation to which we have access reduce at the end of the experimental chain to **PHOTONS** interacting with **ELECTRONS** (or other leptons, or baryons).
- No wonder that the dual space to the space of four-vectors like $k^\mu = [\frac{\omega}{c}, \mathbf{k}]$ or $j^\mu = [\rho c, \mathbf{j}]$, i.e. the space of linear functionals on it, inherits the same symmetry group.

Of Matter and Space-Time

According to the present knowledge, the ultimate undivisible and undestructible constituents of matter, called atoms by ancient Greeks, are in fact the so-called QUARKS, carrying fractional electric charges and baryonic numbers, the two features that appear to be undestructible and conserved under any circumstances.



The ultimate constituents of matter

- **All baryons, e.g. protons and neutrons, are composed of quarks, which cannot be isolated and unobservable in a free and unbound state. Also the mesons which convey the strong interactions between the baryons, are composed of quark-antiquark pairs.**

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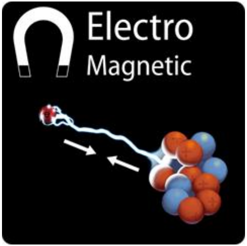
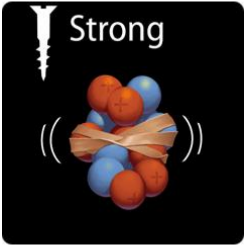
- **As for today, we do not know more fundamental constituents of matter than quarks. It is also amazing that the deep inelastic scattering experiments reveal point-like objects inside the proton. It seems reasonable to think that they are at least as stable as the proton itself.**

Introduction: Fundamental interactions

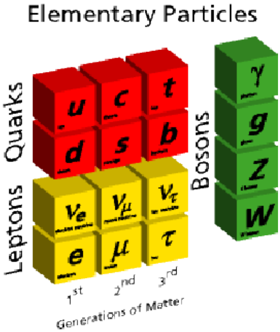
There are three fundamental gauge fields in nature:

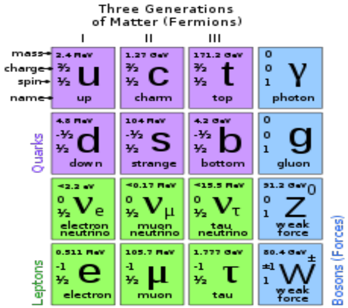




Introduction: Quarks and Leptons

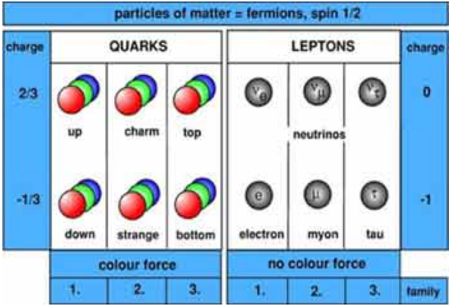
The carriers of elementary charges also go by packs of three: three families of quarks, and three types of leptons.



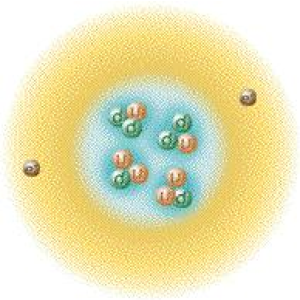


Introduction: Three colors

There is an additional 3-symmetry: the three colors.



The three colors are needed to combine three quarks into a hadron



A schematic representation of a Helium atom.
Each nucleon contains three quarks, of three different colors.

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- **As a result, the wave functions in the Schroedinger picture were required to belong to one of the linear representations of the Lorentz group, which means that they must satisfy the following covariance principle:**

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(\Lambda(x)) = S(\Lambda)\,\psi(x).$$

Quantum covariance

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Quantum covariance

- The nature of the representation $S(\Lambda)$ determines the character of the field considered: spinorial, vectorial, tensorial...
- As in many other fundamental relations, the seemingly simple equation
$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(\Lambda(x)) = S(\Lambda)\,\psi(x).$$
creates a bridge between two totally different realms: the space-time accessible via classical macroscopic observations, and the Hilbert space of quantum states. It can be interpreted in two opposite ways, depending on which side we consider as the cause, and which one as the consequence.

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- or maybe it is already present as symmetry of quantum states, and then implemented and extended to the macroscopic world in classical limit ? In such a case, the covariance principle should be written as follows:
- $$\Lambda^{\mu'}_{\mu}(S) j^{\mu} = j^{\mu'}(\psi') = j^{\mu'}(S(\psi)),$$

  In the above formula
  $$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$$
  is the Dirac current, ψ is the electron wave function.

- **In view of the analysis of the causal chain, it seems more appropriate to write the same transformations with Λ depending on S:**

$$\psi'(x^{\mu'}) = \psi'(\Lambda^{\mu'}_{\nu}(S)x^{\nu}) = S\psi(x^{\nu}) \tag{3}$$

$$\Lambda^{\mu'}_{\mu}(S)x^{\mu}(\psi,\bar{\psi}) = x^{\mu'}(S\psi,\bar{\psi}\bar{S}). \tag{4}$$

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- This form of the same relation suggests that the transition from one quantum state to another, represented by the unitary transformation S is the primary cause that implies the transformation of observed quantities such as the electric 4-current, and as a consequence, the apparent transformations of time and space intervals measured with classical physical devices.

Quantum covariance

- **Although mathematically the two formulations are equivalent, it seems more plausible that the Lorentz group resulting from the averaging of the action of the $SL(2, \mathbf{C})$ in the Hilbert space of states contains less information than the original double-valued representation which is a consequence of the particle-anti-particle symmetry, than the other way round.**

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- **In what follows, we shall draw physical consequences from this approach, concerning the strong interactions in the first place.**

First principles

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- **On a classical level, when theory permits, the symbolical force can be replaced by a more explicit expression in which fields responsible for the forces do appear, like in the case of the Lorentz force.**

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- But the fields acting on a test particle are usually generated by more or less distant charges and currents, according to the formula giving the retarded four-potential $A_\mu(x^\lambda)$:
- $$A_\mu(\mathbf{r},t) = \frac{1}{4\pi c} \int\int\int \frac{j_\mu(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\ \mathbf{r} - \mathbf{r}'\ |} d^3\mathbf{r}'. \qquad (5)$$

  then we get the field tensor given by

  $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

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- **with** $\psi^\dagger = \bar{\psi}^T \gamma^5, \quad \text{where}$

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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- which are supposed to transform under two non-equivalent representations of the $SL(2, \mathbf{C})$ group:

$$\xi_{\alpha'} = S^{\alpha}_{\alpha'} \xi_\alpha, \quad \chi_{\dot{\beta}'} = S^{\dot{\beta}}_{\dot{\beta}'} \chi_{\dot{\beta}'}, \tag{7}$$

- **The electric charge conservation is equivalent to the annulation of the four-divergence of j^μ:**

$$\partial_\mu j^\mu = \left(\partial_\mu \psi^\dagger \gamma^\mu\right) \psi + \psi^\dagger \left(\gamma^\mu \partial_\mu \psi\right) = 0, \tag{8}$$

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- from which we infer that this condition will be satisfied if we have

$$\partial_\mu \psi^\dagger \gamma^\mu = -m\psi^\dagger \;\text{ and }\; \gamma^\mu \partial_\mu \psi = m\psi, \tag{9}$$

which is the Dirac equation.

- **In terms of the spinorial components ξ and χ the Dirac equation can be seen as a pair of two coupled equations which can be written in terms of Pauli's σ-matrices:**

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-

$$\left(-i\hbar \frac{1}{c}\frac{\partial}{\partial t} - mc\right)\chi = i\hbar\sigma \cdot \nabla\xi. \tag{10}$$

- **The relativistic invariance imposed on this equation is usually presented as follows: under a Lorentz transformation Λ the 4-current j^μ undergoes the following change:**

$$j^\mu \to j^{\mu'} = \Lambda^{\mu'}_{\mu} j^\mu. \tag{11}$$

- **This means that the matrices γ^{μ} must transform as components of a 4-vector, too. Parallelly, the components of the bi-spinor ψ must be transformed in a way such as to leave the form of the Dirac equations unchanged: writing symbolically the transformation of $\mid \psi >$ as $\mid \psi' >= S \mid \psi >$, and $< \psi' \mid = < \psi \mid S^{-1}$, we should have**

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-
$$j^{\mu'} = < \psi' \mid \gamma^{\mu'} \mid \psi' > = < \psi \mid S^{-1} \gamma^{\mu'} S \mid \psi > =$$
$$\Lambda^{\mu'}_{\mu} j^{\mu} = \Lambda^{\mu'}_{\mu} < \psi \mid \gamma^{\mu} \mid \psi > \tag{12}$$

from which we infer the transformation rules for gamma-matrices:

$$S^{-1} \gamma^{\mu'} S = \Lambda^{\mu'}_{\mu} \gamma^{\mu}. \tag{13}$$

Pauli's exclusion principle

- **The exclusion principle for the fermions is one of the most important facts of the elementary particle physics. In its first formulation in** 1924 **by Wolfgang Pauli only the states of the electrons in atomic shells were concerned: two electrons with the same quantum numbers can coexist only if their extra quantum number related to the proper spin of the electron, takes on different values out of the only two accessible ones.**

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- Not only does it explain the structure of atoms and the periodic table of elements, but it also guarantees the stability of matter preventing its collapse, as suggested by Ehrenfest, and proved later by Dyson in his seminal papers.

Pauli's exclusion principle

- **In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states.**

Pauli's exclusion principle

- **In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states.**
- **The easiest way to see how the principle works is to apply Dirac's formalism in which wave functions of particles in given state are obtained as products between the "bra" and "ket" vectors.**

Pauli's exclusion principle

Consider the wave function of a particle in the state $| \, x >$,

$$\Phi(x) = < \psi \, | \, x > . \tag{14}$$

A two-particle state of $(| \, x >, | \, y >)$ is a tensor product

$$| \, \psi > = \sum \Phi(x, y) \, (| \, x > \otimes \, | \, y >). \tag{15}$$

If the wave function $\Phi(x, y)$ is anti-symmetric, i.e. if it satisfies

$$\Phi(x, y) = -\Phi(y, x), \tag{16}$$

then $\Phi(x, x) = 0$ and such states have vanishing probability.

Pauli's exclusion principle

Conversely, suppose that $\Phi(x, x)$ does vanish. This remains valid in any basis provided the new basis $\mid x' >, \mid y' >$ was obtained from the former one via unitary transformation.
Let us form an arbitrary state being a linear combination of $\mid x >$ and $\mid y >$,

$$\mid z >= \alpha \mid x > + \beta \mid y >, \quad \alpha, \beta \in \mathbf{C},$$

and let us form the wave function of a tensor product of such a state with itself:

$$\Phi(z, z) = < \psi \mid (\alpha \mid x > + \beta \mid y >) \otimes (\alpha \mid x > + \beta \mid y >), \tag{17}$$

Pauli's exclusion principle

- **which develops as follows:**

$$\alpha^2 \ < \psi \mid x, x > + \alpha\beta < \psi \mid x, y >$$

$$+\beta\alpha \ < \psi \mid y, x > + \beta^2 \ < \psi \mid y, y > =$$

$$= \alpha^2 \, \Phi(x, x) + \alpha\beta \, \Phi(x, y) + \beta\alpha \, \Phi(y, x) + \beta^2 \, \Phi(y, y). \quad (18)$$

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$$= \alpha^2 \, \Phi(x, x) + \alpha\beta \, \Phi(x, y) + \beta\alpha \, \Phi(y, x) + \beta^2 \, \Phi(y, y). \quad (18)$$

- **Now, as $\Phi(x, x) = 0$ and $\Phi(y, y) = 0$, the sum of remaining two terms will vanish if and only if (16) is satisfied, i.e. if $\Phi(x, y)$ is anti-symmetric in its two arguments.**

Pauli's exclusion principle

- **After second quantization, when the states are obtained with creation and annihilation operators acting on the vacuum, the anti-symmetry is encoded in the anti-commutation relations**

$$\psi(x)\psi(y) + \psi(y)\psi(x) = 0 \tag{19}$$

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where $\psi(x) \mid 0 >=\mid x >$.

- **This anti-symmetry is so fundamental that it should be also Lorentz-invariant, or rather act as a Lorentz symmetry generating principle.**

Quantum covariance

- **The Pauli exclusion principle gives a hint about how it might work. In its simplest version, it introduces an anti-symmetric form on the Hilbert space describing electron's states:**

$$\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \quad \alpha, \beta = 1, 2; \quad \epsilon^{12} = 1,$$

Quantum covariance

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- Now, if we require that Pauli's principle must apply independently of the choice of a basis in Hilbert space, i.e. that after a linear transformation we get

$$\epsilon^{\alpha'\beta'} = S^{\alpha'}_{\alpha} S^{\beta'}_{\beta} \epsilon^{\alpha\beta} = -\epsilon^{\beta'\alpha'}, \quad \epsilon^{1'2'} = 1,$$

More explicitly,

$$\epsilon^{\alpha'\beta'} = S^{\alpha'}_{\alpha} S^{\beta'}_{\beta} \epsilon^{\alpha\beta} = -\epsilon^{\beta'\alpha'}, \quad \epsilon^{1'2'} = 1,$$

yields

$$S^{1'}_{1} S^{2'}_{2} - S^{1'}_{2} S^{2'}_{1} = 1$$

which means that the matrix $S^{\alpha'}_{\alpha}$ must have the determinant equal to 1, which defines the $SL(2, \mathbf{C})$ group.

- **Its real dimension is 6 (an arbitrary complex 2×2 matrix depends on four complex parameters, or equivalently, on 8 real partameters; imposing that the determinant must be equal to 1 is equivalent to two real equations, the imaginary part of the determinant being equal to 0, and this leaves just 6 free real parameters.**

- **Its real dimension is** 6 **(an arbitrary complex** 2×2 **matrix depends on four complex parameters, or equivalently, on** 8 **real partameters; imposing that the determinant must be equal to** 1 **is equivalent to two real equations, the imaginary part of the determinant being equal to** 0**, and this leaves just** 6 **free real parameters.**
- **It is also easy to prove that the Lie algebras of both** $SL(2, \mathbf{C})$ **and the Lorentz group do coincide, both satisfying commutation relations defined by the formulae (22).**

The spinor realization of *six* $D = 4$ **Lorentz algebra generators is often represented by two sets containing three generators each, the generators of three independent spatial rotations** J_i **and the three generators of the Lorentz "boosts"** K_l **involving the space-time transformations along the three space axes, defined as follows:**

$$J_i = \frac{i}{2}\,\epsilon_{ijk}\left[\gamma^j, \gamma^k\right], \quad K_l = \frac{1}{2}\,\left[\gamma_l, \gamma_0\right] \tag{20}$$

. **The explicit form in terms of tensorial products of** 2×2 **matrices is then as follows:**

$$J_l = -\frac{i}{2}\mathbb{1}_2 \otimes \sigma_l, \quad K_i = -\frac{1}{2}\,\sigma_1 \otimes \sigma_i. \tag{21}$$

The full set of relations defining the Lorentz algebra is given by the following equations:

$$[J_i, J_k] = \epsilon_{ikl} J_l, \quad [J_i, K_k] = \epsilon_{ikl} K_l,$$
$$[K_i, K_k] = -\epsilon_{ikl} J_l. \tag{22}$$

- **The notion of *isospin* was introduced as early as in 1932 by W. Heisenberg who noticed that at high energies when the strong nuclear interacions prevail, proton and neutron behave almost as two states of the same particle (*"the nucleon"* as far as the electromagnetic forces, much weaker that the nuclear ones, can be neglected.**

- **The notion of *isospin* was introduced as early as in 1932 by W. Heisenberg who noticed that at high energies when the strong nuclear interacions prevail, proton and neutron behave almost as two states of the same particle (** *"the nucleon"* **as far as the electromagnetic forces, much weaker that the nuclear ones, can be neglected.**
- **To acknowledge the new discrete degree of freedom, taking on two values only, one needs a two-component wave function, just like in the case of the half-integer spin of the electron.**

- **In the high energy limit, when neutron and proton become undistinguishable, their state vectors in the Hilbert space of states can be linearly superposed under the condition that the resulting state vector is normalized to 1:**

$$| p' >=| \alpha | p > +\beta | n >|= 1, \; < p' | p' >= 1 \text{ implies } \bar{\alpha}\alpha + \bar{\beta}\beta = 1, \tag{23}$$

- **In the high energy limit, when neutron and proton become undistinguishable, their state vectors in the Hilbert space of states can be linearly superposed under the condition that the resulting state vector is normalized to 1:**

$$\mid p' >=\mid \alpha \mid p > +\beta \mid n >\mid = 1, \; < p' \mid p' >= 1 \text{ implies } \bar{\alpha}\alpha + \bar{\beta}\beta = 1, \tag{23}$$

- **Such transformations form a group which can be represented by matrices of the following special type:**

$$M = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \text{ with } \det M = 1. \tag{24}$$

This real three-dimensional Lie group is called $SU(2)$, and is topologically isomorphic with a 3-sphere.

The Lie algebra of the $SU(2)$ **group** is isomorphic with the Lie algebra $SO(3)$ of three-dimensional Euclidean rotations. Its lowest-dimensional representation is given by the following three 2×2 **matrices**:

$$I_1 = \frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_2 = \frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad I_3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

The two states of the nucleon form a basis of the lowest-dimensional spin $\frac{1}{2}$ **representation** of the $SU(2)$ **group**. Two commuting operators of the $SU(2)$ Lie algebra are I^2 and I_3, **the Casimir operator**, and the third component of the isospin vector.

In the first model of isospin symmetry proposed by Heisenberg, the two states of a nucleon, proton and neutron, are represented by two "iso-spinors", the two-component columns as follows:

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{26}$$

The third component of isospin I_3 acting on basic vectors produce the eigenvalues $+\frac{1}{2}$ for the proton and $-\frac{1}{2}$ for the neutron. A simple rule for the electric charge follows,

$$Q = I_3 + \frac{1}{2}, \tag{27}$$

attributing the electric charge $+1$ to the proton, and 0 to the neutron state, respectively.

It was also conjectured by Yukawa that strong interactions must be mediated by lighter particles called *mesons*, belonging to a three-dimensional representation of the $SU(2)$ group, intertwining between two isospinors belonging to the half-integer representation. The three intermediary particles, called later the π-mesons, can be represented by the following three-component columns, spanning the integer isopin $= 1$ representation:

$$\pi^+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \pi^+ = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \pi^+ = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{28}$$

- The eigenvalues of the I_3 **operator** on these states are $+1$, 0, and -1, respectively, coinciding with their electric charges. Therefore, the **isospin** 1 **pion states** satisfy the simple relation between the eigenvalues of electric charge and the third component of the isospin, $Q = I_3$. To produce a single formula valid for nucleons and pions alike, a new conserved quantity has to be introduced: the *hypercharge* Y.

- The eigenvalues of the I_3 **operator** on these states are $+1$, 0, and -1, respectively, coinciding with their electric charges. Therefore, the **isospin** 1 **pion states** satisfy the simple relation between the eigenvalues of electric charge and the third component of the isospin, $Q = I_3$. To produce a single formula valid for nucleons and pions alike, a new conserved quantity has to be introduced: the *hypercharge* Y.
- Called also the *baryon number*, its eigenvalue is 1 for strongly interacting fermions (nucleons and hyperons discovered later), and 0 for mesons. As a result, one gets the **Gell-Mann-Okubo formula** relating the three quantum numbers characterizing strongly interacting particles:

$$Q = I_3 + \frac{Y}{2}. \tag{29}$$

In Quantum Chromodynamics quarks are considered as fermions, endowed with spin $\frac{1}{2}$. Only *three* quarks or anti-quarks can coexist inside a fermionic baryon (respectively, anti-baryon), and a pair quark-antiquark can form a meson with integer spin.

Besides, they must belong to different *colors*, also a three-valued set. There are two quarks in the first generation, u and d ("up" and "down"), which may be considered as two states of a more general object, just like proton and neutron in $SU(2)$ symmetry are two isospin components of a nucleon doublet.

According to the QCD model, in stable bound state there is place for *two* quarks in the same u-state or d-state, but not three.

Quarks display quite unusual quatum numbers. Conservation and additivity of the baryon number lead to the conclusion that the baryon charge of quarks must be equal to $\frac{1}{3}$, that of anti-quarks being $-\frac{1}{3}$.
The half-integer isospin representation attributes the value $I_3 = +\frac{1}{2}$ to the u-quark (the "up" state) and the value $I_3 = -\frac{1}{2}$ to the d-quark (the "down" state).
Then, according to the Gell-Mann-Okubo formula, the electric charges of quarks are:

$$Q(u) = \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{3}, \quad Q(d) = -\frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = -\frac{1}{3}, \tag{30}$$

and obviously, $Q(\bar{u}) = -\frac{2}{3}, \quad Q(\bar{d}) = \frac{1}{3}$.

- **This suggests that a natural generalization of Pauli's exclusion principle would be that no *three* quarks in the same state can form a stable configuration perceived as one of the strongly interacting particles (even the relatively short-lived resonances).**

- **This suggests that a natural generalization of Pauli's exclusion principle would be that no *three* quarks in the same state can form a stable configuration perceived as one of the strongly interacting particles (even the relatively short-lived resonances).**
- **Now we need statistical properties that would allow the coexistence of two quarks with the same isospin, but not three. By analogy with the Fermi-Dirac statistics based on the Z_2 nilpotent operators, we should introduce the Z_3 nilpotent operators $\theta_A, \ A, B = 1, 2$, such that $\theta_A^2 \neq 0$, but $\theta_A^3 = 0$.**

Let us require then the vanishing of wave functions representing the tensor product of ***three*** **(but not necessarily two) identical states. That is, we require that** $\Phi(x, x, x) = 0$ for any state $\mid x >$. **As in the former case, consider an arbitrary superposition of** three different states, $\mid x >$, $\mid y >$ and $\mid z >$,

$$\mid w >= \alpha \mid x > + \beta \mid y > + \gamma \mid z >$$

and apply the same criterion, $\Phi(w, w, w) = 0$.

We get then, after developing the tensor products,

$$\Phi(w,w,w) = \alpha^3\Phi(x,x,x) + \beta^3\Phi(y,y,y) + \gamma^3\Phi(z,z,z)$$

$$+\alpha^2\beta[\Phi(x,x,y)+\Phi(x,y,x)+\Phi(y,x,x)]+\gamma\alpha^2[\Phi(x,x,z)+\Phi(x,z,x)+\Phi(z,x,x)]$$

$$+\alpha\beta^2[\Phi(y,y,x)+\Phi(y,x,y)+\Phi(x,y,y)]+\beta^2\gamma[\Phi(y,y,z)+\Phi(y,z,y)+\Phi(z,y,y)]$$

$$+\beta\gamma^2[\Phi(y,z,z)+\Phi(z,z,y)+\Phi(z,y,z)]+\gamma^2\alpha[\Phi(z,z,x)+\Phi(z,x,z)+\Phi(x,z,z)]$$

$$+\alpha\beta\gamma[\Phi(x,y,z)+\Phi(y,z,x)+\Phi(z,x,y)+\Phi(z,y,x)+\Phi(y,x,z)+\Phi(x,z,y)] = 0.$$

The terms $\Phi(x,x,x)$, $\Phi(y,y,y)$ and $\Phi(z,z,z)$ do vanish by virtue of the original assumption. Among the remaining terms, the combinations preceded by various independent powers of numerical coefficients α, β and γ, must vanish separately.
This is achieved if the following Z_3 symmetry is imposed on our wave functions:

$$\Phi(x,y,z) = j\,\Phi(y,z,x) = j^2\,\Phi(z,x,y).$$

with j representing the primitive root of 1,

$$j = e^{\frac{2\pi i}{3}}, \;\; j^3 = 1, \;\; j + j^2 + 1 = 0. \tag{31}$$

Note that the complex conjugates of functions $\Phi(x,y,z)$ transform under cyclic permutations of their arguments with $j^2 = \bar{j}$ replacing j in the above formula

$$\Psi(x,y,z) = j^2\,\Psi(y,z,x) = j\,\Psi(z,x,y).$$

Let $A, B, ... = 1, 2, ..N$; we suppose that the generators θ^A span an associative algebra, in which the following cubic relations are imposed:

$$\theta^A\theta^B\theta^C = j\,\theta^B\theta^C\theta^A = j^2\,\theta^C\theta^A\theta^B, \quad \text{with } \; j = e^{\frac{2\pi i}{3}} \tag{32}$$

We also introduce a similar set of *conjugate* generators $\bar{\theta}^{\dot{A}}$, $\dot{A}, \dot{B}, ... = 1, 2, ..., N$, satisfying similar condition with j^2 replacing j:

$$\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}} = j^2\,\bar{\theta}^{\dot{B}}\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{A}} = j\,\bar{\theta}^{\dot{C}}\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}, \tag{33}$$

We complete the constitutive relations by the commutation between the generators θ and their conjugates $\bar{\theta}$:

$$\theta^A\bar{\theta}^{\dot{B}} = -j\bar{\theta}^{\dot{B}}\theta^A. \tag{34}$$

The cubic analogue of binary anti-commutation relations defines a Z_3-graded generalization of Grassmann algebra $\mathcal{A}_{Z_3}$ The natural Z_3 grading attributes the Z_3-grade 1 to θ's and the Z_3-grade 2 to $\bar{\theta}$'s. Under the associative multiplication the grades add up modulo 3, so that e.g. the product $\theta^A \theta^B$ has the Z_3-grade 2, the product $\theta^A \bar{\theta}^{\dot{B}}$ has the Z_3-grade 0, etc.

The following two properties of the Z_3-graded analogue of Grassmann algebra are useful for modelling basic properties of quarks. First, it is easy to prove that only quadratic and cubic expressions do not vanish; all products of four and more generators must vanish. Indeed, let us consider a fourth-order expression, $\theta^A\theta^B\theta^C\theta^D$. We have, using the associativity property, the following identities:

$$\theta^A\theta^B\theta^C\theta^D = j\,\theta^B\theta^C\theta^A\theta^D = j^2\,\theta^B\theta^A\theta^D\theta^C = j^3\,\theta^A\theta^D\theta^B\theta^C = j^4\,\theta^A\theta^B\theta^C\theta^D, \tag{35}$$

and because $j^4 = j \neq 1$, the only solution is $\theta^A\theta^B\theta^C\theta^D = 0$.

Secondly, foreseeing that cubic combinations of quarks result in composite particles perceived as fermions (e.g. proton and neutron), the corresponding operators should anti-commute. This is true indeed, as the following simple computation shows:

$$(\theta^A\theta^B\theta^C)(\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}}) = (-j)^3\bar{\theta}^{\dot{D}}(\theta^A\theta^B\theta^C)(\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}}) = \dots$$

$$= (-j)^9(\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}})(\theta^A\theta^B\theta^C) = -(\bar{\theta}^{\dot{D}}\bar{\theta}^{\dot{E}}\bar{\theta}^{\dot{F}})(\theta^A\theta^B\theta^C), \qquad (36)$$

The conjugate generators $\bar{\theta}^{\dot{B}}$ span an algebra $\bar{\mathcal{A}}$ isomorphic with $\mathcal{A}$.

With two quark states, u and d, we get only two independent cubic combinations and three independent quadratic combinations. Considering the formulae (32) and (33) with only two generators and replacing θ^1 by u and θ^2 by d, we see that the only non-vanishing cubic monomials are

$$udu = jduu = j^2 uud, \quad dud = judd = j^2 ddu, \quad \bar{u}\bar{d}\bar{u} = j^2 \bar{d}\bar{d}\bar{u} = j^2 = j\bar{u}\bar{u}\bar{d}, \tag{37}$$

corresponding to the proton and neutron, anti-proton and anti-neutron states. Due to the equation (35), all combinations of order four and higher do vanish identically. The only quadratic combinations satisfying relations (34) are as follows:

$$u\bar{d} = -j\bar{d}u, \quad \bar{u}d = -j^2 d\bar{u}, \quad u\bar{u} - d\bar{d}, \quad u\bar{u} - d\bar{d}. \tag{38}$$

corresponding to three pions , π^+, π^- and π^0.

By analogy with the derivation of the $SL(2, \mathbf{C})$ symmetry from the skew Z_2-symmetry of Pauli's exclusion principle, let us show that certain complex representation of the same symmetry group can be derived from the Z_3-graded generalization of the exclusion principle defined above. Let us suppose that as usual Dirac fermions, quark should be described by a four-component column composed of two Pauli-like spinors. According to the ternary Z_3-skew symmetric anti-commutation hypothesis, these spinors are operator-valued. Let us denote them in the usual way:

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad \begin{pmatrix} d^1 \\ d^2 \end{pmatrix}, \quad \begin{pmatrix} \bar{u}^{\dot{1}} \\ \bar{u}^{\dot{2}} \end{pmatrix}, \quad , \quad \begin{pmatrix} \bar{d}^{\dot{1}} \\ \bar{d}^{\dot{2}} \end{pmatrix}. \tag{39}$$

where u^1 is the operator corresponding to the spin-up state of the u-quark, u^2 is the spin-down state of the u-quark, etc., with the conjugate operator valued spinors denoted by means of dotted indices: $\bar{u}^{\dot{1}}$, $\bar{u}^{\dot{2}}$, etc.

The cubic combinations supposed to produce the observable fermionic states will contain the mixed products of the components of both u and d quarks, which should also obey the Pauli exclusion principle. This means that although there can be two u or two d quarks out of three, they cannot display the same spin direction. Therefore the only possible products must be tthe following:

$$u^1 u^2 d^1, \quad u^1 u^2 d^2, \quad d^1 d^2 u^1, \quad d^1 d^2 u^2, \tag{40}$$

and of course the four similar expressions made of the conjugate quarks with dotted indices:

$$\bar{u}^{\dot{1}} \bar{u}^{\dot{2}} \bar{d}^{\dot{1}}, \quad \bar{u}^{\dot{1}} \bar{u}^{\dot{2}} \bar{d}^{\dot{2}}, \quad \bar{d}^{\dot{1}} \bar{d}^{\dot{2}} \bar{u}^{\dot{1}}, \quad \bar{d}^{\dot{1}} \bar{d}^{\dot{2}} \bar{u}^{\dot{2}}. \tag{41}$$

We notice that there are four independent cubic terms made of the combinations (uud) and $(\bar u \bar u \bar d)$, and also four independent cubic expressions made of the combinations (udd) and $(\bar u \bar d \bar d)$.
This is exactly the number of components one needs to describe two independent Dirac fermions, the proton and the neutron. This is so because the order in cubic products does not matter: according to the ternary anti-commutation laws (32, 33), all permutations are linearly dependent:

$$u^1 u^2 d^1 = j u^2 d^1 u^1 = j^2 d^1 u^1 u^2, \qquad d^1 d^2 u^1 = j d^2 u^1 d^1 = j^2 u^1 d^1 d^2,$$
$$\bar u^{\dot 1} \bar u^{\dot 2} \bar d^{\dot 1} = j^2 \bar u^{\dot 2} \bar d^{\dot 1} \bar u^{\dot 1} = j \bar d^{\dot 1} \bar u^{\dot 1} \bar u^{\dot 2}, \quad \bar d^{\dot 1} \bar d^{\dot 2} \bar u^{\dot 2} = j^2 \bar d^{\dot 2} \bar u^{\dot 2} \bar d^{\dot 1} = j \bar u^{\dot 2} \bar d^{\dot 1} \bar d^{\dot 2}. \tag{42}$$

We can therefore produce a more symmetric expression containing all possible permutations as follows:

$$u^1u^2d^1 = \frac{1}{3}\left[u^1u^2d^1 + ju^2d^1u^1 + j^2d^1u^1u^2\right],$$

$$u^1d^2d^1 = \frac{1}{3}\left[u^1d^2d^1 + jd^2d^1u^1 + j^2d^1u^1d^2\right],$$

$$u^2u^1d^2 = \frac{1}{3}\left[u^2u^1d^2 + ju^1d^2u^2 + j^2d^2u^2u^1\right],$$

$$d^2d^1u^2 = \frac{1}{3}\left[d^2d^1u^2 + jd^1u^2d^2 + j^2u^2d^2d^1\right],$$

and

$$\bar{u}^{\dot{1}}\,\bar{u}^{\dot{2}}\bar{d}^{\dot{1}} = \frac{1}{3}\left[\bar{u}^{\dot{1}}\,\bar{u}^{\dot{2}}\,\bar{d}^{\dot{1}} + j^2\;\bar{u}^{\dot{2}}\,\bar{d}^{\dot{1}}\,\bar{u}^{\dot{1}} + j\;\bar{d}^{\dot{1}}\,\bar{u}^{\dot{2}}\,\bar{u}^{\dot{1}}\right],$$

$$\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}} = \frac{1}{3}\left[\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}} + j^2\;\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}}\,\bar{u}^{\dot{1}} + j\;\bar{d}^{\dot{1}}\,\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}}\right],$$

$$\bar{u}^{\dot{2}}\,\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}} = \frac{1}{3}\left[\bar{u}^{\dot{2}}\,\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}} + j^2\;\bar{u}^{\dot{1}}\,\bar{d}^{\dot{2}}\,\bar{u}^{\dot{2}} + j\;\bar{d}^{\dot{2}}\,\bar{u}^{\dot{2}}\,\bar{u}^{\dot{1}}\right],$$

$$\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}}\,\bar{u}^{\dot{2}} = \frac{1}{3}\left[\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}}\,\bar{u}^{\dot{2}} + j^2\;\bar{d}^{\dot{1}}\,\bar{u}^{\dot{2}}\,\bar{d}^{\dot{2}} + j\;\bar{u}^{\dot{2}}\,\bar{d}^{\dot{2}}\,\bar{d}^{\dot{1}}\right].$$

This is similar to the "averaging" the product of two anti-commuting spinors using the skew-symmetric 2-form $\epsilon_{\alpha\beta}$: $\alpha, \beta = 1, 2,\ \epsilon_{12} = 1$. If the variables χ^α anti-commute, i.e. if $\chi^\alpha \chi^\beta = -\chi^\beta \chi^\alpha$, then we can write

$$\chi^1 \chi^2 = \frac{1}{2} \left[\epsilon_{\alpha\beta}\, \chi^\alpha \chi^\beta \right]. \tag{43}$$

By analogy with this Z_2 skew-symmetric case, we can introduce the Z_3 skew-symmetric 3-form playing a similar role; however, due to the existence of two independent combinations of indices, (121) **and** (212), **two independent Z_3-skew symmetric 3-forms must be introduced, with an extra upper index labeling them. For the sake of generality, let us consider some abstract Z_3-graded algebra spanned by two generators $\theta^A,\ A, B, = 1, 2$ and their conjugates $\bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}$, satisfying the set of constitutive relations** (32, 33 and 34).

We introduce the following two 3-forms ρ^{α}_{ABC} and their conjugates $\bar{\rho}^{\dot{\beta}}_{\dot{D}\dot{E}\dot{F}}$, with $\alpha, \dot{\beta} = 1, 2$ and $A, \dot{B} = 1, 2$, whose components are given explicitly below:

$$\rho^1_{121} = 1, \;\; \rho^1_{211} = j^2, \;\; \rho^1_{112} = j, \quad \rho^2_{212} = 1, \;\; \rho^2_{122} = j^2, \;\; \rho^2_{221} = j. \tag{44}$$

$$\bar{\rho}^{\dot{1}}_{\dot{1}\dot{2}\dot{1}} = 1, \;\; \bar{\rho}^{\dot{1}}_{\dot{2}\dot{1}\dot{1}} = j, \;\; \bar{\rho}^{\dot{1}}_{\dot{1}\dot{1}\dot{2}} = j^2, \quad \bar{\rho}^{\dot{2}}_{\dot{2}\dot{1}\dot{2}} = 1, \;\; \bar{\rho}^{\dot{2}}_{\dot{1}\dot{2}\dot{2}} = j, \;\; \bar{\rho}^{\dot{2}}_{\dot{2}\dot{2}\dot{1}} = j^2, \tag{45}$$

Quite obviously, the 3-forms ρ^{α}_{ABC} and $\bar{\rho}^{\dot{\beta}}_{\dot{D}\dot{E}\dot{F}}$ satisfy the following Z_3-symmetry properties:

$$\rho^1_{121} = j^2\,\rho^1_{211} = j\,\rho^1_{112}, \quad \rho^2_{212} = j^2\,\rho^2_{122} = j\,\rho^2_{221}, \tag{46}$$

and

$$\bar{\rho}^{\dot{1}}_{\dot{1}\dot{2}\dot{1}} = j\,\bar{\rho}^{\dot{1}}_{\dot{2}\dot{1}\dot{1}} = j^2\,\bar{\rho}^{\dot{1}}_{\dot{1}\dot{1}\dot{2}} \quad \bar{\rho}^{\dot{2}}_{\dot{2}\dot{1}\dot{2}} = j\,\bar{\rho}^{\dot{2}}_{\dot{1}\dot{2}\dot{2}} = j^2\,\bar{\rho}^{\dot{2}}_{\dot{2}\dot{2}\dot{1}} \tag{47}$$

The constitutive ternary commutation formulae take on the following shortened form:

$$u^1 u^2 d^1 = \frac{1}{3}\,\rho^1_{ABC} u^A u^B d^C, \quad u^2 u^1 d^2 = \frac{1}{3}\,\rho^2_{ABC} u^A u^B d^C,$$

$$\bar{u}^{\dot{1}} \bar{u}^{\dot{2}} \bar{d}^{\dot{1}} = \frac{1}{3}\,\bar{\rho}^{\dot{1}}_{\dot{A}\dot{B}\dot{C}} \bar{u}^{\dot{A}} \bar{u}^{\dot{B}} \bar{d}^{\dot{C}}, \quad \bar{u}^{\dot{2}} \bar{u}^{\dot{1}} \bar{d}^{\dot{2}} = \frac{1}{3}\,\bar{\rho}^{\dot{2}}_{\dot{A}\dot{B}\dot{C}} \bar{u}^{\dot{A}} \bar{u}^{\dot{B}} \bar{d}^{\dot{C}}, \qquad (48)$$

mapping the 12 degrees of freedom of the three quarks (u, u, d) onto the four degrees of freedom of the proton, supposed to be a $1/2$-spin fermion, and

$$d^1 u^2 d^1 = \frac{1}{3}\,\rho^1_{ABC} d^A u^B d^C, \quad d^2 u^1 d^2 = \frac{1}{3}\,\rho^2_{ABC} u^A u^B d^C,$$

$$\bar{d}^{\dot{1}} u^{\dot{2}} d^{\dot{1}} = \frac{1}{3}\,\bar{\rho}^{\dot{1}}_{\dot{A}\dot{B}\dot{C}} d^{\dot{A}} u^{\dot{B}} d^{\dot{C}}, \quad \bar{d}^{\dot{2}} u^{\dot{1}} d^{\dot{2}} = \frac{1}{3}\,\bar{\rho}^{\dot{2}}_{\dot{A}\dot{B}\dot{C}} d^{\dot{A}} u^{\dot{B}} d^{\dot{C}} \qquad (49)$$

mapping the 12 degrees of freedom of the three quarks (d, d, u) onto the four degrees of freedom of the neutron.

- **The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.**

- The constitutive cubic relations between the generators of the Z_3 **graded algebra** can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.
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- The constitutive cubic relations between the generators of the Z_3 **graded algebra** can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis.
- Let $U^{A'}_{A}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U^{B'}_{B}\, \theta^B$.
- We are looking for the solution of the covariance condition for the ρ-matrices:

$$S^{\alpha'}_{\beta}\; \rho^{\beta}_{ABC} = U^{A'}_{A}\; U^{B'}_{B}\; U^{C'}_{C}\; \rho^{\alpha'}_{A'B'C'}. \qquad (50)$$

Now, $\rho^1_{121} = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ-matrix on the right-hand side is $\rho^{1'}_{A'B'C'}$, which has only three components,

$$\rho^{1'}_{1'2'1'} = 1, \quad \rho^{1'}_{2'1'1'} = j^2, \quad \rho^{1'}_{1'1'2'} = j,$$

which leads to the following equation:

$$S^{1'}_1 = U^{1'}_1 \, U^{2'}_2 \, U^{1'}_1 + j^2 \, U^{2'}_1 \, U^{1'}_2 \, U^{1'}_1 + j \, U^{1'}_1 \, U^{1'}_2 \, U^{2'}_1 = U^{1'}_1 \, (U^{2'}_2 \, U^{1'}_1 - U^{2'}_1 \, U^{1'}_2), \tag{51}$$

because $j^2 + j = -1$.

For the alternative choice $\alpha' = 2'$ the ρ-matrix on the right-hand side is $\rho^{2'}_{A'B'C'}$, whose three non-vanishing components are

$$\rho^{2'}_{2'1'2'} = 1, \quad \rho^{2'}_{1'2'2'} = j^2, \quad \rho^{2'}_{2'2'1'} = j.$$

The corresponding equation becomes now:

$$S^{2'}_1 = U^{2'}_1 \, U^{1'}_2 \, U^{2'}_1 + j^2 \, U^{1'}_1 \, U^{2'}_2 \, U^{2'}_1 + j \, U^{2'}_1 \, U^{2'}_2 \, U^{1'}_1 = U^{2'}_1 \, (U^{1'}_2 \, U^{2'}_1 - U^{1'}_1 \, U^{2'}_2), \tag{52}$$

The remaining two equations are obtained in a similar manner.

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears on the right-hand side.

$$S^{2'}_{1} = -U^{2'}_{1}\,[det(U)], \tag{53}$$

The remaining two equations are obtained in a similar manner, resulting in the following:

$$S^{1'}_{2} = -U^{1'}_{2}\,[det(U)], \qquad S^{2'}_{2} = U^{2'}_{2}\,[det(U)]. \tag{54}$$

The determinant of the 2×2 complex matrix $U^{A'}_{B}$ appears everywhere on the right-hand side. Taking the determinant of the matrix $S^{\alpha'}_{\beta}$ one gets immediately

$$\det(S) = [\det(U)]^3. \tag{55}$$

The matrices $S^{\alpha'}_{\beta}$ belong to the $SL(2, \mathbf{C})$ group if the resulting cubic combinations of quarks (proton and neutron) behave like Dirac spinors; therefore we have

$$\det(S^{\alpha'}_{\beta}) = 1. \tag{56}$$

This means that also

$$\left[\det(U^{A'}_{B})\right]^3 = 1, \tag{57}$$

which means that

$$\det(U) = 1, \; j, \; \text{or } j^2. \tag{58}$$

- **We see that the invariance group of the Z_3-graded algebra of isospin quark components satisfying Z_3-symmetric cubic commutation relations is a Z_3-covering of the $SL(2, \mathbf{C})$ group. The Lie algebra of this group is a Z_3-covering of the Lorentz algebra.**

- We see that the invariance group of the Z_3-graded algebra of isospin quark components satisfying Z_3-symmetric cubic commutation relations is a Z_3-covering of the $SL(2, \mathbf{C})$ group. The Lie algebra of this group is a Z_3-covering of the Lorentz algebra.
- There is another representation of Z_3 symmetry corresponding to an extra degree of freedom of quarks, the *colour*. Incorporating the colour, which is akin to three-valued spin due to its exclusivity - no quarks displaying the same colour can be present in a stable bound state - leads to a Z_3-graded generalization of the Dirac equation.

A Z_3-graded analog of Pauli's exclusion principle and the Z_3-graded Dirac's equation were introduced in our papers in 2017, 2018, 2019.

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