On the solutions of Knizhnik-Zamolodchikov differential equations by noncommutative Picard-Vessiot theory

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Combinatorics and Arithmetic for Physics: special days
November 30 - Decembre 2, 2021, IHES, Bures-sur-Yvette.

Outline

INTRODUCTION

Knizhnik-Zamolodchikov differential equations

Let $(\mathcal{H}(\widetilde{\mathbb{C}_*^n}), 1_{\mathcal{H}(\widetilde{\mathbb{C}_*^n})})$ be the ring of holomorphic functions over the universal covering of the configuration space of n points, *i.e.*

$$\mathbb{C}_*^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}.$$

Let $\mathcal{H}(\widetilde{\mathbb{C}_*^n})\langle\langle \mathcal{T}_n \rangle\rangle$ be the ring of noncommutative series over the alphabet $\mathcal{T}_n := \{t_{i,j}\}_{1 \leq i < j \leq n}$ and with coefficients in $\mathcal{H}(\widetilde{\mathbb{C}_*^n})$.

The following noncommutative differential equation is so called KZ_n

$$\mathbf{d}F(z) = \Omega_n(z)F(z), \quad \text{where} \quad \Omega_n(z) := \sum_{1 \leq i < j \leq n} \frac{t_{i,j}}{2\mathrm{i}\pi} d\log(z_i - z_j)$$

for which solutions can be computed by convergent iterations, for the discrete topology[1] of pointwise convergence over $\mathcal{H}(\widetilde{\mathbb{C}_*^n})\langle\langle \mathcal{T}_n \rangle\rangle$.

Example (trivial case)

For $n = 2$, one has $\mathcal{T}_2 = \{t_{1,2}\}$ and a solution of the equation

$$\mathbf{d}F(z) = \Omega_2(z)F(z), \quad \text{where} \quad \Omega_2(z) = (t_{1,2}/2\mathrm{i}\pi) d\log(z_1 - z_2),$$

is $F(z_1, z_2) = e^{(t_{1,2}/2\mathrm{i}\pi)\log(z_1 - z_2)} = (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi} \in \mathcal{H}(\widetilde{\mathbb{C}_*^2})\langle\langle \mathcal{T}_2 \rangle\rangle$.

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1. $\forall S, T \in \mathcal{H}(\widetilde{\mathbb{C}_*^n})\langle\langle \mathcal{T}_n \rangle\rangle, d(S, T) = 2^{\varpi(S-T)}$, where ϖ denotes the valuation.

Quadratic relations among $\{t_{i,j}\}_{1\le i<j\le n}$

According to Drinfel'd, KZ_n is completely integrable if[2]

$$d\Omega_n(z) - \Omega_n(z)\wedge\Omega_n(z) = 0.$$

It turns out that this condition induces the following quadratic relations in $\{t_{i,j}\}_{1\le i<j\le n}$:

$$\mathcal{R}_n = \begin{cases} [t_{i,k}+t_{j,k}, t_{i,j}] = 0 & \text{for distinct } i,j,k \quad \text{and } 1\le i<j<k\le n, \\ [t_{i,j}+t_{i,k}, t_{j,k}] = 0 & \text{for distinct } i,j,k \quad \text{and } 1\le i<j<k\le n, \\ [t_{i,j}, t_{k,l}] = 0 & \text{for distinct } i,j,k,l \quad \text{and } \begin{cases} 1\le i<j\le n, \\ 1\le k<l\le n, \end{cases} \end{cases}$$

generating the Lie ideal $\mathcal{J}_{\mathcal{R}_n}$.
Solutions of KZ_n belong now to $\mathcal{H}(\widetilde{\mathbb{C}_*^n})\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$.

2. Cartier uses a similar criterion with matrices in place of letters $t_{i,j}\in\mathcal{T}_n$.

NONCOMMUTATIVE SERIES WITH HOLOMORPHIC COEFFICIENTS

Differential ring of holomorphic functions

- $\mathcal{A} = (\mathcal{H}(\mathcal{V}), \partial_1, \ldots, \partial_n)$, the differential ring of holomorphic functions on a simply connected manifold $\mathcal{V}$ of $\mathbb{C}^n (n > 0)$ and equipped $1_{\mathcal{H}(\mathcal{V})}$ as the neutral element.

  For any $f \in \mathcal{H}(\mathcal{V})$, one has $df = (\partial_1 f)dz_1 + \ldots + (\partial_n f)dz_n$.
- Let $\mathcal{C}$ be a sub differential ring of $\mathcal{A}$ (*i.e.* $\partial_i \mathcal{C} \subset \mathcal{C}$, for $1 \leq i \leq n$) and let $\varsigma \rightsquigarrow z$ denote a path over a simply connected manifold $\mathcal{V}$, *i.e.* the parametrized curve $\gamma : [0,1] \longrightarrow \mathcal{V}$ such that

  $$\gamma(0) = \varsigma = (\varsigma_1, \ldots, \varsigma_n) \quad \text{and} \quad \gamma(1) = z = (z_1, \ldots, z_n).$$
- For any integers i, j such that $1 \leq i < j \leq n$, let $\omega_{i,j}$ denote the 1-differential forms [3], in $\Omega^1(B)$, $\omega_{i,j} = d\xi_{i,j}$, with $\xi_{i,j} \in \mathcal{C}$.

Example ($\xi_{i,j}(z) = \log(z_i - z_j), 1 \leq i < j \leq n$)

Let $\mathcal{C}_0 := \mathbb{C}[\{(\partial_1 \xi_{i,j})^{\pm 1}, \ldots, (\partial_n \xi_{i,j})^{\pm 1}\}_{1 \leq i < j \leq n}]$.

Then $\mathcal{C}_0$ is a sub differential ring of $\mathcal{A}$.

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3. Over $\mathcal{V}$, the holomorphic function $\xi_{i,j}$ is called a primitive for $\omega_{i,j}$ which is said to be a exact form and then is a closed form (*i.e.* $d\omega_{i,j} = 0$).

Notations

- $(\mathcal{T}_n{}^*, 1_{\mathcal{T}_n{}^*})$ is the free monoid generated by $\mathcal{T}_n$. $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$ (resp. $\mathcal{A}\langle\mathcal{T}_n\rangle$) is the set of series (resp. polynomials) over $\mathcal{T}_n$ with coefficients in $\mathcal{A}$. $\mathcal{L}yn\mathcal{T}_n$ (resp. $\mathcal{L}yn\mathcal{T}$) is the set of Lyndon words over $\mathcal{T}_n$ (resp. $\mathcal{T}$).
- $T_k := \{t_{j,k}\}_{1\le j\le k-1}, \mathcal{T} := \{T_2, \ldots, T_n\}$ s.t. $\mathcal{T}_k = T_k \sqcup \mathcal{T}_{k-1}, k \le n$. $|\mathcal{T}_n| = n(n-1)/2$ and $|T_n| = n-1$. If $n \ge 4$ then $|\mathcal{T}_{n-1}| \ge |T_n|$.

Example

- $\mathcal{T}_5 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{1,5}, t_{2,3}, t_{2,4}, t_{2,5}, t_{3,4}, t_{3,5}, t_{4,4}\}$, one has $T_5 = \{t_{1,5}, t_{2,5}, t_{3,5}, t_{4,5}\}$ and $\mathcal{T}_4$.
- $\mathcal{T}_4 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}$, one has $T_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}$ and $\mathcal{T}_3$.
- $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$, one has $T_3 = \{t_{1,3}, t_{2,3}\}$ and $\mathcal{T}_2 = \{t_{1,2}\}$.

- In $(\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle, \partial_1, \ldots, \partial_n)$, for any $S \in \mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle$, one defines

$$\partial_i S = \sum_{w\in\mathcal{T}_n^*} (\partial_i \langle S|w\rangle) w \quad \text{and} \quad \mathbf{d}S = \sum_{i=1}^{n} (\partial_i S) dz_i.$$

$\mathrm{Const}(\mathcal{A}) = \mathbb{C}.1_{\mathcal{H}(\Omega)}$ and $\mathrm{Const}(\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle) = \mathbb{C}\langle\langle\mathcal{T}_n\rangle\rangle$.

Diagonal series

$\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle$ is the set of Lie polynomials over $\mathcal{T}_n$ with coefficients in $\mathcal{A}$ and is equipped with the basis $\{P_l\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ over which are constructed the PBW basis $\{P_w\}_{w\in\mathcal{T}_n^*}$ of $\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)$ and its dual, $\{S_w\}_{w\in\mathcal{T}_n^*}$, containing the pure transcendence basis $\{S_l\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ of[4] $(\mathcal{A}\langle\mathcal{T}_n\rangle, ⧢, 1_{\mathcal{T}_n^*})$.

Example (in KZ_3, $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and $t_{1,2} \prec t_{1,3} \prec t_{2,3}$)

$\forall k \geq 0, i = 1$ or 2, $\quad t_{1,2}^k t_{i,3} \in \mathcal{L}yn\mathcal{T}_3, \quad P_{t_{1,2}^k t_{i,3}} = \mathrm{ad}_{t_{1,2}}^k t_{i,3}, S_{t_{1,2}^k t_{i,3}} = t_{1,2}^k t_{i,3}$.

In $(\mathcal{A}\langle\mathcal{T}_n\rangle, \mathrm{conc}, 1_{\mathcal{T}_n^*}, \Delta_{⧢}, \mathrm{e})$, the diagonal series is defined by

$$\mathcal{D} := \mathcal{M}^*, \quad \text{with} \quad \mathcal{M} := \sum_{t\in\mathcal{T}_n} t \otimes t,$$

and is the unique solution of the equations

$$\nabla S = \mathcal{M} S \quad \text{and} \quad \nabla S = S\mathcal{M},$$

where ∇S denotes $S - 1_{\mathcal{T}_n^*} \otimes 1_{\mathcal{T}_n^*}$, for $S \in \mathcal{A}\langle\mathcal{T}_n\rangle \hat{\otimes} \mathcal{A}\langle\mathcal{T}_n\rangle$. Then

$$\mathcal{D} = \Big(\prod_{l\in\mathcal{L}yn\mathcal{T}_{n-1}}^{\searrow} \prod_{\substack{l=l_1 l_2 \\ l_2\in\mathcal{L}yn\mathcal{T}_{n-1}, l_1\in\mathcal{L}yn T_n}}^{\searrow} \prod_{l\in\mathcal{L}yn T_n}^{\searrow} \Big) e^{S_l \otimes P_l}, \quad \text{for} \quad n > 2.$$

4. in which one defines $\Delta_{⧢} x = x \otimes 1_{\mathcal{T}_n^*} + 1_{\mathcal{T}_n^*} \otimes x$, or equivalently, $u ⧢ 1_{\mathcal{T}_n^*} = 1_{\mathcal{T}_n^*} ⧢ u = u$ and $xu ⧢ yv = x(u ⧢ yv) + y(xu ⧢ v)$.

Example of lexicographic ordering (in KZ_n, $n \geq 4$)

Let us consider the following total order over T_k :

$$t_{1,k} \succ \ldots \succ t_{k-1,k}, \quad \text{for} \quad n \geq k \geq 2,$$

and over $\mathcal{T}$:

$$T_2 \succ \ldots \succ T_n \quad \text{and then} \quad \mathcal{L}ynT_2 \succ \ldots \succ \mathcal{L}ynT_n.$$

With this ordering, one has

$$\mathcal{L}yn\mathcal{T}_{n-1} \succ \mathcal{L}ynT_n.\mathcal{L}yn\mathcal{T}_{n-1} \succ \mathcal{L}ynT_n.$$

More generally, for any $(t_1, t_2) \in T_{k_1} \times T_{k_2}, 2 \leq k_1 < k_2 \leq n$, one has

$$t_1 t_2 \in \mathcal{L}yn\mathcal{T}_n \quad \text{and} \quad t_2 \succ t_1 t_2 \succ t_1.$$

Hence,

- For any $l \in \mathcal{L}ynT_{k-1}$ and $t \in T_k, 2 \leq k \leq n$, one has $lt \in \mathcal{L}yn\mathcal{T}_n$ and $l \prec lt \prec t$.
- For any $l_1 \in \mathcal{L}ynT_{k_1}$ and $l_2 \in \mathcal{L}ynT_{k_2}, 2 \leq k_1 < k_2 \leq n$, one has $l_1 l_2 \in \mathcal{L}yn\mathcal{T}_n$ and $l_1 \prec l_1 l_2 \prec l_2$.
- For any $l_1 \in \mathcal{L}ynT_k$ and $l_2 \in \mathcal{L}yn\mathcal{T}_{k-1}, 2 \leq k \leq n$, one has $l_1 l_2 \in \mathcal{L}yn\mathcal{T}_n$ and $l_1 \prec l_1 l_2 \prec l_2$.
- For any $t \in T_k, x \in \mathcal{T}_{k-1}, 2 \leq k_1 < k_2 \leq n$ and $i \geq 0$, one has $t \prec x$ and $t^i x \in \mathcal{L}yn\mathcal{T}_k$ and then $P_{t^i x} = \mathrm{ad}_t^i x$ and $S_{t^i x} = t^i x$.

More about notations

Let us back to the relations

$$\mathcal{R}_n = \begin{cases} [t_{i,k}+t_{j,k},t_{i,j}]=0 & \text{for distinct } i,j,k \quad \text{and } 1\le i<j<k\le n,\\ [t_{i,j}+t_{i,k},t_{j,k}]=0 & \text{for distinct } i,j,k \quad \text{and } 1\le i<j<k\le n,\\ [t_{i,j},t_{k,l}]=0 & \text{for distinct } i,j,k,l \quad \text{and } \begin{cases}1\le i<j\le n,\\ 1\le k<l\le n,\end{cases}\end{cases}$$

generating the Lie ideal $\mathcal{J}_{\mathcal{R}_n}$.

- The monoid (resp. the set of Lyndon words) generated by $\mathcal{T}_n$ satisfying the relations $\mathcal{R}_n$ is denoted by $\langle \mathcal{T}_n^*; \mathcal{J}_{\mathcal{R}_n}\rangle$ (resp. $\langle \mathcal{L}yn\mathcal{T}_n; \mathcal{J}_{\mathcal{R}_n}\rangle$).
- The set of noncommutative polynomials (resp. series) with coefficients in $\mathcal{A}$, over $\mathcal{T}_n$, satisfying $\mathcal{R}_n$, is denoted by $\mathcal{A}\langle\mathcal{T}_n\rangle/\mathcal{J}_{\mathcal{R}_n}$ (resp. $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$).
- The set of Lie polynomials (resp. Lie series) with coefficients in $\mathcal{A}$, over $\mathcal{T}_n$, satisfying $\mathcal{R}_n$, is denoted by $\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle/\mathcal{J}_{\mathcal{R}_n}$ (resp. $\mathcal{L}ie_{\mathcal{A}}\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$).
- $H_{⧢}(\mathcal{T}_n)/\mathcal{J}_{\mathcal{R}_n}$ denotes $(\mathcal{A}\langle\mathcal{T}_n\rangle/\mathcal{J}_{\mathcal{R}_n}, \mathtt{conc}, \Delta_{⧢}, 1_{\mathcal{T}_n^*})$.

Combinatorial aspects with infinitesimal braid like relations

Let us consider the Lie ideal $\mathcal{I}_n$ generated by $\{\mathrm{ad}^k_{T_n} t_{i,j}\}^{k\geq 0}_{t_{i,j}\in\mathcal{T}_{n-1}}$.
By the PBW theorem, the enveloping algebra $\mathcal{U}(\mathcal{I}_n)$ is freely generated by $\{\mathrm{ad}^{k_1}_{T_n} t_{i_1,j_1}\ldots\mathrm{ad}^{k_p}_{T_n} t_{i_p,j_p}\}^{k_1,\ldots,k_p\geq 0,p\geq 0}_{t_{i_1,l_1},\ldots,t_{i_p,l_p}\in\mathcal{T}_{n-1}}$ and by the Lazard elimination, for any $n>2$, one also has

$$\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle=\mathcal{I}_n\oplus\mathcal{L}ie_{\mathcal{A}}\langle T_n\rangle.$$

Lemma

For any $n>2$, one has

1. $\mathcal{I}_n/\mathcal{J}_{\mathcal{R}_n}=\{0\}$ *and then* $\mathcal{U}(\mathcal{I}_n)/\mathcal{J}_{\mathcal{R}_n}=\{0\}$.
2. $\mathcal{U}(\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle)/\mathcal{J}_{\mathcal{R}_n}=\mathcal{A}\langle T_n\rangle/\mathcal{J}_{\mathcal{R}_n}$ *and then* $[T_{n-1},T_n]/\mathcal{J}_{\mathcal{R}_n}=\{[t_{i,n-1},t_{i,n}]\}_{1\leq i\leq n-2},\ldots,[T_2,T_n]/\mathcal{J}_{\mathcal{R}_n}=\{[t_{1,2},t_{1,n}]\}$.
3. $\{P_l\}_{l\in\langle\mathcal{L}ynT_n;\mathcal{J}_{\mathcal{R}_n}\rangle}=T_n\cup\{[t_{i,n},t_{j,n}]\}_{1\leq i<j\leq n-1}\cup$ $\{[t_{k,n},[t_{i,n},t_{j,n}]],[t_{l,n},[t_{j,n},t_{k,n}]]\}_{1\leq l<i<j<k\leq n-1}\cup$ $\{P_l\}_{l\in\langle\mathcal{L}yn^{\geq 4}T_n;\mathcal{J}_{\mathcal{R}_n}\rangle}$.

BACKGROUND ON NONCOMMUTATIVE PV THEORY

Iterated integrals and Chen series

The iterated integral associated, of the 1-differential forms $\{\omega_{i,j}\}_{1\leq i<j\leq n}$ and along the path $\varsigma \rightsquigarrow z$, is given by $\alpha_{\varsigma}^{z}(1_{\mathcal{T}_n^*}) = 1_{\mathcal{H}(\mathcal{V})}$ and, for any $w = t_{i_1,j_1}t_{i_2,j_2}\ldots t_{i_k,j_k} \in \mathcal{T}_n^*$,

$$\alpha_{\varsigma}^{z}(w) := \int_{\varsigma}^{z} \omega_{i_1,j_1}(s_1) \int_{\varsigma}^{s_1} \omega_{i_2,j_2}(s_2) \ldots \int_{\varsigma}^{s_{k-1}} \omega_{i_k,j_k}(s_k) \in \mathcal{H}(\mathcal{V}),$$

where $(\varsigma, s_1 \ldots, s_{k-1}, z)$ is a subdivision of $\varsigma \rightsquigarrow z$.

The Chen series, of the differential forms $\{\omega_{i,j}\}_{1\leq i<j\leq n}$ and along a path $\varsigma \rightsquigarrow z$, is the following noncommutative generating series

$$C_{\varsigma\rightsquigarrow z} := \sum_{w\in\mathcal{T}_n^*} \alpha_{\varsigma}^{z}(w)w \in \mathcal{H}(\mathcal{V})\langle\langle\mathcal{T}_n^*\rangle\rangle.$$

Proposition

1. $\forall u, v$ *in* $\mathcal{T}_n^*, \alpha_{\varsigma}^{z}(u ⧢ v) = \alpha_{\varsigma}^{z}(u)\alpha_{\varsigma}^{z}(v)$ *(Chen's lemma).*
2. $\forall t \in \mathcal{T}_n, k \geq 0, \alpha_{\varsigma}^{z}(t^k) = (\alpha_{\varsigma}^{z}(t))^k/k!$ *and then* $\alpha_{\varsigma}^{z}(t^*) = e^{\alpha_{\varsigma}^{z}(t)}$.
3. *For any compact* $K \subset \mathcal{V}$, *there is* $c > 0$ *and a morphism of monoids* $\mu : \mathcal{T}_n^* \longrightarrow \mathbb{R}_{\geq 0}$ *s.t.* $\|\langle C_{\varsigma\rightsquigarrow z}|w\rangle\|_K \leq c\mu(w)\,|w|!^{-1}$, *for* $w \in \mathcal{T}_n^*$, *and then* $C_{\varsigma\rightsquigarrow z}$ *is said to be exponentially bounded from above.*

Basic triangular theorem over a differential ring

Recall that $\mathcal{A} = (\mathcal{H}(\mathcal{V}), \partial_1, \ldots, \partial_n)$ and $\mathcal{C}$ be a sub differential ring of $\mathcal{A}$.

Lemma

The following assertions are equivalent[5]

1. *The following map is injective*

$$\begin{array}{rcl} (\mathcal{A}\langle\mathcal{T}_n\rangle, ⧢, 1_{\mathcal{T}_n^*}) & \longrightarrow & (\mathcal{H}(\mathcal{V}), *, 1_{\mathcal{H}(\mathcal{V})}), \\ w & \longmapsto & \alpha_\varsigma^z(w). \end{array}$$

2. $\{\alpha_\varsigma^z(w)\}_{w\in\mathcal{T}_n^*}$ *is linearly free over* $\mathcal{C}$.
3. $\{\alpha_\varsigma^z(l)\}_{l\in\mathcal{L}yn\mathcal{T}_n}$ *is algebraically free over* $\mathcal{C}$.
4. $\{\alpha_\varsigma^z(t)\}_{t\in\mathcal{T}_n}$ *is algebraically free over* $\mathcal{C}$.
5. $\{\alpha_\varsigma^z(t)\}_{t\in\mathcal{T}_n\cup\{1_{\mathcal{T}_n^*}\}}$ *is linearly free over* $\mathcal{C}$.

5. This is the abstract form, over ring, of (Deneufchâtel, Duchamp, HNM & Solomon, 2011).

Noncommutative differential equations

$$(NCDE) \quad \mathbf{d}S = M_n S, \quad \text{where}^6 \quad M_n = \sum_{1\le i<j\le n} \omega_{i,j} t_{i,j}.$$

Proposition

1. *$C_{\varsigma\rightsquigarrow z}$, satisfying $(NCDE)$, is group-like and $\log C_{\varsigma\rightsquigarrow z}$ is primitive :*

$$C_{\varsigma\rightsquigarrow z} = \prod_{l\in\mathcal{L}yn\mathcal{T}_n}^{\searrow} e^{\alpha_\varsigma^z(S_l)P_l} \quad \text{and} \quad \log C_{\varsigma\rightsquigarrow z} = \sum_{w\in\mathcal{T}_n^*} \alpha_\varsigma^z(w)\pi_1(w),$$

where $\pi_1(w) = \sum_{k\ge 1} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in\mathcal{T}_n\mathcal{T}_n^*} \langle w|u_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup u_k\rangle u_1\ldots u_k.$

2. *Let $C\in\mathbb{C}\langle\langle\mathcal{T}_n\rangle\rangle, \langle C|1_{\mathcal{T}_n^*}\rangle = 1$. Then $C_{\varsigma\rightsquigarrow z}C$ satisfies $(NCDE)$. Moreover, $C_{\varsigma\rightsquigarrow z}C$ is group-like if and only if C is group-like.*

From this, it follows that the differential Galois group of $(NCDE)$ + group-like solutions is[7] the group $\{e^C\}_{C\in\mathcal{L}ie_{\mathbb{C}.1_\Omega}\langle\langle\mathcal{X}\rangle\rangle}$. Which leads to the definition of the PV extension related to $(NCDE)$ as $\widehat{\mathcal{C}_0.\mathcal{X}}\{C_{z_0\rightsquigarrow z}\}$.

6. $M_n\in\Omega^1(\mathcal{V})\langle\mathcal{T}_n\rangle$ and $\Delta_{\sqcup\!\sqcup} M_n = 1_{\mathcal{T}_n^*}\otimes M_n + M_n\otimes 1_{\mathcal{T}_n^*}$.
7. In fact, the Hausdorff group (group of characters) of $(\mathcal{A}\langle\mathcal{T}_n\rangle, \sqcup\!\sqcup, 1_{\mathcal{T}_n^*})$.

ALGORITHMIC AND COMPUTATIONAL ASPECTS OF SOLUTIONS OF KZ_n BY DEVISSAGE

KZ_3 : Simplest non-trivial case (1/4)

One has $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ and

$$\Omega_3(z) = \frac{1}{2\mathrm{i}\pi}\left(t_{1,2}\frac{d(z_1 - z_2)}{z_1 - z_2} + t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right).$$

Solution of $\mathbf{d}F(z) = \Omega_3(z)F(z)$ can be computed as limit of the sequence $\{F_l\}_{l\geq 0}$, in $\mathcal{H}(\widetilde{\mathbb{C}_*^3})\langle\langle\mathcal{T}_3\rangle\rangle$, by convergent Picard's iteration :

$$F_0(z) = 1_{\mathcal{H}(\widetilde{\mathbb{C}_*^n})} \quad \text{and} \quad F_l(z) = \int_0^z \Omega_3(s)F_{l-1}(s).$$

Let us compute, by another way, a solution of $\mathbf{d}F(z) = \Omega_3(z)F(z)$ as the limit of the sequence $\{V_l\}_{l\geq 0}$, in $\mathcal{H}(\widetilde{\mathbb{C}_*^3})\langle\langle\mathcal{T}_3\rangle\rangle$, iteratively obtained by

$$\begin{aligned}
V_0(z) &= e^{(t_{1,2}/2\mathrm{i}\pi)\log(z_1 - z_2)},\\
V_l(z) &= \int_0^z e^{(t_{1,2}/2\mathrm{i}\pi)(\log(z_1 - z_2) - \log(s_1 - s_2))}\tilde{\Omega}_2(s)V_{l-1}(s)\\
&= V_0(z)\int_0^z e^{-(t_{1,2}/2\mathrm{i}\pi)\log(s_1 - s_2)}\tilde{\Omega}_2(s)V_{l-1}(s),
\end{aligned}$$

$$\text{with } \tilde{\Omega}_2(z) = \frac{1}{2\mathrm{i}\pi}\left(t_{1,3}\frac{d(z_1 - z_3)}{z_1 - z_3} + t_{2,3}\frac{d(z_2 - z_3)}{z_2 - z_3}\right).$$

KZ_3 : Simplest non-trivial case (2/4)

Explicit solution is $F = V_0 G$, where $V_0(z) = (z_1 - z_2)^{t_{1,2}/2\mathrm{i}\pi}$ and

$$G(z) = \sum_{\substack{t_{i_1,j_1}\ldots t_{i_m,j_m}\in\{t_{1,3},t_{2,3}\}^*\\ m\geq 0}} \int_0^z \omega_{i_1,j_1}(s_1)\varphi^{s_1}(t_{i_1,j_1})\ldots\int_0^{s_{m-1}} \omega_{i_m,j_m}(s_m)\varphi^{s_m}(t_{i_m,j_m}),$$

where $\omega_{1,3}(z) = d\log(z_1 - z_3)$ and $\omega_{2,3}(z) = d\log(z_2 - z_3)$ and φ is the following automorphism of Lie algebra, $\mathcal{L}ie_{\mathcal{H}(\widetilde{\mathbb{C}_*^n})}\langle\mathcal{T}_3\rangle$,

$$\varphi^z = e^{\mathrm{ad}_{-(t_{1,2}/2\mathrm{i}\pi)\log(z_1-z_2)}} = \sum_{k\geq 0}\frac{\log^k(z_1-z_2)}{(-2\mathrm{i}\pi)^k k!}\mathrm{ad}_{t_{1,2}}^k.$$

Since $t_{1,2} \prec t_{1,3} \prec t_{2,3}$ and, for $k \geq 0$ and $i = 1$ or 2, $t_{1,2}^k t_{i,3} \in \mathcal{L}yn\mathcal{T}_3$ then

$$P_{t_{1,2}^k t_{i,3}} = \mathrm{ad}_{t_{1,2}}^k t_{i,3} \quad\text{and}\quad S_{t_{1,2}^k t_{i,3}} = t_{1,2}^k t_{i,3}$$

and then

$$\varphi^z(t_{i,3}) = \sum_{k\geq 0}\frac{\log^k(z_1-z_2)}{(-2\mathrm{i}\pi)^k k!}P_{t_{1,2}^k t_{i,3}},\quad \check{\varphi}^z(t_{i,3}) = \sum_{k\geq 0}\frac{\log^k(z_1-z_2)}{(-2\mathrm{i}\pi)^k k!}S_{t_{1,2}^k t_{i,3}},$$

where $\check{\varphi}$ (adjoint to φ) is the following automorphism of $(\mathcal{A}\langle\mathcal{T}_3\rangle, ш, 1_{\mathcal{T}_3^*})$

$$\check{\varphi}^z = e^{-(t_{1,2}/2\mathrm{i}\pi)\log(z_1-z_2)} = \sum_{k\geq 0}\frac{\log^k(z_1-z_2)}{(-2\mathrm{i}\pi)^k k!}t_{1,2}^k.$$

KZ_3 : Simplest non-trivial case (3/4)

Belonging to $\mathcal{H}(\widetilde{\mathbb{C}^3_*})\langle\langle\mathcal{T}_3\rangle\rangle$, G satisfies $\mathbf{d}G(z) = \bar{\Omega}_2(z)G(z)$, where

$$\bar{\Omega}_2(z) = \frac{1}{2\mathrm{i}\pi}\left(\varphi^z(t_{1,3})\frac{d(z_1-z_3)}{z_1-z_3} + \varphi^z(t_{2,3})\frac{d(z_2-z_3)}{z_2-z_3}\right).$$

In the affine plan $(P_{1,2}) : z_1 - z_2 = 1$, one has

$$\log(z_1 - z_2) = 0 \quad \text{and then} \quad \varphi \equiv \mathrm{Id}.$$

Setting $x_0 = t_{1,3}/2\mathrm{i}\pi, x_1 = -t_{2,3}/2\mathrm{i}\pi$ and $z_1 = 1, z_2 = 0, z_3 = s$, one has

$$\bar{\Omega}_2(z) = \frac{1}{2\mathrm{i}\pi}\left(t_{1,3}\frac{d(z_1-z_3)}{z_1-z_3} + t_{2,3}\frac{d(z_2-z_3)}{z_2-z_3}\right) = x_1\frac{ds}{1-s} + x_0\frac{ds}{s}.$$

KZ_3 admits then the noncommutative generating series of polylogarithms, L, as the actual solution satisfying the Drinfel'd asymptotic conditions. Via L and the homographic substitution $g : z_3 \longmapsto (z_3 - z_2)/(z_1 - z_2)$, mapping $\{z_2, z_1\}$ to $\{0, 1\}$, $\mathrm{L}((z_3 - z_2)/(z_1 - z_2))$ is a particular solution of KZ_3, in $(P_{1,2})$. So is $\mathrm{L}((z_3 - z_2)/(z_1 - z_2))(z_1 - z_2)^{(t_{1,2}+t_{1,3}+t_{2,3})/2\mathrm{i}\pi}$.

KZ_3 : Simplest non-trivial case (4/4)

Denoting $(X^*, 1_{X^*})$ the monoid generated by $X = \{x_0, x_1\}$, recall that

$$\mathrm{L}(s) := \sum_{w \in X^*} \mathrm{Li}_w(s) w \in \mathcal{H}(\widetilde{\mathbb{C} \setminus \{0,1\}})\langle\langle X \rangle\rangle,$$

where $\mathrm{Li}_\bullet$ is the character of $(\mathcal{H}(\widetilde{\mathbb{C} \setminus \{0,1\}})\langle X \rangle, ш, 1_{X^*})$ defined by

$$\mathrm{Li}_{1_{X^*}} = 1_{\mathcal{H}(\widetilde{\mathbb{C} \setminus \{0,1\}})}, \quad \mathrm{Li}_{x_0}(s) = \log(s), \quad \mathrm{Li}_{x_1}(s) = \log(1-s)$$

and, for any $x_i w \in \mathcal{L}ynX \setminus X$,

$$\mathrm{Li}_{x_i w}(s) = \int_0^s \omega_i(\sigma) \mathrm{Li}_w(\sigma), \quad \text{where} \quad \begin{cases} \omega_0(s) = ds/s, \\ \omega_1(s) = ds/(1-s). \end{cases}$$

$\{\mathrm{Li}_l\}_{l \in \mathcal{L}ynX}$ (resp. $\{\mathrm{Li}_w\}_{w \in X^*}$) are $\mathbb{C}$-algebraically (resp. linearly) free.

By the Friedrichs crirerion, L is group like. Thus,

$$\mathrm{L}(s) = \prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mathrm{Li}_{S_l}(s) P_l} \quad \text{and then} \quad \begin{cases} \lim_{z \to 0} \mathrm{L}(s) e^{-x_0 \log z} & = & 1, \\ \lim_{z \to 1} e^{x_1 \log(1-z)} \mathrm{L}(s) & = & \Phi_{KZ}, \end{cases}$$

where Φ_{KZ} is the following constant group like series

$$\Phi_{KZ} := \prod_{l \in \mathcal{L}ynX \setminus X}^{\searrow} e^{\mathrm{Li}_{S_l}(1) P_l} \in \mathbb{R}\langle\langle X \rangle\rangle, \quad \text{for} \quad \begin{cases} x_0 = & t_{1,2}/2\mathrm{i}\pi, \\ x_1 = & -t_{2,3}/2\mathrm{i}\pi. \end{cases}$$

admitting $\{\mathrm{Li}_l(1)\}_{l \in \mathcal{L}ynX \setminus X}$ as convergent locale coordinates.

Solutions of $(NCDE)$ in $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$ (1/2)

Let the solution of $(NCDE)$ be computed by $\{V_m(\varsigma,z)\}_{m\ge 0}$ satisfying

$$V_m(\varsigma,z)=\sum_{t_{i,j}\in\mathcal{T}_{n-1}}\int_\varsigma^z\Big(\underset{t\in T_n}{\sqcup\!\sqcup}\,e^{[\alpha_\varsigma^z(t)-\alpha_\varsigma^s(t)]t}\Big)\omega_{i,j}(s)t_{i,j}V_{m-1}(\varsigma,s)\Big),$$

$$V_0(\varsigma,z)=\underset{t\in T_n}{\sqcup\!\sqcup}\,e^{\alpha_\varsigma^z(t)t}=\sum_{i_1,\ldots,i_{n-1}\ge 0}((\alpha_\varsigma^z(t_{1,n}^{i_1})t_{1,n}^{i_1})\sqcup\!\sqcup\ldots\sqcup\!\sqcup((\alpha_\varsigma^z(t_{n-1,n}^{i_{n-1}})t_{n-1,n}^{i_{n-1}}).$$

Then V_0 satisfies the partial differential equation

$$\partial_n f=N_{n-1}f,\quad\text{where}\quad N_{n-1}=\sum_{k=1}^{n-1}\omega_{k,n}t_{k,n}$$

and, for any $m\ge 1$, on obtains explicitly

$$V_m(\varsigma,z)=\sum_{w=t_{i_1,j_1}\ldots t_{i_m,j_m}\in\mathcal{T}_{n-1}^*}\int_\varsigma^z\omega_{i_1,j_1}(s_1)\cdots\int_\varsigma^{s_{m-1}}\omega_{i_m,j_m}(s_m)\kappa_w(z,s_1,\cdots,s_m),$$

where

$$\begin{aligned}V_0(\varsigma,z)^{-1}\kappa_w(z,s_1,\cdots,s_m)&=\prod_{p=1}^m e^{\mathrm{ad}_{-\sum_{t\in T_n}\alpha_\varsigma^{s_p}(t)t}}t_{i_p,j_p}\\&=\sum_{q_1,\cdots,q_k\ge 0}\prod_{p=1}^m\frac{1}{q_p!}\mathrm{ad}^{q_p}_{-\sum_{t\in T_n}\alpha_\varsigma^{s_p}(t)t}t_{i_p,j_p}.\end{aligned}$$

Solutions of $(NCDE)$ in $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$ (2/2)

Hence, $V_0(\varsigma,z)^{-1}\kappa_w(z,s_1,\cdots,s_m)=\varphi^{(\varsigma,s_1)}_{t_{\bullet,n}}(t_{i_1,j_1})\ldots\varphi^{(\varsigma,s_m)}_{t_{\bullet,n}}(t_{i_m,j_m})$,
where $\varphi_{t_{\bullet,n}}$ is an automorphisms of $\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle$ defined on letters s.t.

over T_n, $\varphi_{t_{\bullet,n}}\equiv\mathrm{Id}$ and over $\mathcal{T}_{n-1}$, $\varphi^{(\varsigma,z)}_{t_{\bullet,n}}(t_{i,j})=e^{\mathrm{ad}_{-\alpha_\varsigma^{(\varsigma,z)}(t_{i,n})t_{i,n}}}t_{i,j}$.

It can be extended as an injective conc-morphism of $\widehat{\mathcal{A}\langle\mathcal{T}_n\rangle}$ s.t. its adjoint, denoted by $\check{\varphi}_{\bullet,n}$ and restricted in $(\mathcal{A}\langle\mathcal{T}_n\rangle,ш,1_{\mathcal{T}_n^*})$, is an automorphism.
One has $\varphi_{t_{\bullet,n}}(\widehat{\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle})\subseteq\widehat{\mathcal{L}ie_{\mathcal{A}}\langle\mathcal{T}_n\rangle}$ and $\check{\varphi}_{t_{\bullet,n}}(\widehat{\mathcal{A}\langle\mathcal{T}_n\rangle})\subseteq\widehat{\mathcal{A}\langle\mathcal{T}_n\rangle})$.

Theorem

$(NCDE)$ admits $V_0(\varsigma,z)G(\varsigma,z)$ as solution and $G(\varsigma,z)$ is obtained by the Picard's iteration of

$$\mathbf{d}S=M^{t_{\bullet,n}}_{n-1}S,\quad \textit{where}\quad M^{t_{\bullet,n}}_{n-1}(z)=\sum_{1\le i<j\le n-1}\omega_{i,j}(z)\varphi^{(\varsigma,z)}_{t_{\bullet,n}}(t_{i,j}).$$

It can be also obtained, in $\mathcal{A}\langle\langle\mathcal{T}_n\rangle\rangle/\mathcal{J}_{\mathcal{R}_n}$, as follows

$$G(\varsigma,z)=\sum_{w\in\mathcal{T}^*_{n-1}}\alpha^z_\varsigma(\check{\varphi}^z_{t_{\bullet,n}}(w))w=\prod_{l\in\mathcal{L}yn\mathcal{T}_{n-1}}^{\searrow}e^{\alpha^z_\varsigma(\check{\varphi}^z_{t_{\bullet,n}}(S_l))P_l}.$$

There is a holomorphic function in $\mathcal{H}(\mathcal{V})$, $g_{t_{\bullet,n}}$, s.t.

$$M^{t_{\bullet,n}}_{n-1}(z)=\sum_{1\le i<j\le n-1}g^*_{t_{\bullet,n}}\omega_{i,j}(z)t_{i,j}.$$

Solutions of KZ_n $(n \geq 4)$

Now, let $\mathcal{V} = \widetilde{\mathbb{C}_*^n}$, where $C_*^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n | z_i \neq z_j \text{ for } i \neq j\}$ and let us consider the affine plans $(P_{i,j}) : z_i - z_j = 1, 1 \leq i < j \leq n-1$.

Theorem $(\omega_{i,j}(z) = d\log(z_i - z_j), t_{i,j} \leftarrow t_{i,j}/2\mathrm{i}\pi)$

For $z_n \to z_{n-1}$, solution of $(NCDE)$ is in the form $f(z)G(z_1, \ldots, z_{n-1})$ s.t.

1. $f(z) \sim (z_{n-1} - z_n)^{t_{n-1,n}}$ *satisfying* $\partial_n f = N_{n-1} f$, *where*[8]
$$N_{n-1}(z) = \sum_{k=1}^{n-1} t_{k,n} \frac{dz_n}{z_n - z_k} = \sum_{k=1}^{n-1} t_{k,n} \frac{ds}{s - s_k}, \quad \text{with} \begin{cases} s &= z_n, \\ s_k &= z_n - z_k. \end{cases}$$
2. $G(z_1, \ldots, z_{n-1})$ *satisfies* $\mathbf{d}S = M_{n-1}^{t_{\bullet,n}} S$, *where*
$$M_{n-1}^{t_{\bullet,n}}(z) = \sum_{1 \leq i < j \leq n-1} (z_i - z_{n-1})^{-\mathrm{ad}_{t_{i,n}} t_{i,j}} d\log(z_i - z_j).$$
Moreover $M_{n-1}^{t_{\bullet,n}}$ *exactly coincides with* M_{n-1} *in* $\bigcap_{i=1}^{n-1}(P_{i,n-1})$.

Conversely, if f satisfies $\partial_n f = N_{n-1} f$ and $G(z_1, \ldots, z_{n-1})$ satisfies $\mathbf{d}S = M_{n-1}^{t_{\bullet,n}} S$ then $f(z)G(z_1, \ldots, z_{n-1})$ satisfies $(NCDE)$.

---

8. At this stage, z_n is variate, moving towards z_{n-1} while $\{z_k\}_{1 \leq k < n}$ are fixed (and then $d(z_n - z_k) = dz_n$).

Other example of non-trivial case : KZ_4 ($t_{i,j} \leftarrow t_{i,j}/2i\pi$)

$\mathcal{T}_4 = \{t_{1,2}, t_{1,3}, t_{1,4}, t_{2,3}, t_{2,4}, t_{3,4}\}$, $T_4 = \{t_{1,4}, t_{2,4}, t_{3,4}\}$, $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$.

$$\varphi^z_{T_4} = e^{\mathrm{ad}_{-\sum_{t\in T_4} \alpha^z_\varsigma(t)t}} \quad \text{and} \quad \check{\varphi}^z_{T_4} = e^{-\sum_{t\in T_4} \alpha^z_\varsigma(t)t}.$$

Hence,

$$\begin{aligned}
\varphi^z_{t_\bullet,4}(t_{1,4}) &= (z_1 - z_4)^{-ad_{t_{1,4}}} \quad \text{and} \quad \check{\varphi}^z_{t_\bullet,4}(t_{1,4}) = (z_1 - z_4)^{-t_{1,4}},\\
\varphi^z_{t_\bullet,4}(t_{2,4}) &= (z_2 - z_4)^{-\mathrm{ad}_{t_{2,4}}} \quad \text{and} \quad \check{\varphi}^z_{t_\bullet,4}(t_{2,4}) = (z_2 - z_4)^{-t_{2,4}},\\
\varphi^z_{t_\bullet,4}(t_{3,4}) &= (z_3 - z_4)^{-\mathrm{ad}_{t_{3,4}}} \quad \text{and} \quad \check{\varphi}^z_{t_\bullet,4}(t_{3,4}) = (z_3 - z_4)^{-t_{3,4}}.
\end{aligned}$$

For $z_4 \to z_3$, $F(z) = V_0(z)G(z_1, z_2, z_3)$, where $V_0(z) = e^{\sum_{i\leq i\leq 3} t_{i,4}\log(z_i - z_4)}$ and $G(z_1, z_2, z_3)$ satisfies $\mathbf{d}S = M_3^{t_\bullet,4} S$ with

$$M_3^{t_\bullet,4}(z) = \varphi^z_{t_\bullet,4}(t_{1,2})\frac{d(z_1 - z_2)}{z_1 - z_2} + \varphi^z_{t_\bullet,4}(t_{1,3})\frac{d(z_1 - z_3)}{z_1 - z_3} + \varphi^z_{t_\bullet,4}(t_{2,3})\frac{d(z_2 - z_3)}{z_2 - z_3}.$$

Considering $(P_{1,4}) : z_1 - z_4 = 1$, $(P_{2,4}) : z_2 - z_4 = 1$, $(P_{3,4}) : z_3 - z_4 = 1$, one has, in the intersection $(P_{1,4}) \cap (P_{2,4}) \cap (P_{3,4})$,

$$\log(z_1 - z_4) = \log(z_2 - z_4) = \log(z_3 - z_4) = 0 \quad \text{and} \quad \varphi_{t_\bullet,4} \equiv \mathrm{Id}$$

and then $V_0 = 1_{\mathcal{H}(\mathcal{V})}$ and $M_3^{t_\bullet,4}$ exactly coincides with M_3.

Solutions of KZ_n ($n \geq 4$) with asymptotic conditions

Let $F : (\mathbb{C}\langle\mathcal{T}_n\rangle, ш, 1_{\mathcal{T}_n^*}) \to (\mathcal{H}(\mathcal{V}), *, 1_{\mathcal{H}(\mathcal{V})})$ be the character defined by $F_{1_{\mathcal{T}_n^*}} = 1_{\mathcal{H}(\mathcal{V})}$, $\forall t_{i,j} \in \mathcal{T}_n$, $F_{t_{i,j}}(z) = \log(z_i - z_j)$, $\forall t_{i,j} w \in \mathcal{L}yn\mathcal{T}_n \setminus \mathcal{T}_n$,

$$F_{t_{i,j}w}(z) = \int_0^z \omega_{i,j}(s) F_w(s), \quad \text{where} \quad \omega_{i,j}(z) = d\log(z_i - z_j).$$

Corollary ($\omega_{i,j}(z) = d\log(z_i - z_j)$, $t_{i,j} \leftarrow t_{i,j}/2\mathrm{i}\pi$)

1. $\{F_t\}_{t \in \mathcal{T}_n \cup \{1_{\mathcal{T}_n^*}\}}$ *are $\mathcal{C}_0$-linearly free.*
2. F, *being the graph of F, is group like and then $\log F$ is primitive :*

$$\mathrm{F} := \sum_{w \in \mathcal{T}_n^*} F_w w = \prod_{l \in \mathcal{L}yn\mathcal{T}_n}^{\searrow} e^{F_{S_l} P_l} \quad \text{and} \quad \log \mathrm{F} = \sum_{w \in \mathcal{T}_n^*} F_w \pi_1(w),$$

where $\pi_1(w) = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \ldots, u_k \in \mathcal{T}_n \mathcal{T}_n^*} \langle w | u_1 ш \ldots ш u_k \rangle u_1 \ldots u_k.$

3. F *is unique solution of* $\mathbf{d}S = M_n S$ *(and then $C_{\varsigma \rightsquigarrow z} = \mathrm{F}(z)\mathrm{F}^{-1}(\varsigma)$) s.t.*

$$\mathrm{F}(z) \sim_{\substack{z_i \rightsquigarrow z_{i-1} \\ 1 < i \leq n}} (z_{i-1} - z_i)^{t_{i-1,i}} G_i(z_1, \ldots, i-1, i+1, \ldots, z_n)$$

and $G_i(z_1, \ldots, i-1, i+1, \ldots, z_n)$ satisfies $\mathbf{d}S = M_{n-1}^{t_{\bullet,i}} S$, where

$$M_{n-1}^{t_{\bullet,n}}(z) = \sum_{1 \leq i < j \leq n-1} (z_i - z_{n-1})^{-\operatorname{ad}_{t_{i,n}} t_{i,j}} d\log(z_i - z_j).$$

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