# Anomalous statistics of extreme random processes


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- Anomalous fluctuations of “elongated” 2D paths above curved domains

- Spectral density of sparse matrices and 1D random walk trapping in a Poissonian field

- Ultrametric organization of spectral density of random operators and number-theoretic properties of Dedekind $\eta$–function.

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# 2D Random paths above voids of various shapes

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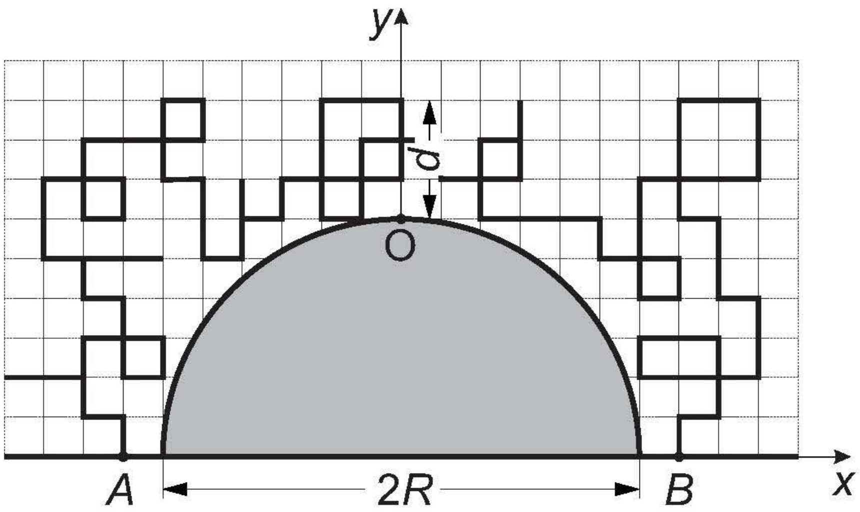


Semicircle:

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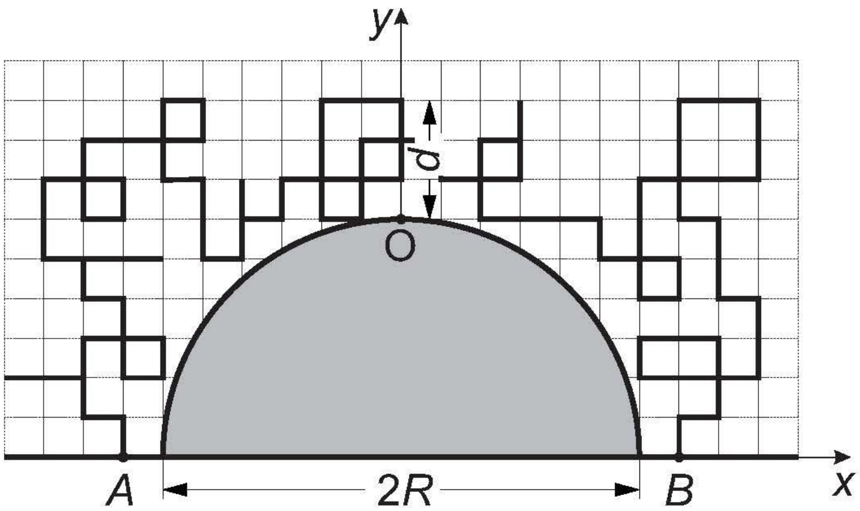


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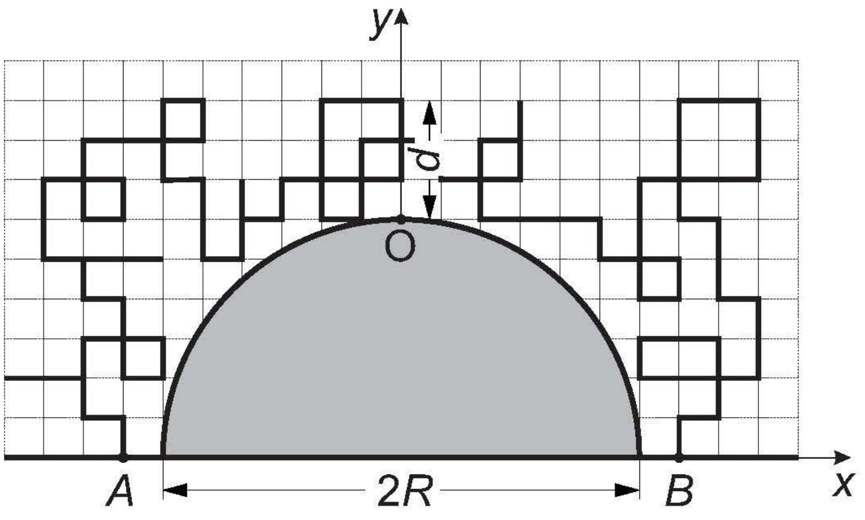


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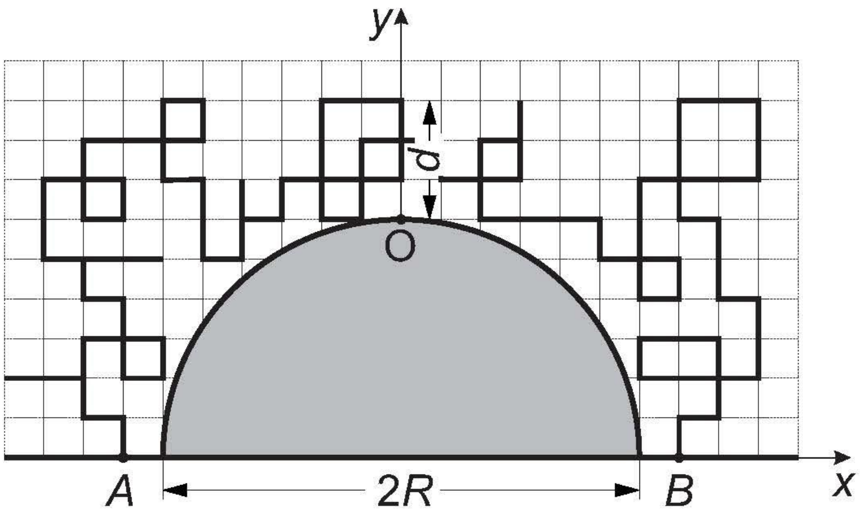


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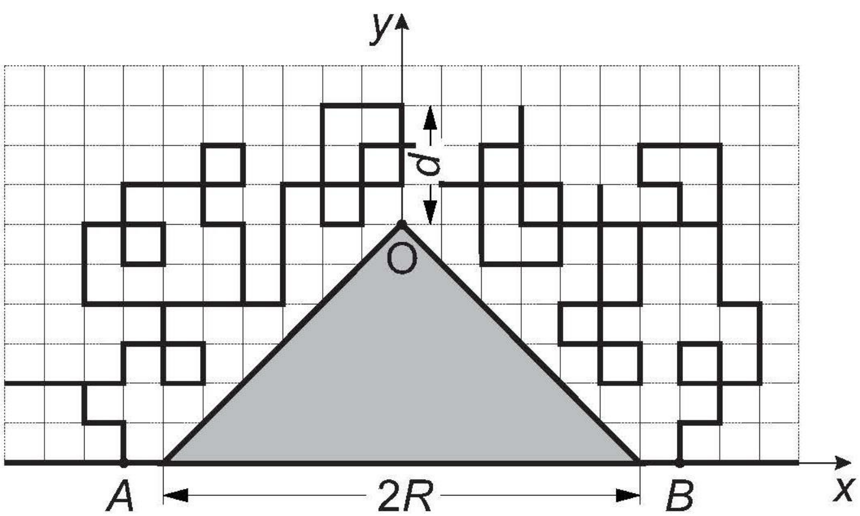


Triangle:

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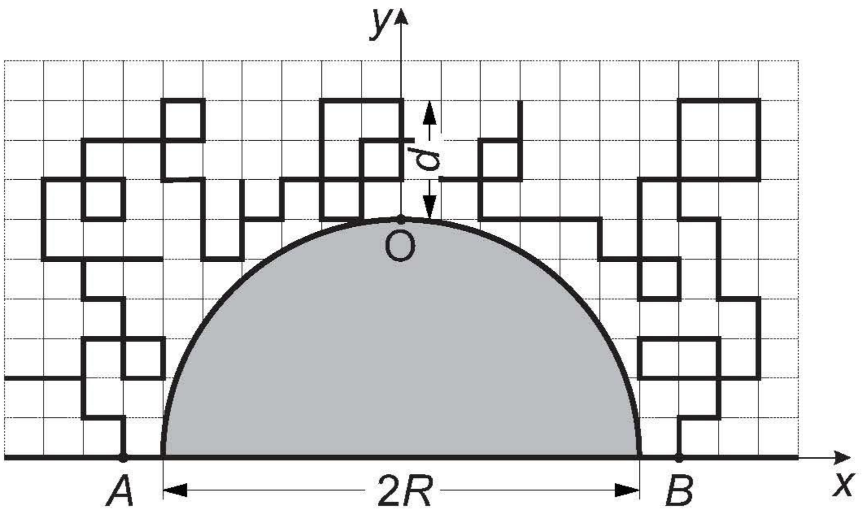


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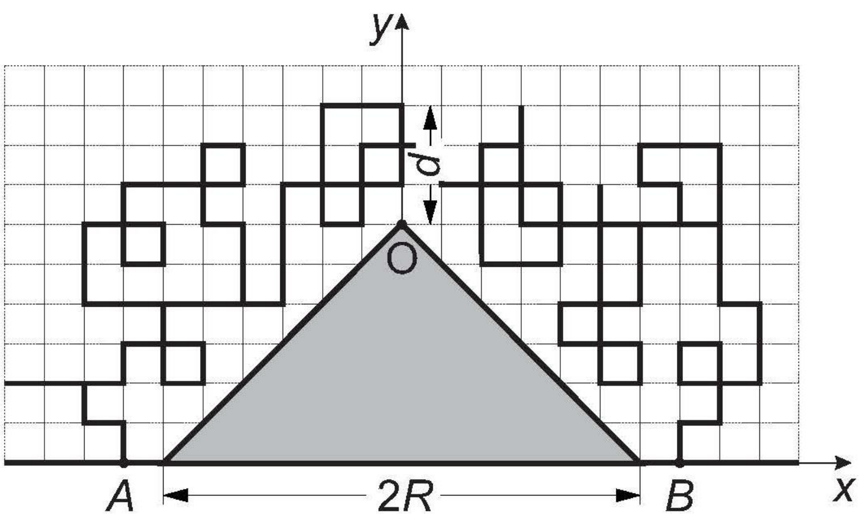


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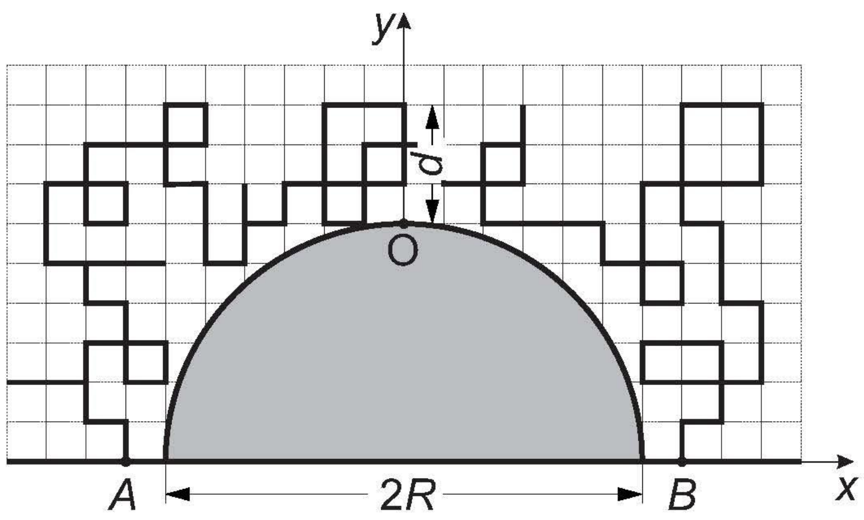


Semicircle:

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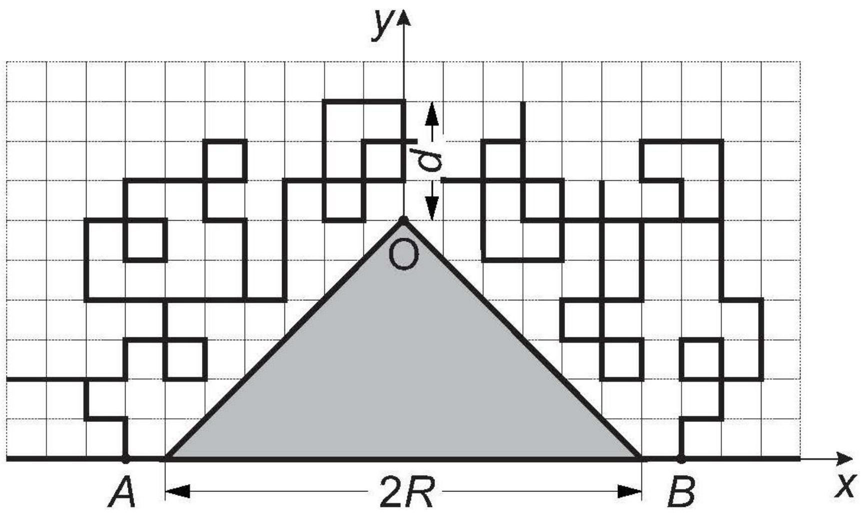
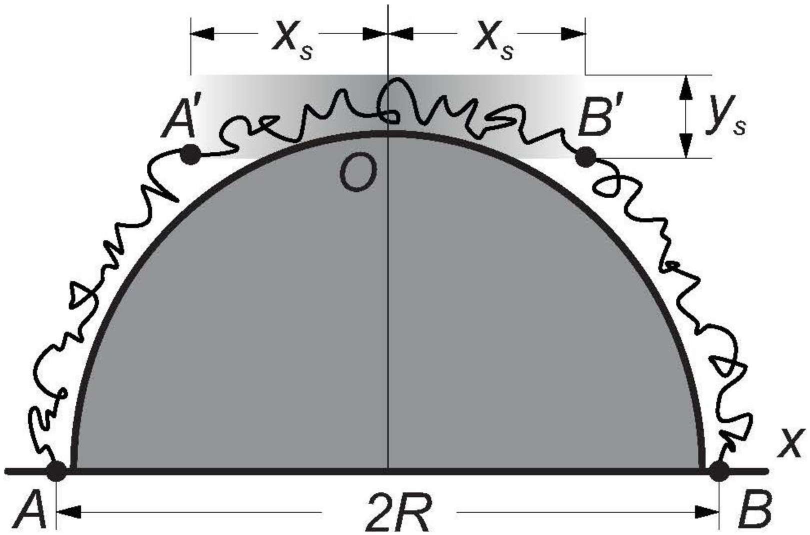


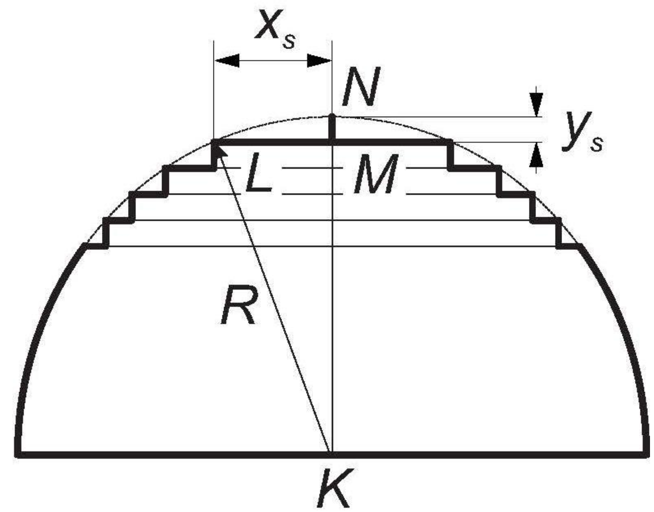

Triangle:

$$N \sim R^2 \rightarrow D \sim \sqrt{N}$$

$$N = cR \rightarrow D \sim \text{const}$$

# Stretched ($N = c\,R$) paths above the semicircle




Linearizing curved shape, we get: $x = \sqrt{R^2 - (R - y)^2} \approx \sqrt{2Ry}$

Above the flat line one has a random walk $y \sim \sqrt{x}$

thus $x \sim \sqrt{R\sqrt{x}}$, and finally, $x \sim R^{2/3}, \quad y \sim R^{1/3}$

Stretching above algebraic curve $\frac{y}{R} \approx \left(\frac{x}{R}\right)^{\eta}$ provides generic scaling

$$y(R) \sim R^{\gamma}; \qquad \gamma = (\eta - 1)/(2\eta - 1)$$

Exponent $\gamma = 1/3$ emerges for uniformly curved surface

# Fluctuations of inflated random loops

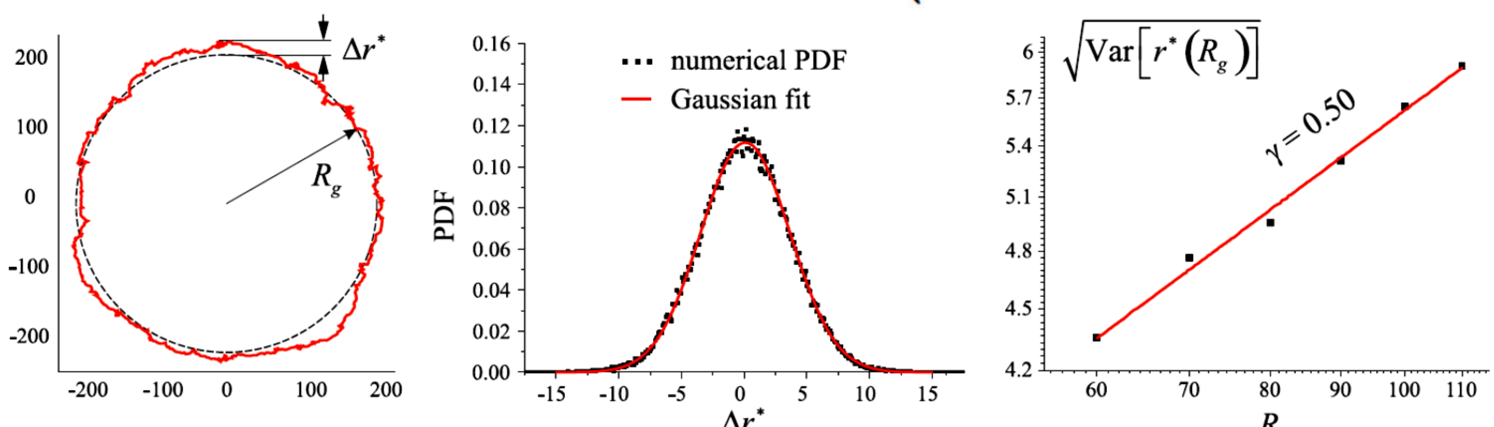
Consider a motion of a charged particle in constant transversal magnetic field $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$, Hamiltonian is $H = \frac{1}{a^2}(\nabla + iq\mathbf{A})^2$

Lamor frequency $\omega = \frac{B|q|}{m}$, select a charge $q = \frac{\pi}{2T} = \frac{\pi}{2N} = \frac{c\pi}{2R_g} \sim \frac{1}{R_g}$

Strong "elongation" of paths: $R_g = cN \ \left(c < \frac{1}{2\pi}\right)$

$$\frac{\partial P(r,\phi,t)}{\partial t} = \left(\nabla^2 + iq(\nabla\mathbf{A} + \mathbf{A}\nabla) - q^2B^2\right) P(r,\phi,t) \qquad P(r,\phi,t) = Q(r,t)$$

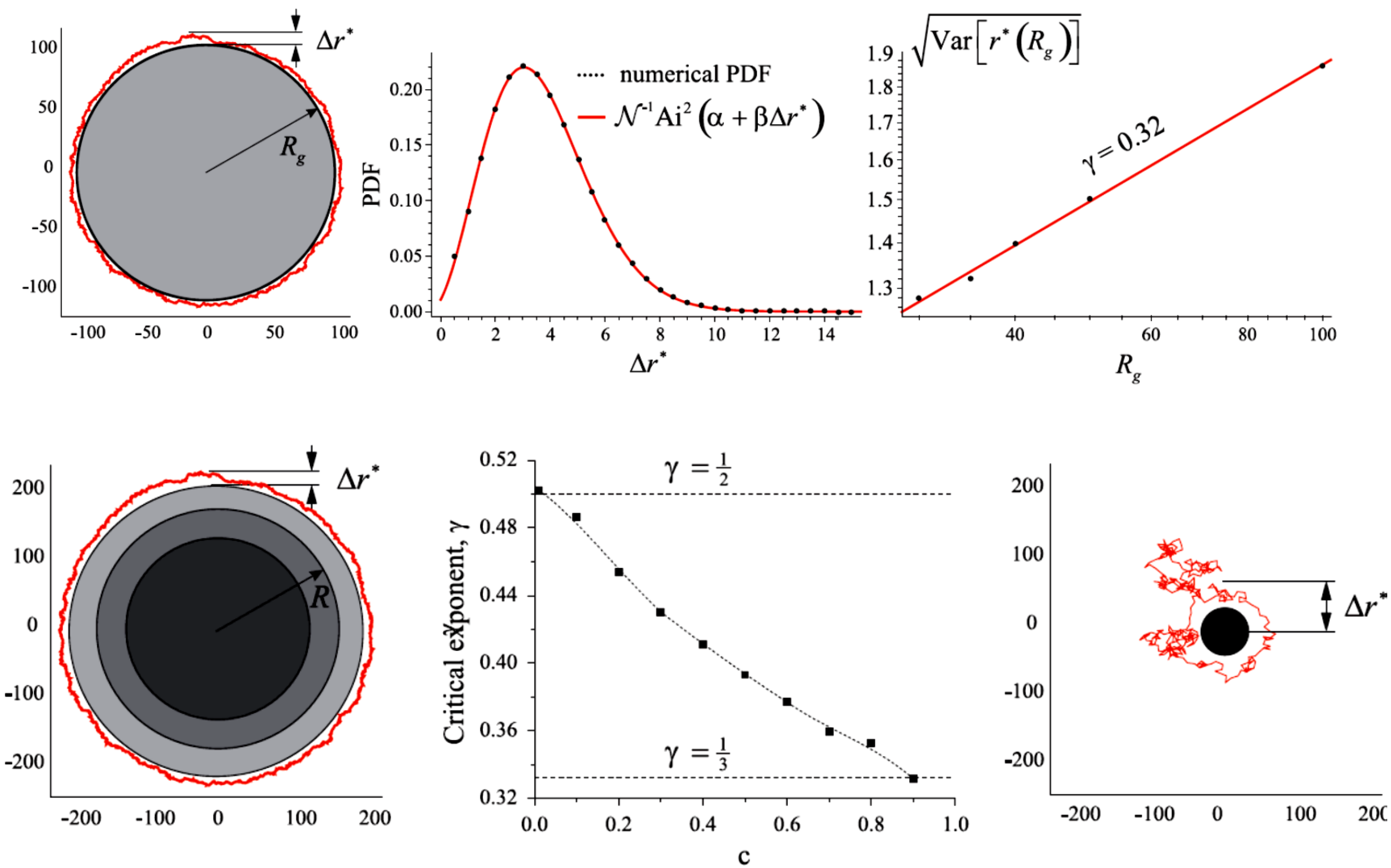
$$Z(r,R_g) = \mathcal{N}^{-1}\, Q(r,t)^2 = \mathcal{N}^{-1} \exp\left(-\frac{c\pi\,(r-R_g)^2}{2R_g}\frac{1+e^{-4\pi}}{1-e^{-4\pi}}\right)$$



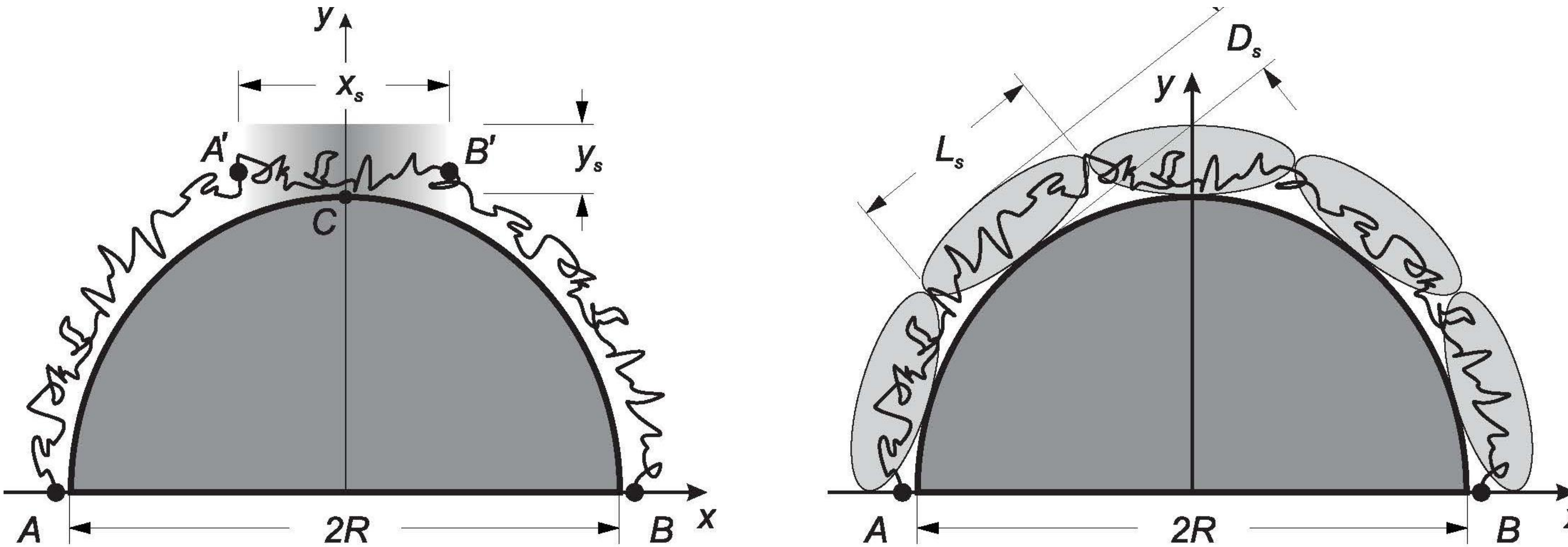
# Fluctuations of inflated random loops “leaning” on an impermeable disc

$$\Delta r^*(N) \propto N^{\gamma(c)}$$

$$Z(r) = \mathcal{N}^{-1} R_g^{-1} \mathrm{Ai}^2\left(\left(\frac{2}{R_g}\right)^{1/3}(r-R_g)+a_1\right)$$



# Free energy of stretched paths: scaling approach




Blob's width $D_s \sim R^{1/3}$, blob's length $L_s \sim R^{2/3}$

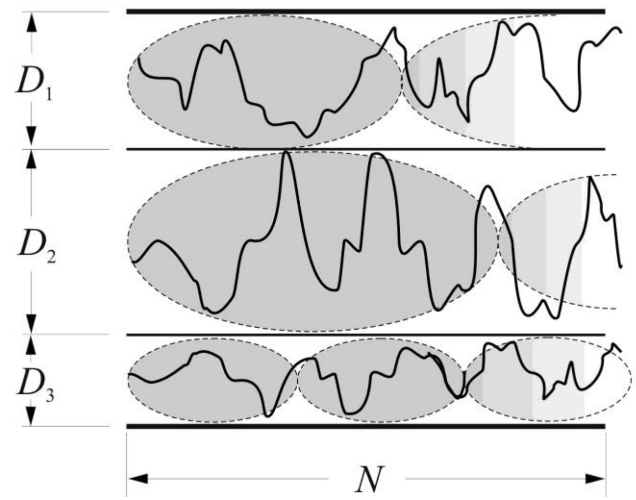
Free energy of a chain stretched above semicircle

$$F \sim \frac{N}{L_s} \sim \frac{R}{R^{2/3}} \sim R^{1/3}$$

Gibbs measure $W(R) = e^{-F(R)} \sim e^{-\alpha R^{1/3}}$

# Lifshitz tail in optimal fluctuation for survival probability in a one-dimensional Poissonian field of random segments

Estimate survival probability in an ensemble of random intervals $D$ with the distribution $Q(D) \sim p^D$




Free energy to be minimized over $D$:

$$F(N,D) = F_e(N) - \ln Q(D)$$

where $F_e(D,N) \sim \dfrac{N}{D^2}; \quad \ln Q(D) \sim D \ln p \quad (\ln p = \beta)$

Correspondingly, $\bar{D}(N) \sim N^{1/3}$ and survival probability (Balagurov, Vaks, 1974) is

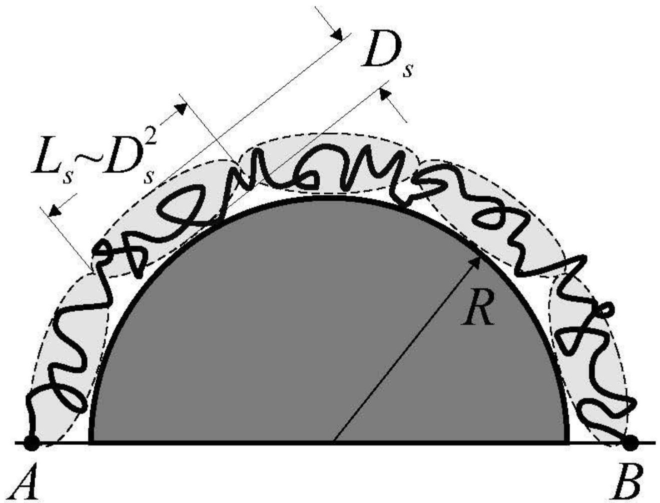
$$P(N) = e^{-F(R,\bar{D})} \sim e^{-\beta^{2/3} N^{1/3}}$$

The inverse Laplace transform gives the Lifshitz tail of 1D Anderson localization

$$P(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} P(N) e^{sN} \sim e^{-\beta/\sqrt{s}}$$

# Comparison of disorder-free stretched KPZ exponent above convex boundary and Lifshitz tail in a Poissonian field

Gibbs measure of stretched path in curved channel



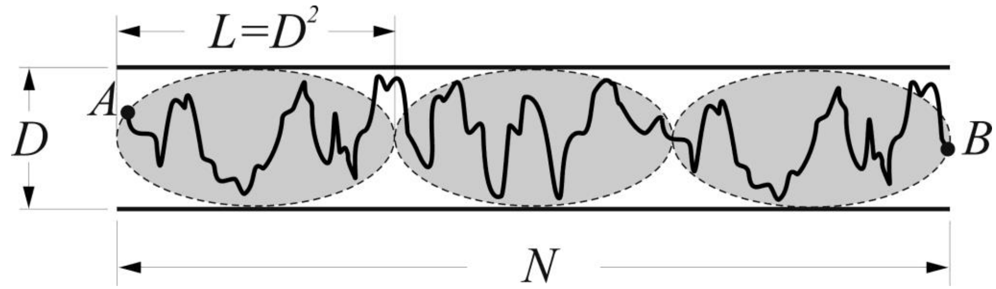

$$F(N,D) \sim \frac{N}{D^2} + \frac{(R+D)^2}{N}\Bigg|_{D<<R} \sim \frac{N}{D^2} + \frac{RD}{N}$$

Tube width at $R \sim cN$ is $\bar{D}(N) \sim N^{1/3}$

No disorder, however path is stretched above semicircle

$$W(R \sim N) \sim e^{-\alpha N^{1/3}}$$

Survival probability in 1D trapping in Poissonian field




Free energy has to be minimized over $D$

$$F(N,D) = F_e(N) - \ln Q(D) \sim \frac{N}{D^2} - \beta D$$

In Poissonian disorder one gets

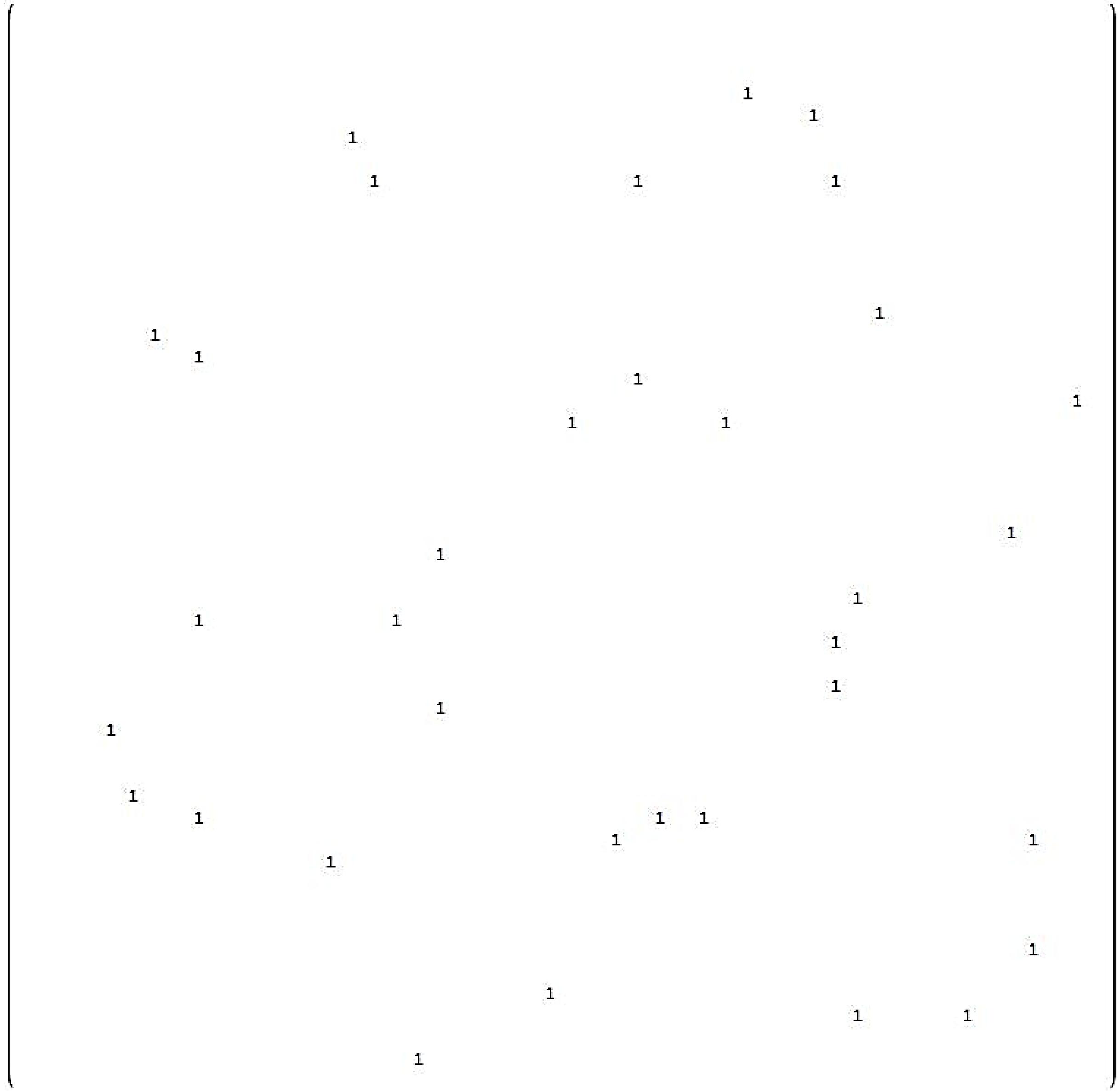
$$\bar{D}(N) \sim N^{1/3}$$

The survival probability is

$$P(N) = e^{-F(R,\bar{D})} \sim e^{-\beta^{2/3} N^{1/3}}$$

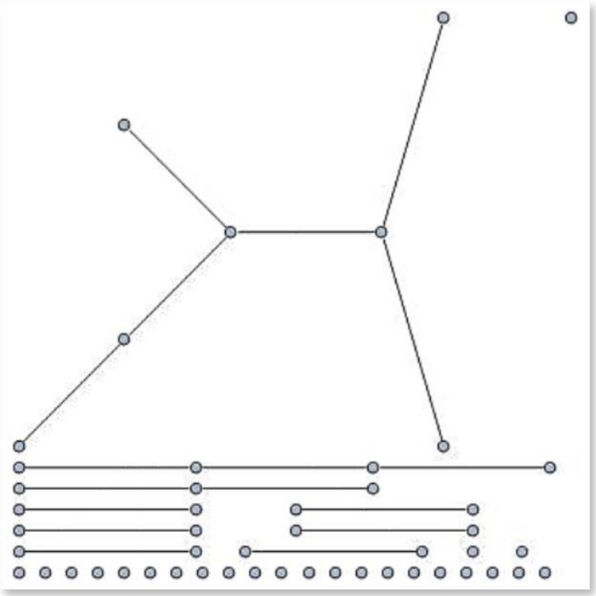
# Spectral density of sparse matrices and 1D random walk trapping in a Poissonian field

# Spectral statistics of sparse random matrices



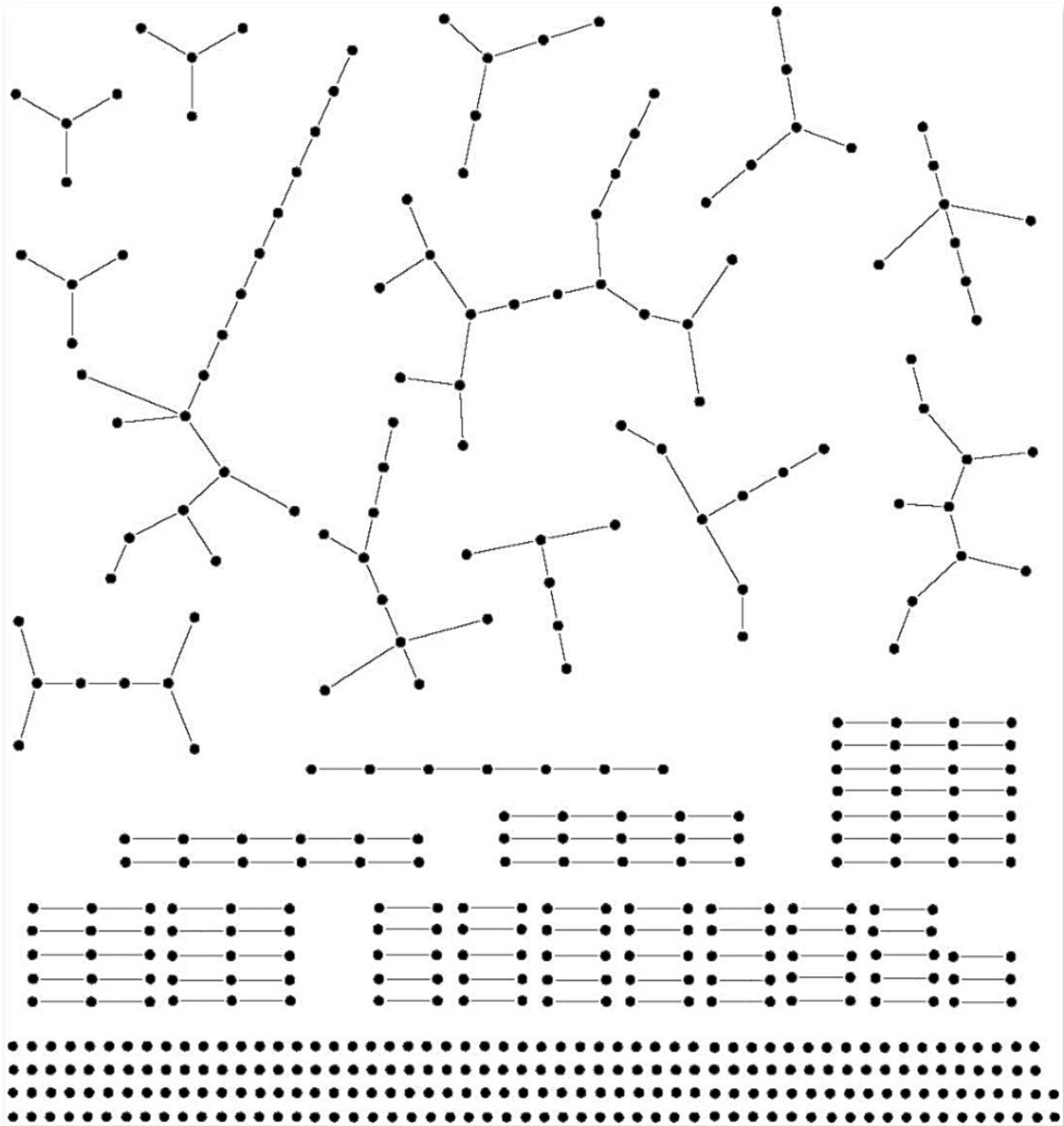
$$a_{ij} = \begin{cases} 1 & \text{prob } r \\ 0 & \text{prob } 1-r \end{cases}$$

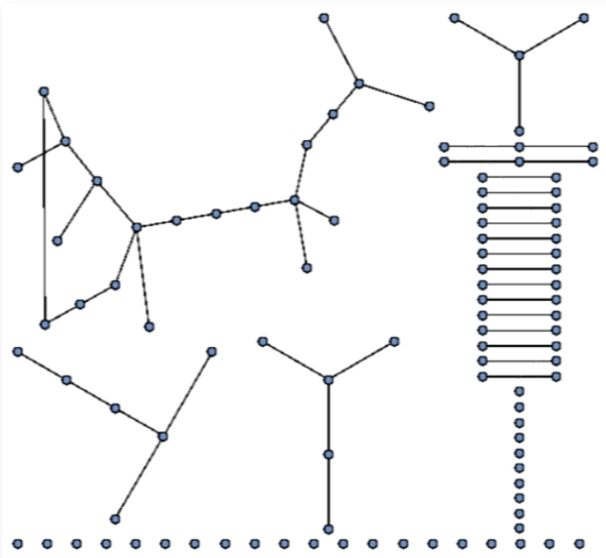
$$N \times N = 50 \times 50$$

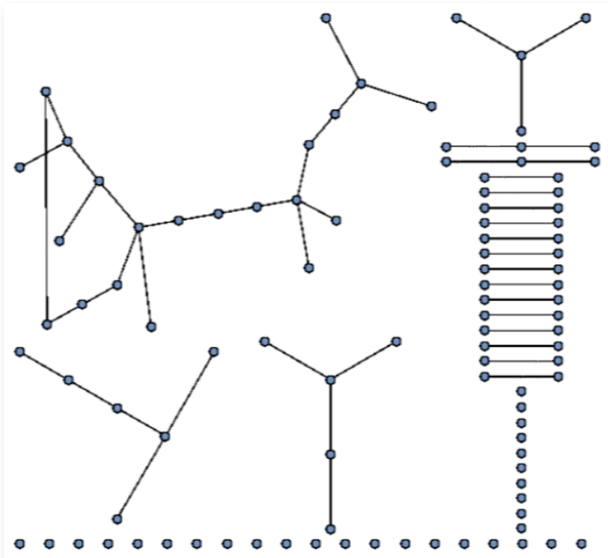


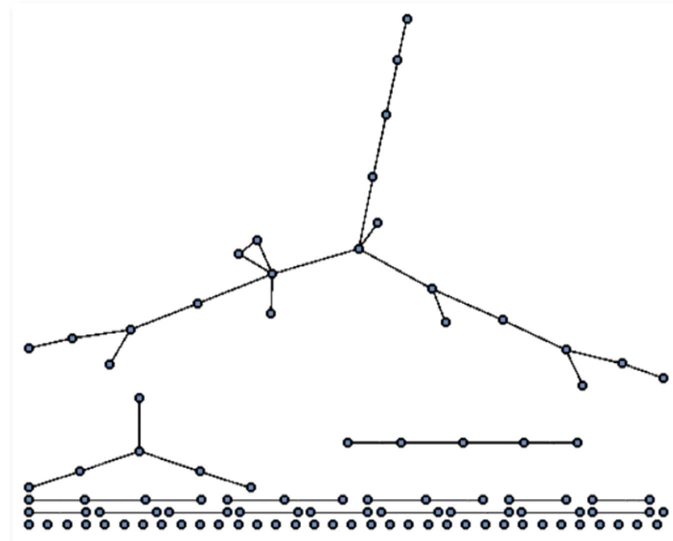
$r = 1/N$, is the percolation threshold ($N \gg 1$)

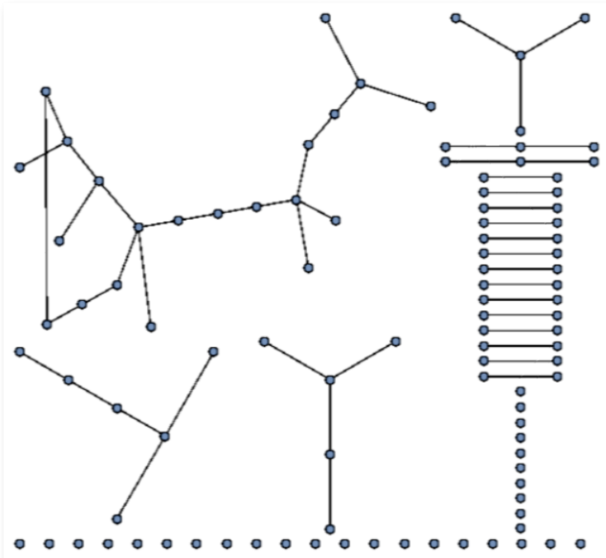
# Sample of collection of subgraphs for one realization of adjacency matrix, $r = 1/N$ ($N = 500$)

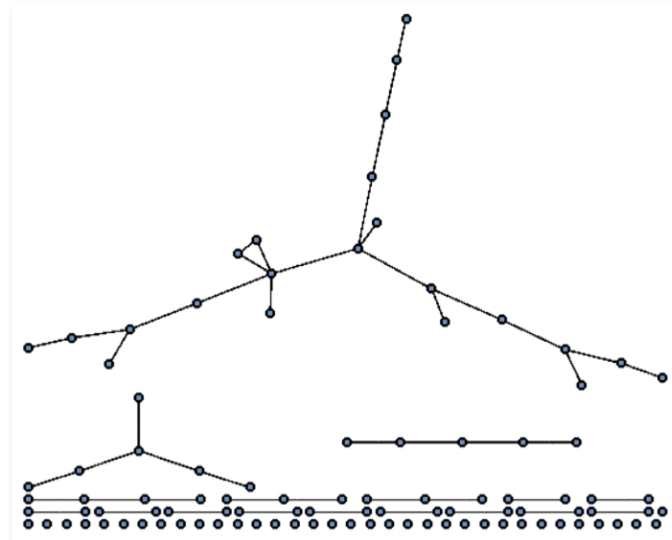


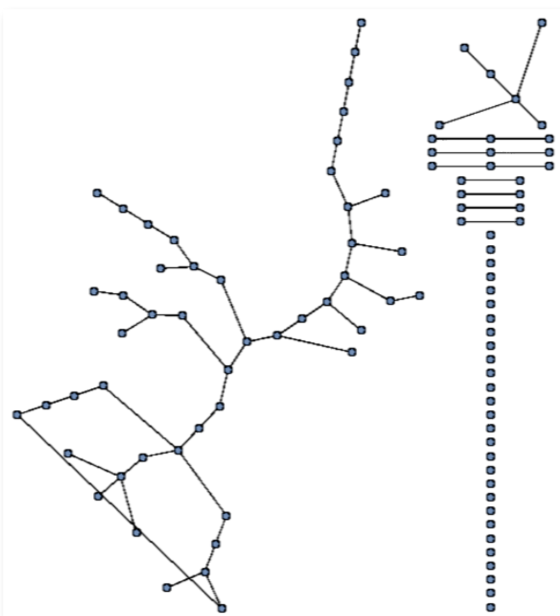


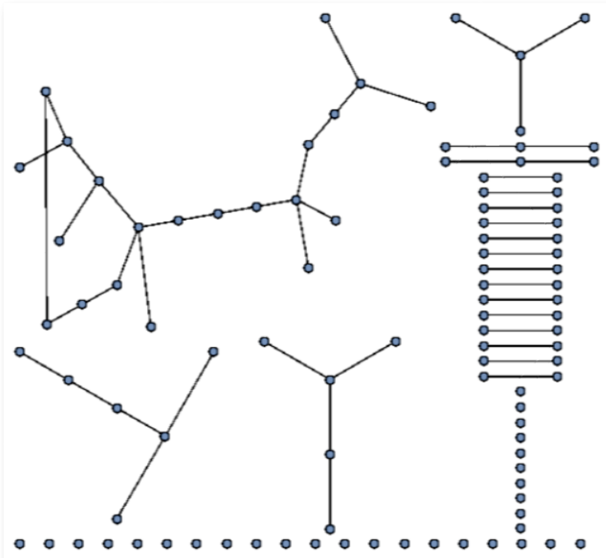


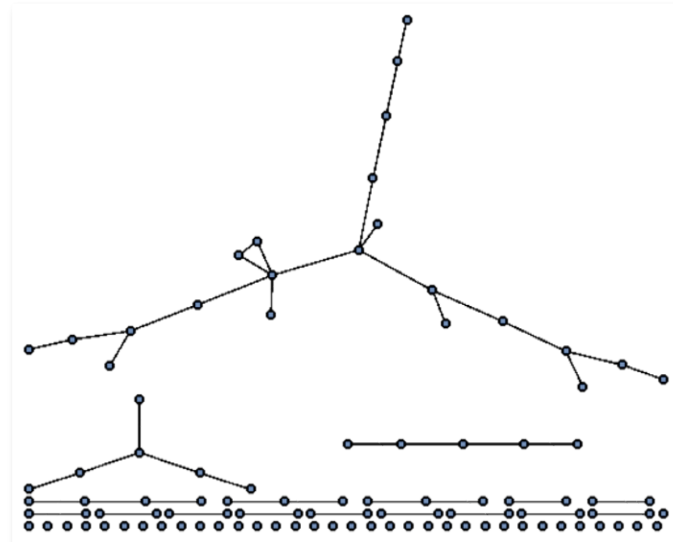


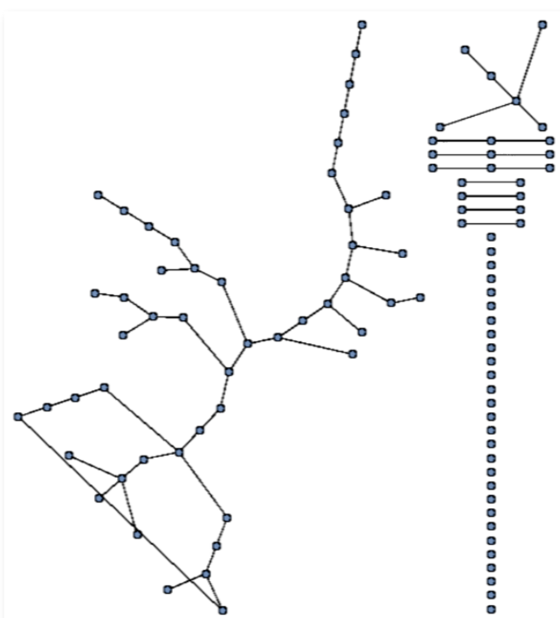


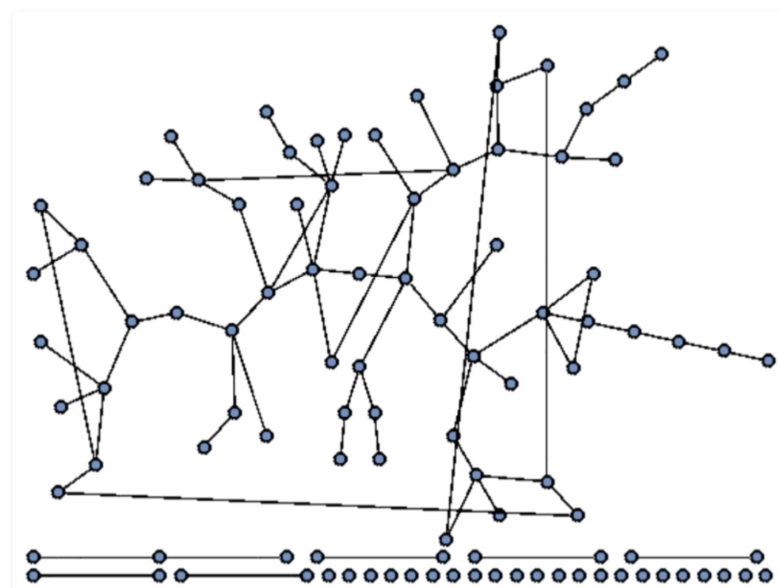


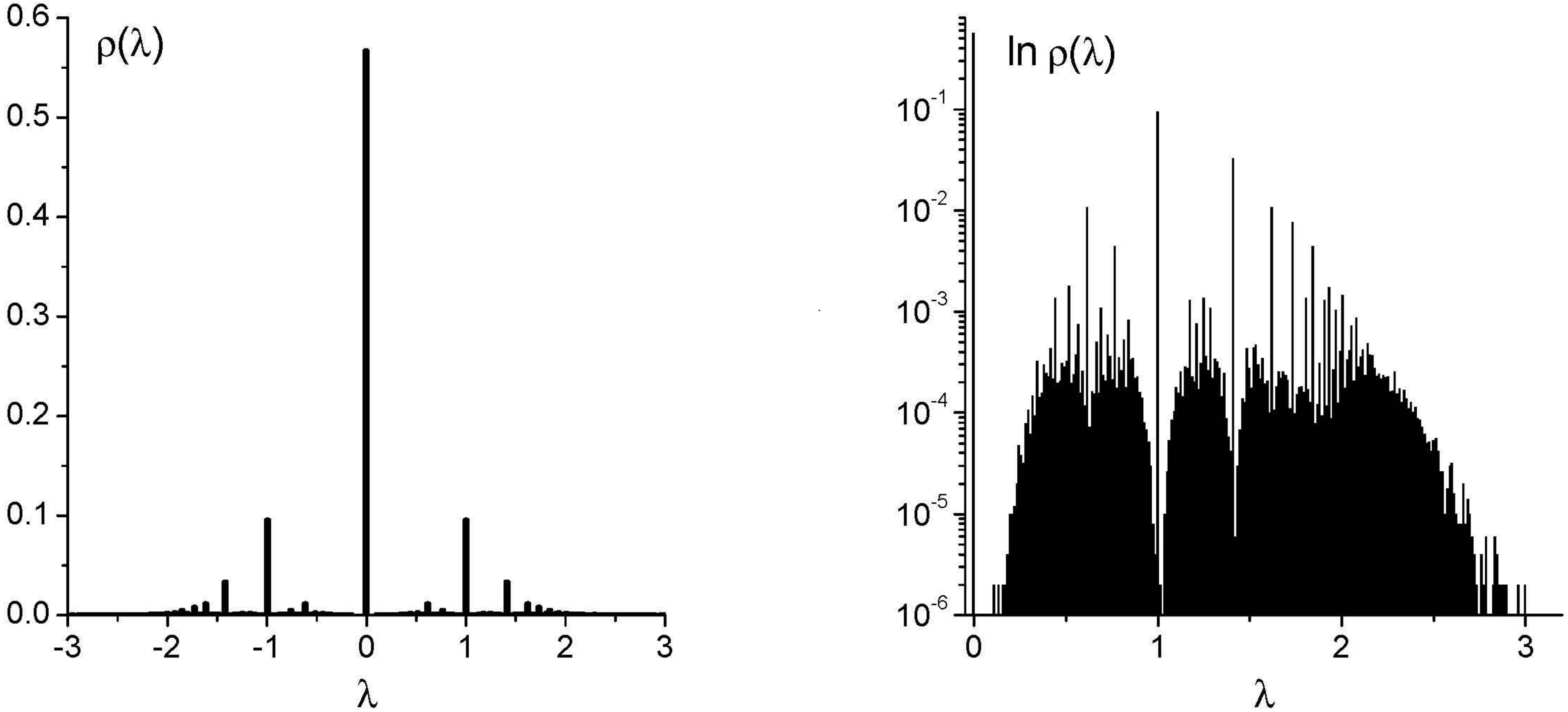













Spectral density $\rho(\lambda)$ of sparse random adjacency matrix has regular hierarchical structure

At percolation threshold $\sim\left(2-e^{-1}\right)/(e-1)\times 100\%$ ~ 95% of subgraphs are linear chain with distribution in length $P(L) \sim e^{-L}$

(V. Avetisov, P. Krapivsky, S.N., 2016)

# Samples of eigenvalue densities of sparse networks of different physical/biological nature

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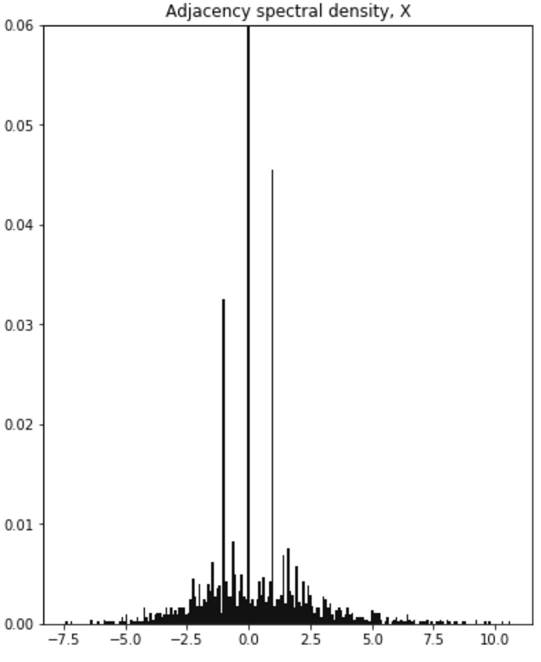
Typical spectral statistics of adjacency matrix of *X chromosome*




eigenvalue

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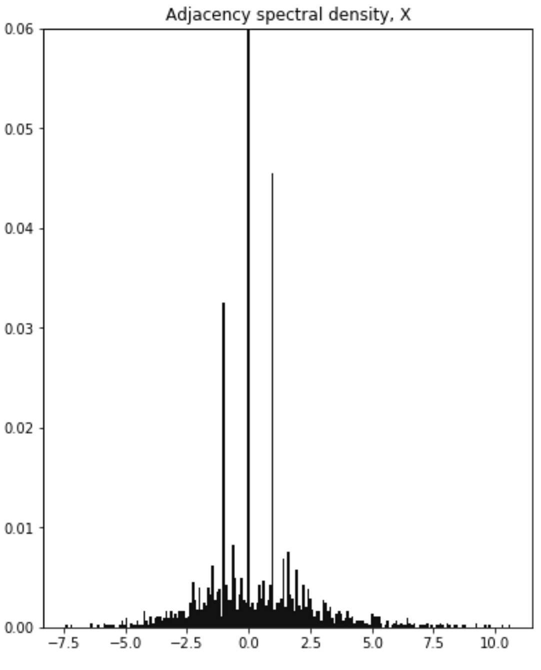
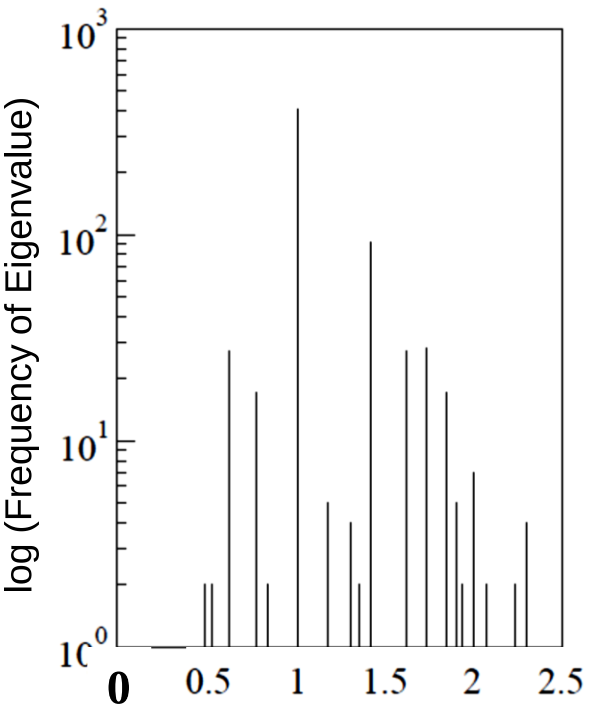



eigenvalue

Spectral statistics of protein-protein interaction network in *Drosophyla melanogaster*

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C. Kamp, K. Christensen (Phys.Rev E, 2005)

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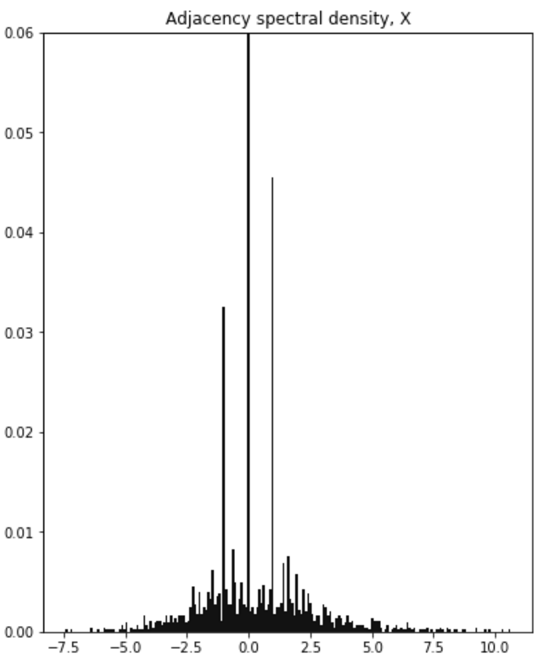


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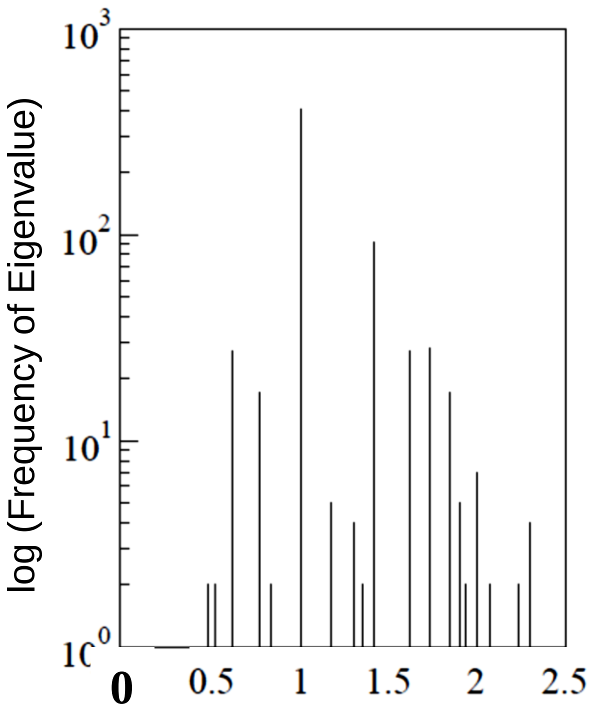


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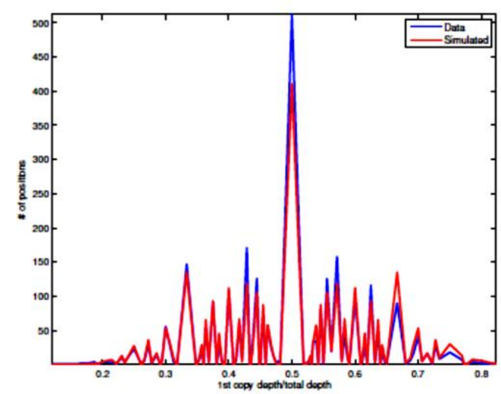
C. Kamp, K. Christensen (Phys.Rev E, 2005)

Distributions of ratios from heterozygous single nucleotide polymorphisms data from the *sequencing of a cancer genome*




V. Trifonov, L. Pasqualucch, R. Dalla-Favera, R.Rabadan (Sci. Rep, 2011)

# Spectral statistics of Schrödinger-like random operators

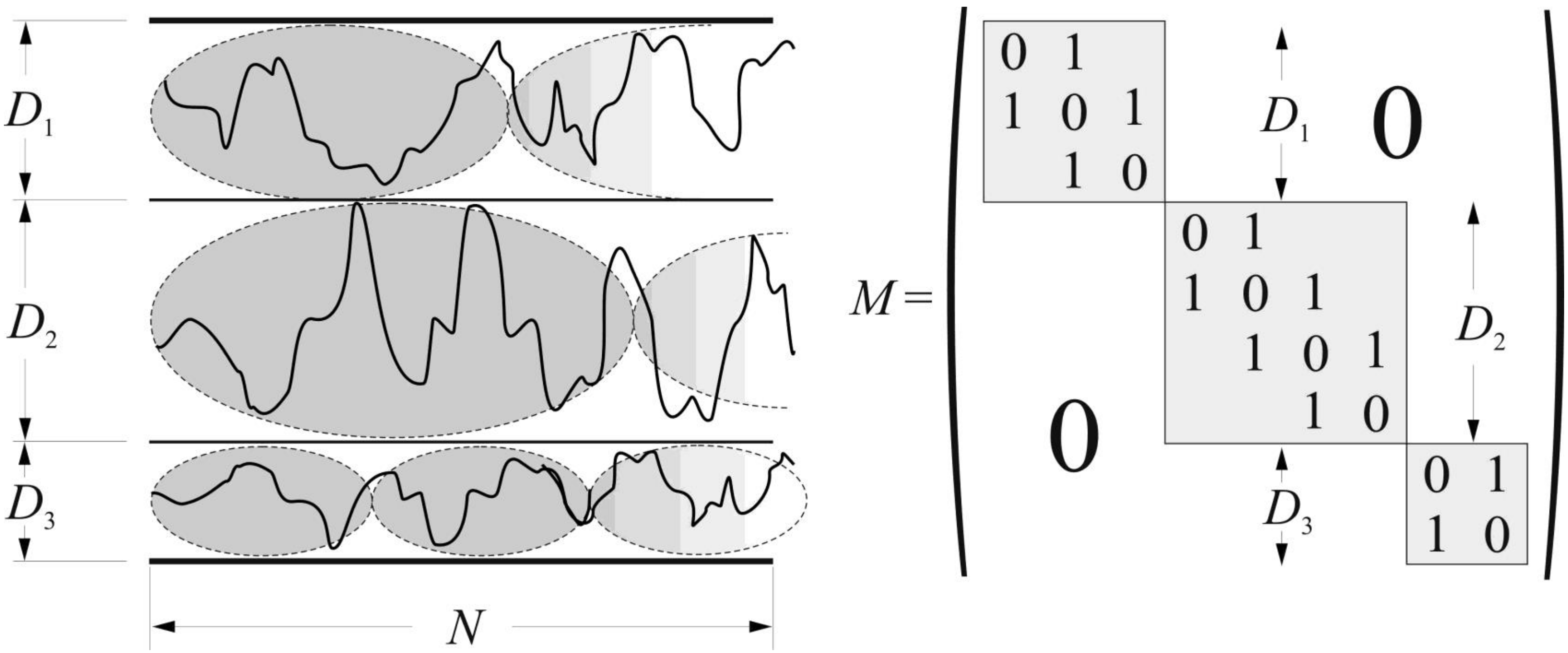
Consider an ensemble of two (three)-diagonal matrices

$$A_N = \begin{pmatrix} 0 & x_1 & 0 & \cdots & 0 \\ x_1 & 0 & x_2 & & \\ 0 & x_2 & 0 & & \\ \vdots & & & & \\ & & & & x_{N-1} \\ 0 & & & x_{N-1} & 0 \end{pmatrix}$$

where the matrix elements are:

$$x_k = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q = 1 - p \end{cases}$$

# Adjacency matrix splits into cells with the distribution $Q(D) \sim p^D$ $(0 < p < 1)$

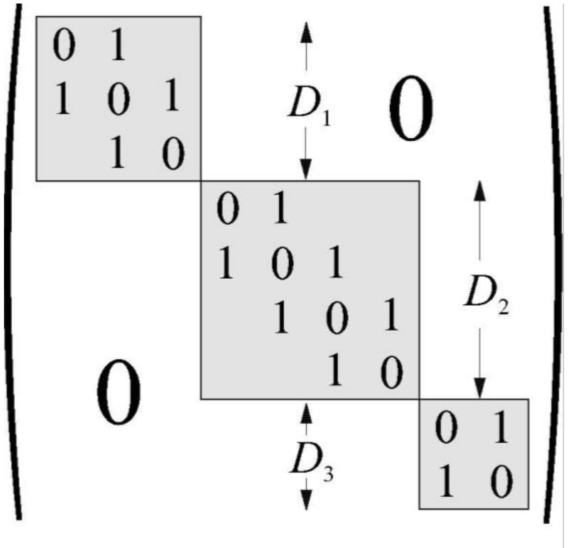



Matrix $N$ x $N$ $(N >> 1)$

Set of eigenvalues in the cell of size $D$ is

$$\lambda_k(D) = -2\cos\frac{\pi k}{D+1} \quad (k = 1, \dots, D)$$

# Spectral density of adjacency matrix




Spectral density, $\rho(\lambda)$, of ensemble of random matrices is:

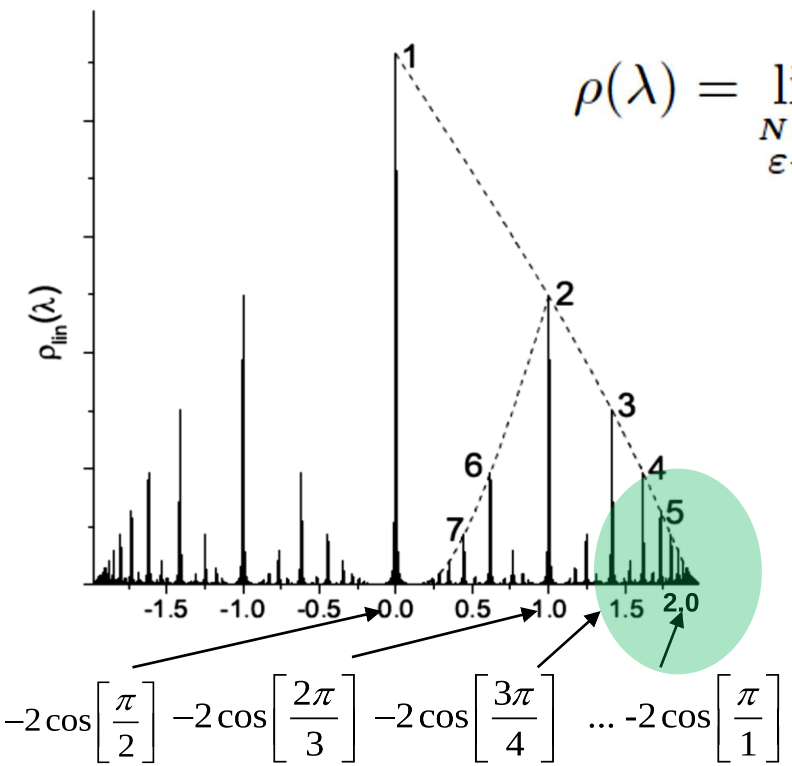
$$\rho(\lambda) = \lim_{N\to\infty} \frac{1}{N} \left\langle \sum_{n=1}^{N} \sum_{k=1}^{D} \delta(\lambda - \lambda_{k,D}) \right\rangle$$

$$= \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{\varepsilon}{\pi N} \sum_{n=1}^{N} Q_D \sum_{k=1}^{D} \mathrm{Im} \frac{1}{\lambda - \lambda_{k,D} - i\varepsilon}$$

Counting contributions from exponentially weighted cells, we get:

$$\rho(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{1}{\pi N} \sum_{n=1}^{N} p^D \sum_{k=1}^{D} \frac{\varepsilon}{\left(\lambda - 2\cos\frac{\pi k}{D+1}\right)^2 + \varepsilon^2}$$

# Lifshitz tail of the spectral density $\rho(\lambda)$ at the **spectral edge**




$$\rho(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{1}{\pi N} \sum_{n=1}^{N} p^n \sum_{k=1}^{n} \frac{\varepsilon}{\left(\lambda - 2\cos\frac{\pi k}{n+1}\right)^2 + \varepsilon^2}$$

$$\lambda = -2\cos\frac{\pi D}{D+1}\Big|_{D\to\infty} \to 2 - \frac{\pi^2}{D^2},$$

$$\rho^{S_1}(\lambda_k)\Big|_{k\to\infty} \to p^D$$

Lifshitz tail of 1D Anderson localization

$$\rho(\lambda \to 2) \to p^{\pi/\sqrt{2-\lambda}}$$

# Ultrametric organization of spectral density of random operators and number-theoretic properties of Dedekind $\eta-$ function.

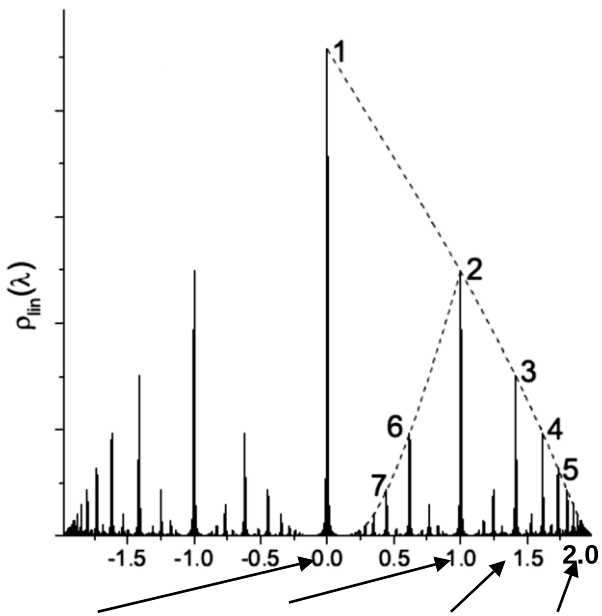
# Definition of the ultrametric space

The pairwise distance, $d(x_1, x_2)$ between elements $x_1$ and $x_2$ is *ultrametric* if it meets three requirements:

- It is non-negative
  $d(x_1, x_2) > 0$ for $x_1 \neq x_2$ and $d(x_1, x_2) = 0$ for $x_1 = x_2$
- It is symmetric
  $d(x_1, x_2) = d(x_2, x_1)$
- It obeys *the strong triangle inequality*,
  $d(x_1, x_2) < \max\{d(x_1, x_3), d(x_3, x_2)\}$

instead of the ordinary triangle inequality typical for Euclidean spaces, $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$

# Positions of peaks are defined by composition rules for Farey numbers (Ford circles)




$$-2\cos\left[\frac{\pi}{2}\right] \quad -2\cos\left[\frac{2\pi}{3}\right] \quad -2\cos\left[\frac{3\pi}{4}\right] \quad \ldots \; -2\cos\left[\frac{\pi}{1}\right]$$

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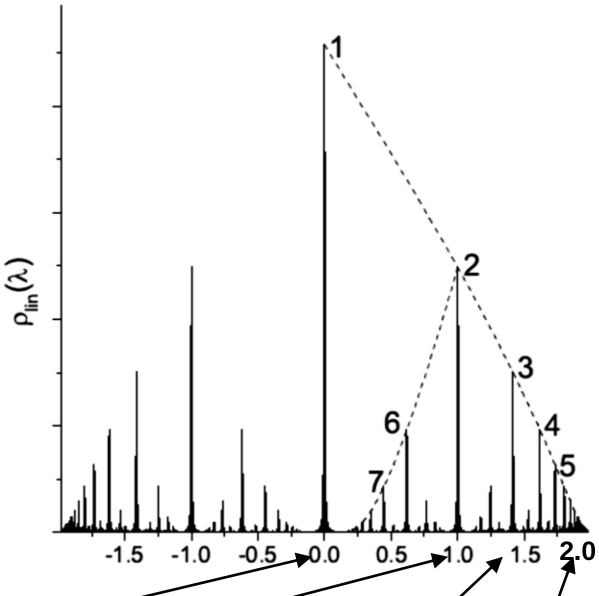


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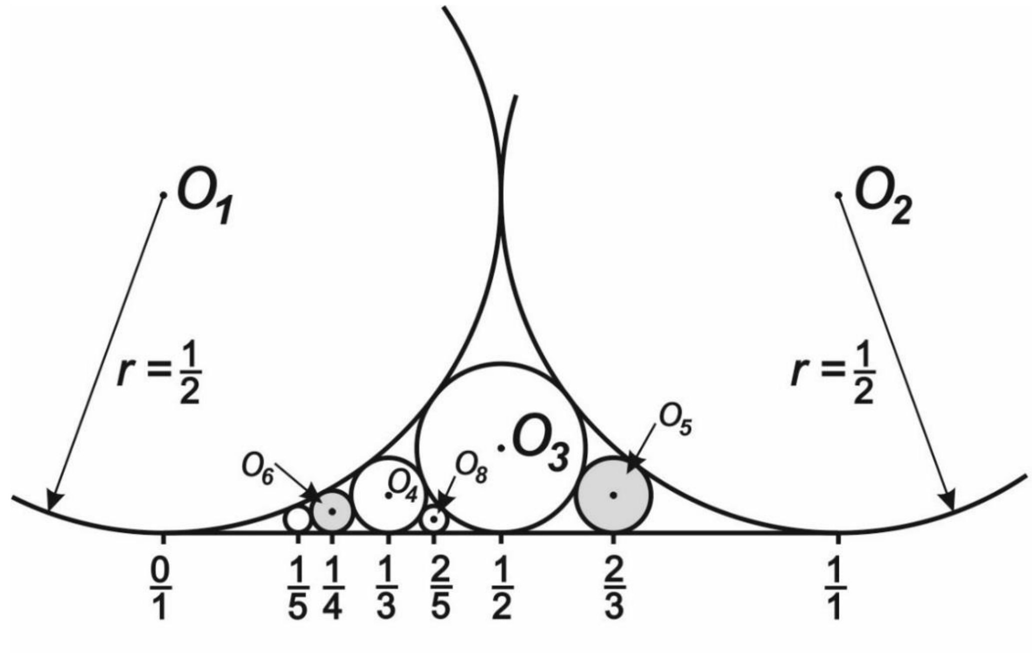

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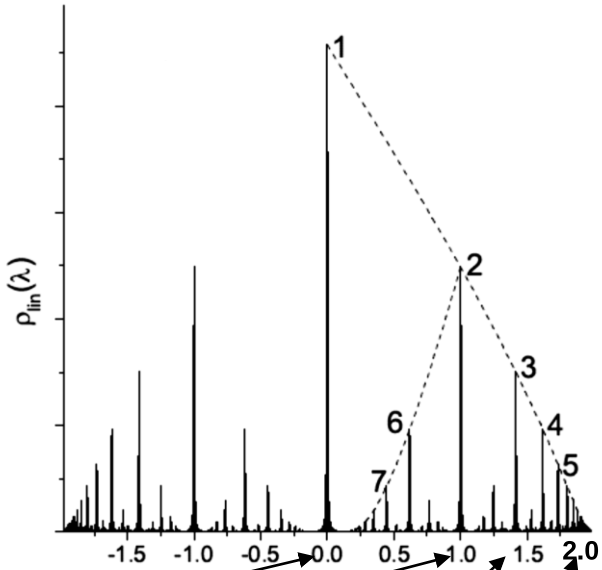
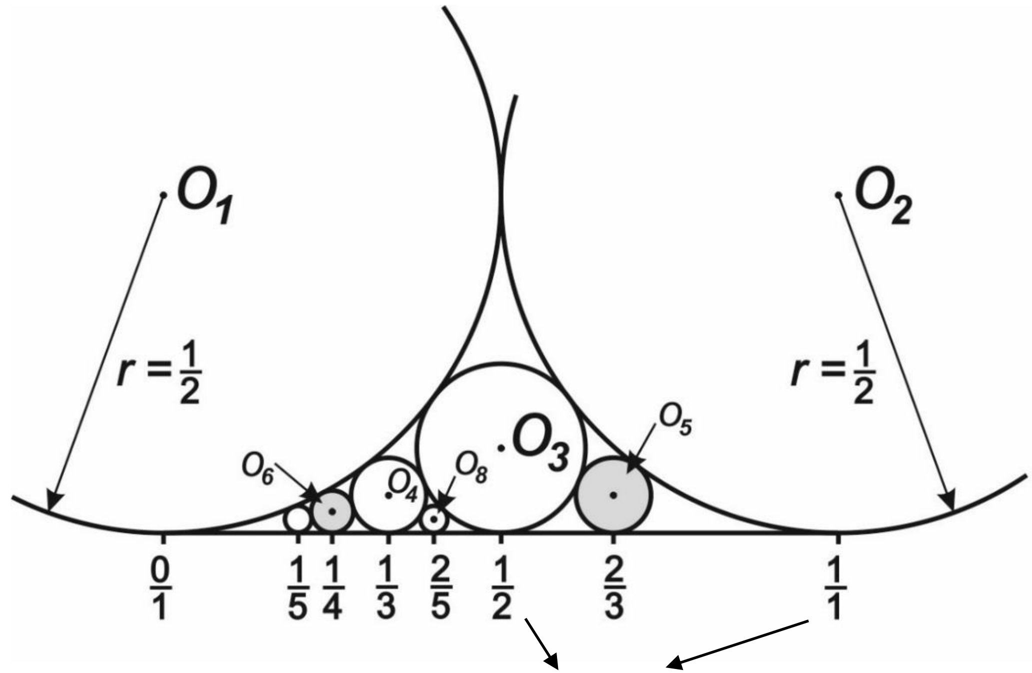



$$-2\cos\left[\frac{\pi}{2}\right] \quad -2\cos\left[\frac{2\pi}{3}\right] \quad -2\cos\left[\frac{3\pi}{4}\right] \quad \ldots\ -2\cos\left[\frac{\pi}{1}\right]$$




$$\frac{1}{2} \oplus \frac{1}{1} = \frac{2}{3}$$

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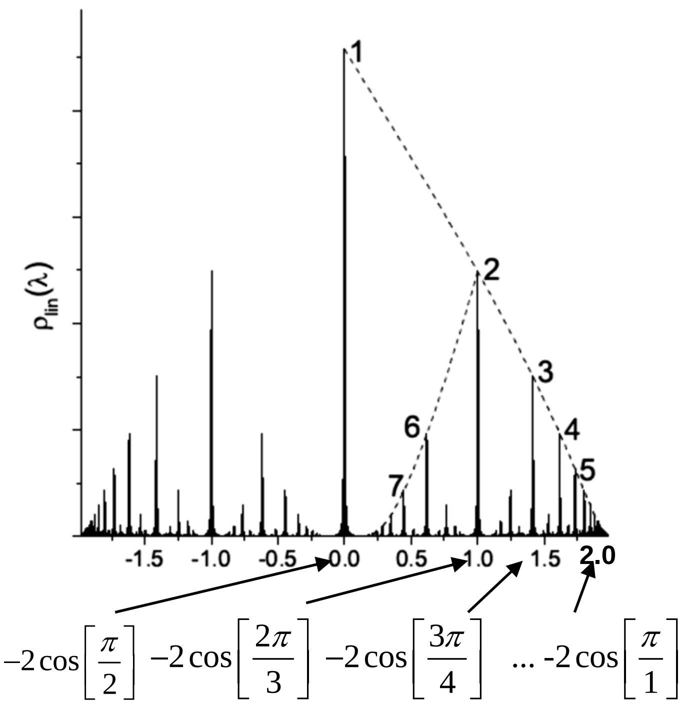


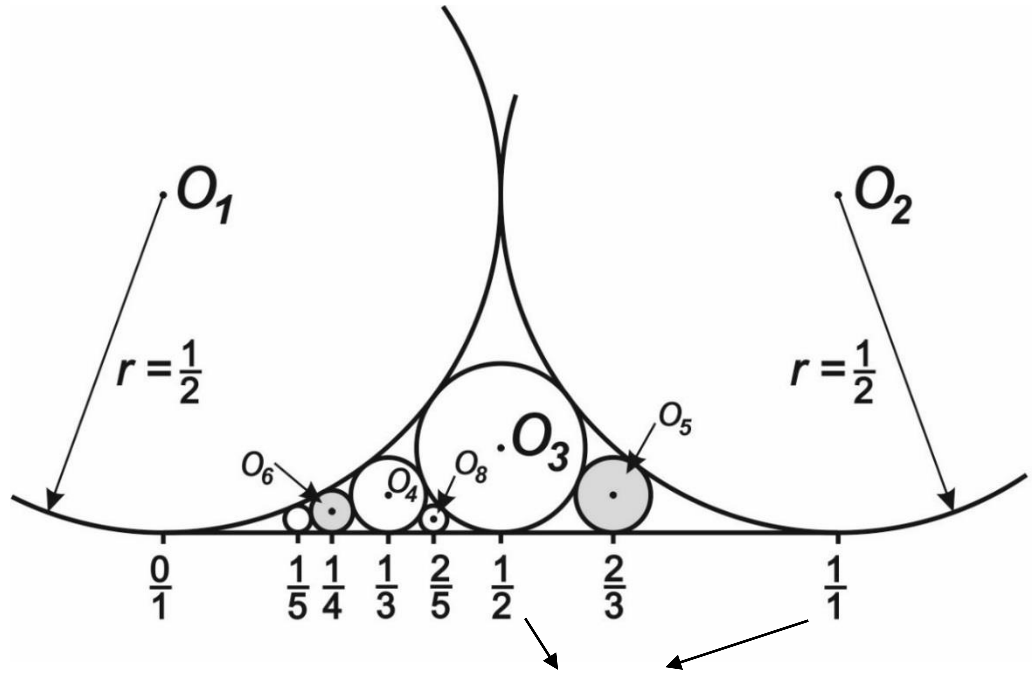


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$$\frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}} \oplus \frac{p_{n+1}}{q_{n+1}} = \frac{p_{n-1} + p_{n+1}}{q_{n-1} + q_{n+1}}$$

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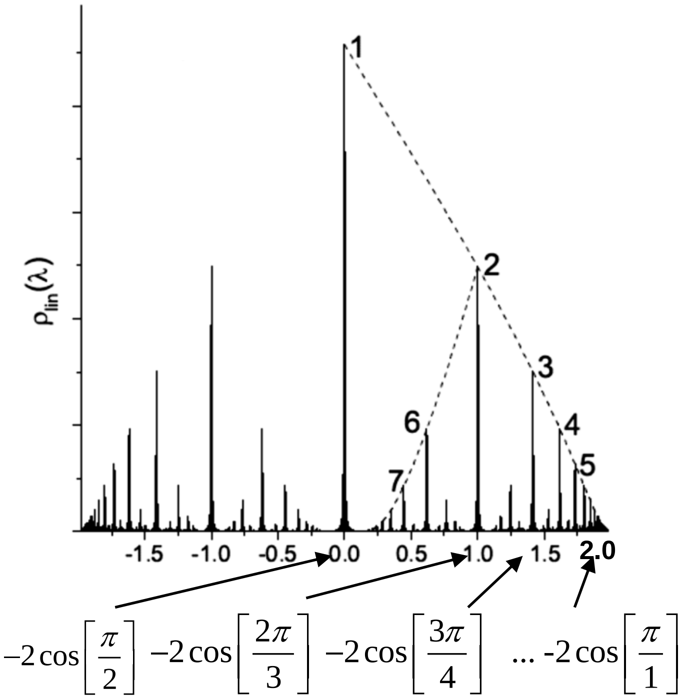


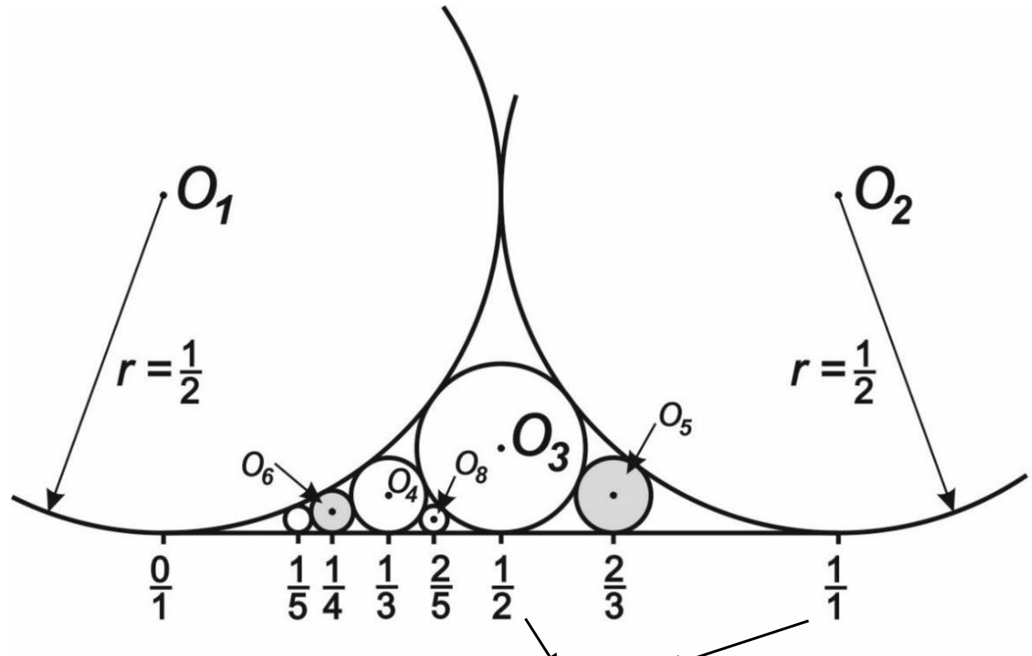


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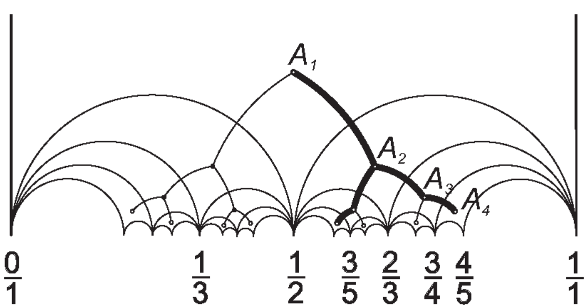


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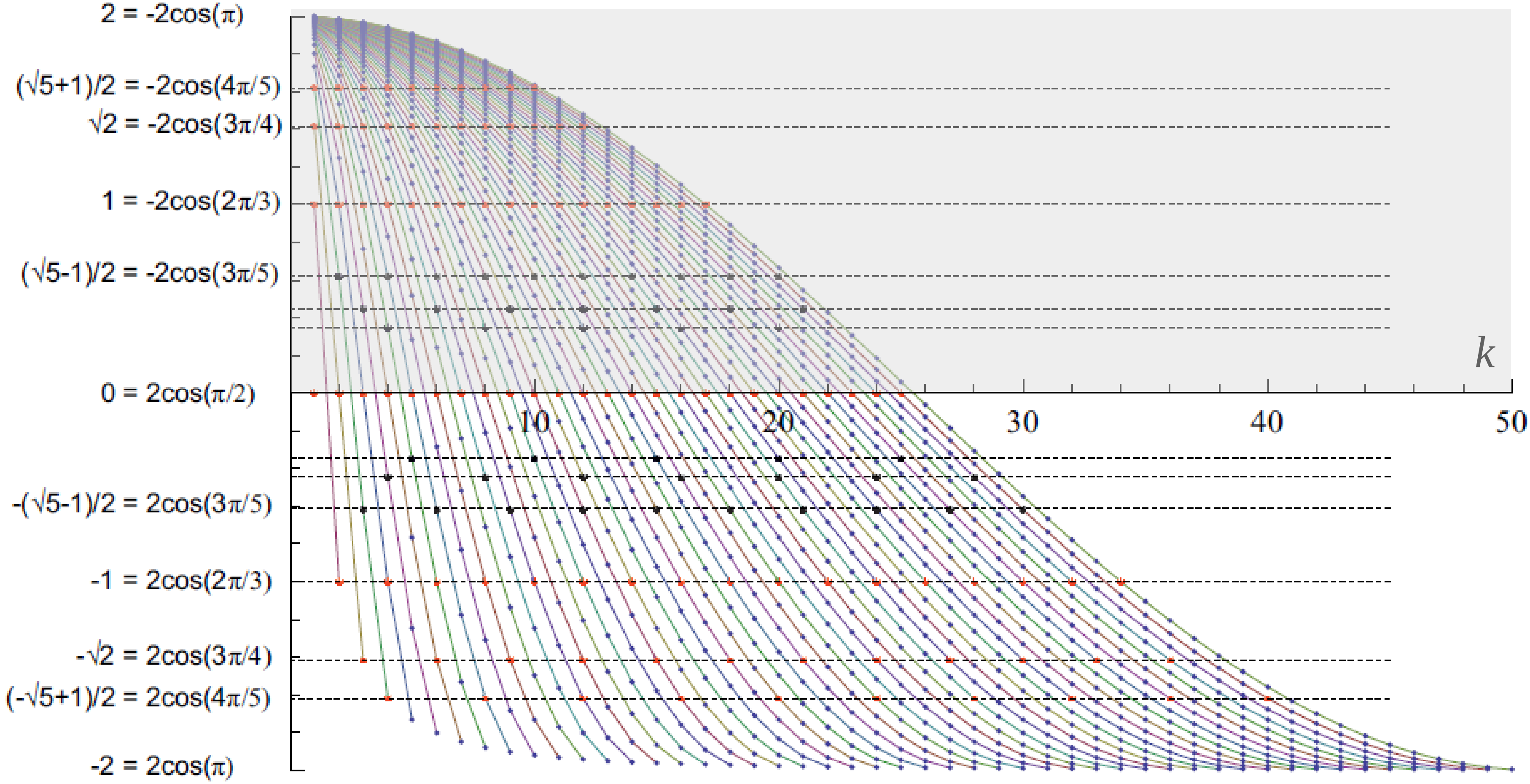
# How to compute the amplitude of a peak (degeneracies of eigenvalues)

$$\lambda_k(n) = -2\cos\frac{\pi k}{n+1}$$

$$\rho(\lambda) = \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \frac{1}{\pi N} \sum_{n=1}^{N} p^n \sum_{k=1}^{n} \frac{\varepsilon}{\left(\lambda - 2\cos\frac{\pi k}{n+1}\right)^2 + \varepsilon^2}$$



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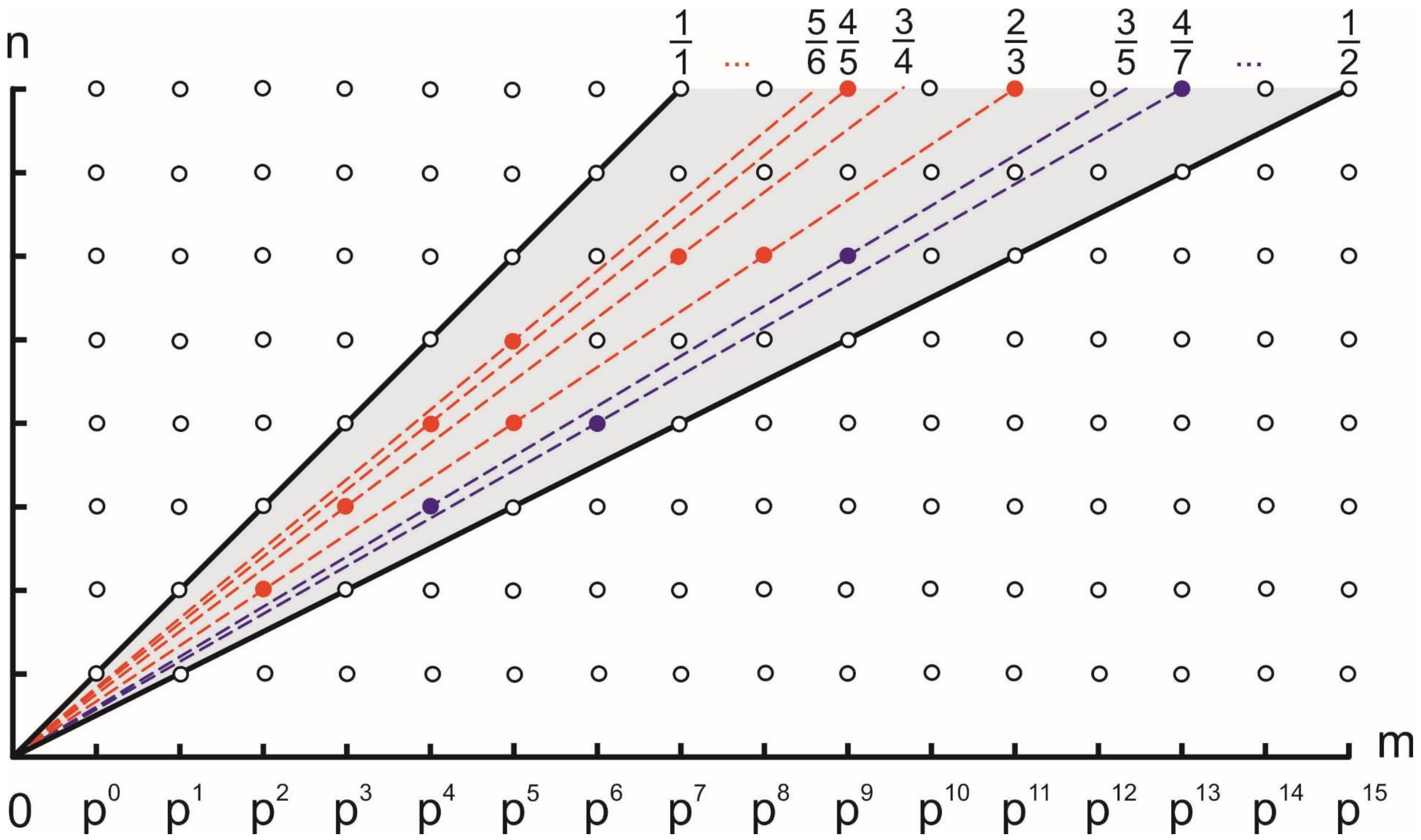
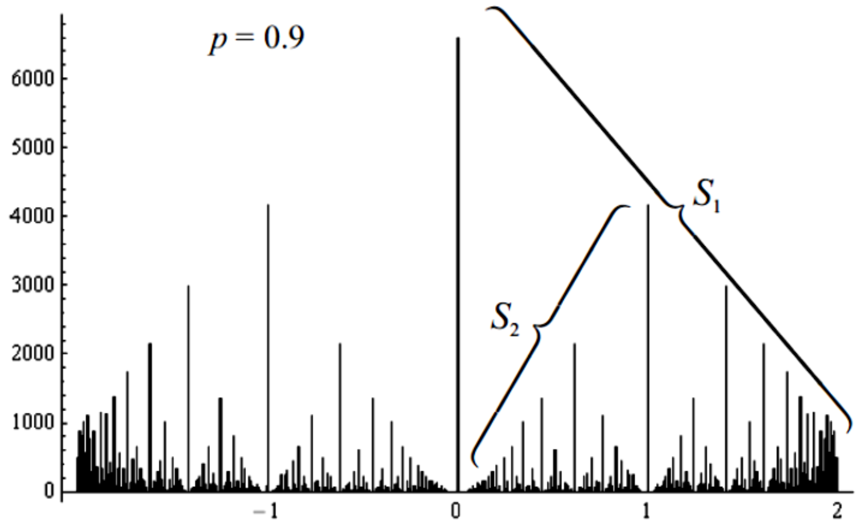
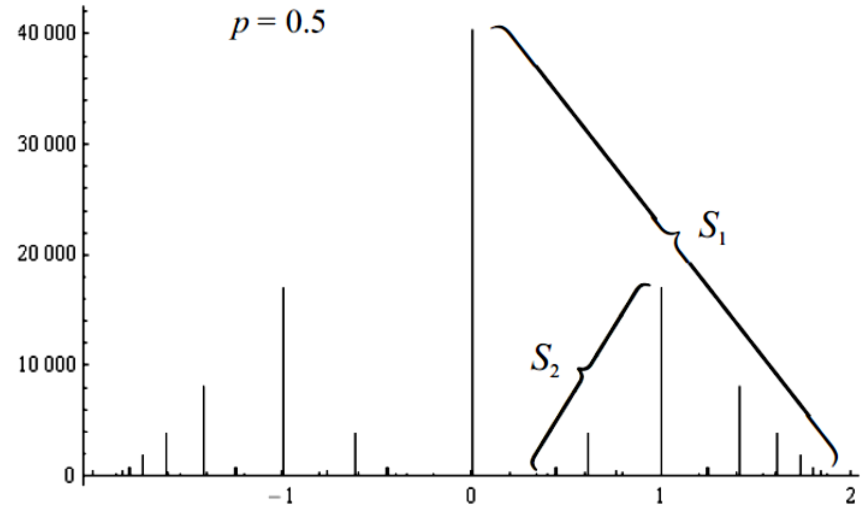


# Sample spectral densities for $p = 0.9$ and $p = 0.5$



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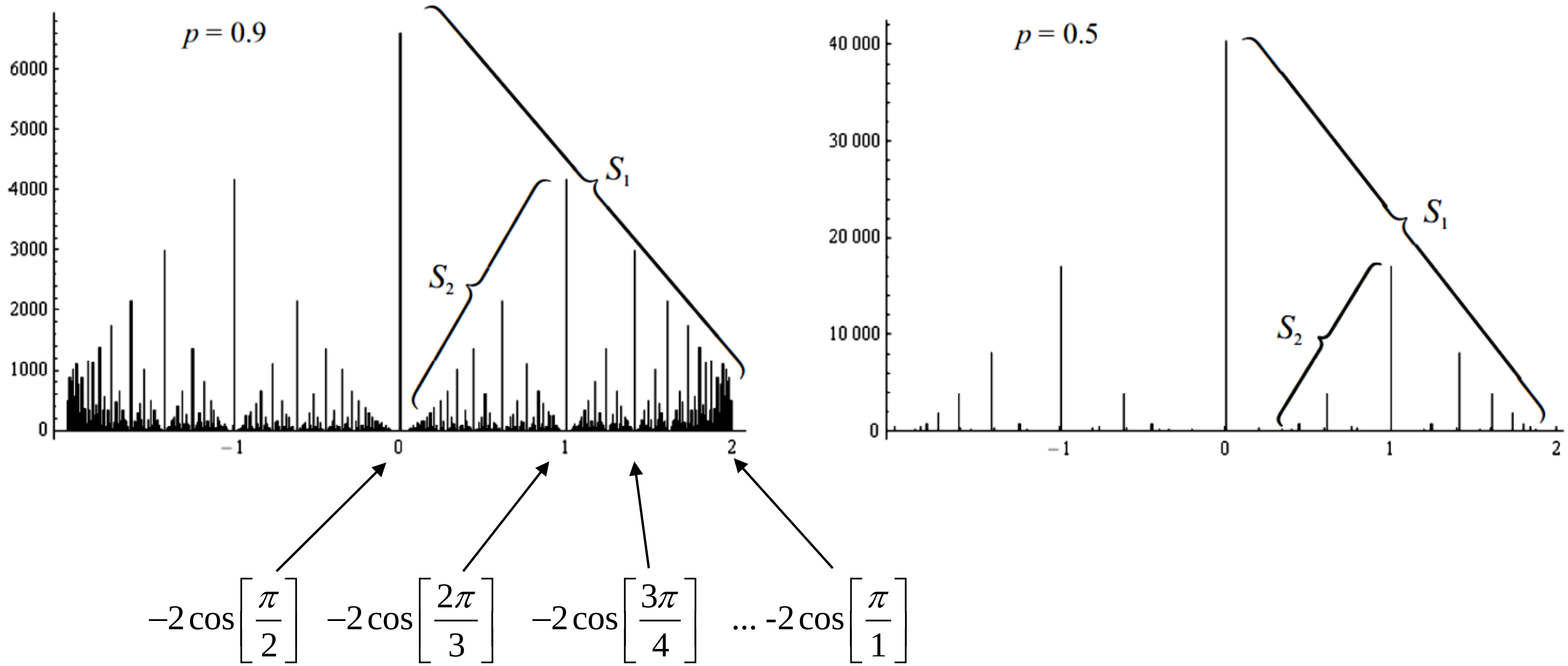

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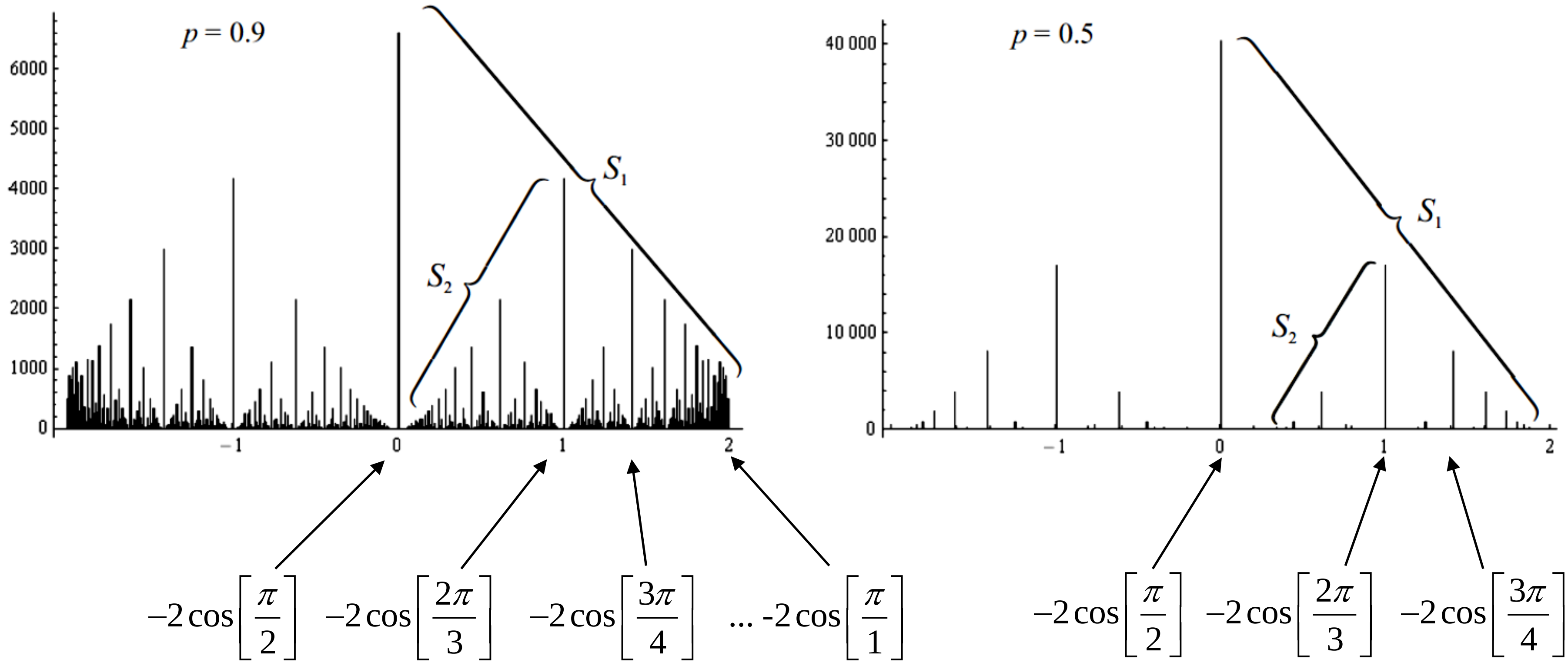

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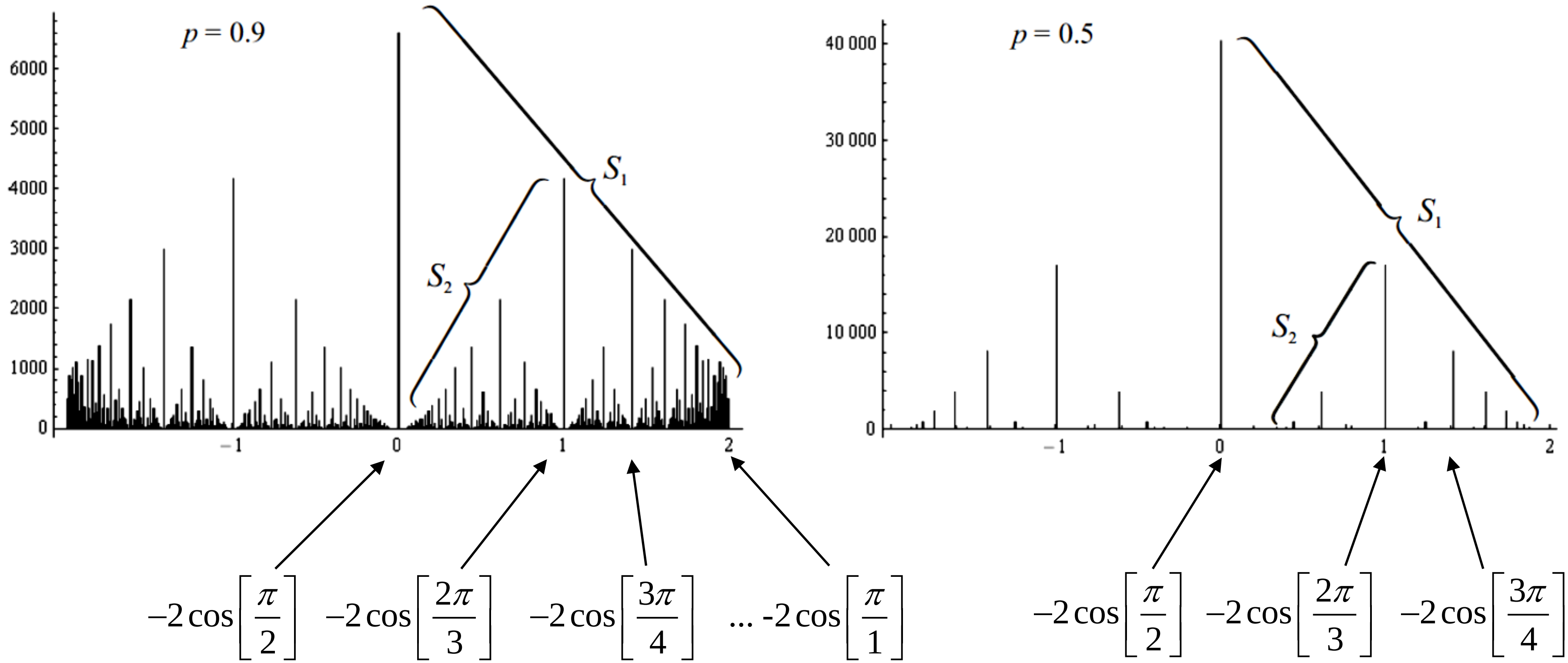


$$S_1 : \begin{cases} \lambda_1 = -2\cos\frac{\pi}{2} & \rho_{S_1}(\lambda_1) = p^1 + p^3 + p^5 + p^9 + \ldots \\ \lambda_2 = -2\cos\frac{2\pi}{3} & \rho_{S_1}(\lambda_2) = p^2 + p^5 + p^8 + p^{11} + \ldots \\ & \ldots \\ \lambda_n = -2\cos\frac{\pi n}{n+1} & \rho_{S_1}(\lambda_n) = \sum_{s=1}^{\infty} p^{(n+1)s-1} = \frac{p^n}{1-p^{n+1}} \quad (k = 1, 2, \ldots) \end{cases}$$

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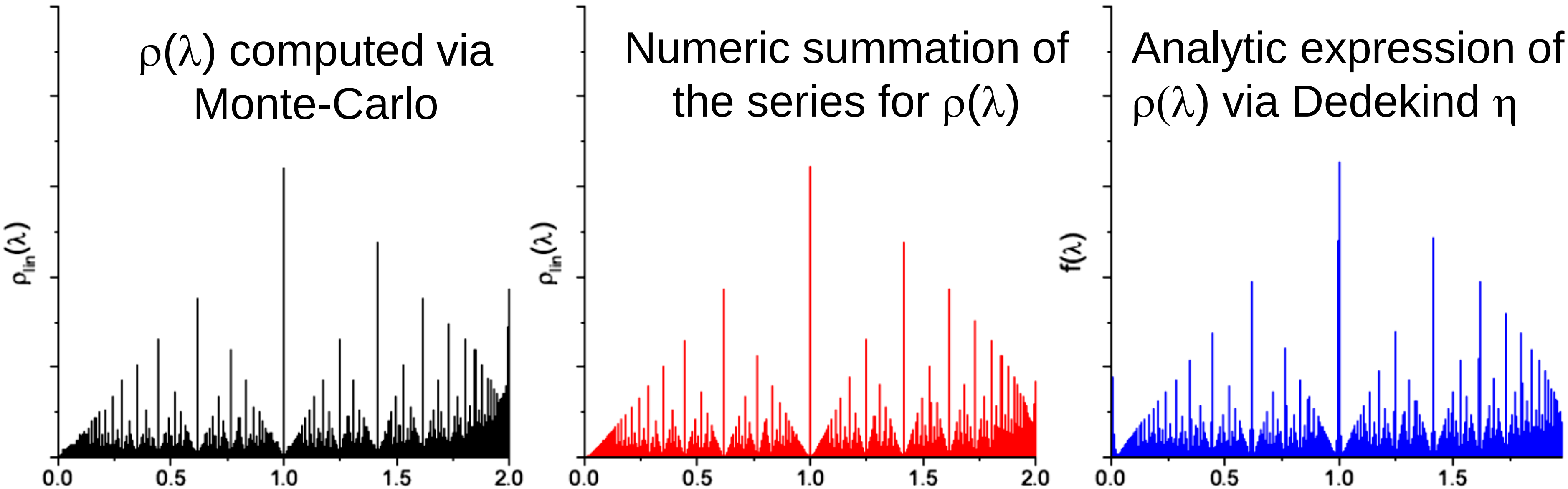

# Reminder: Dedekind η–function

$$\eta(z) = e^{\pi i z/12} \prod_{n=0}^{\infty} (1 - e^{2\pi i n z}) ; \quad z = x + iy \quad (y > 0)$$

$$\eta(z+1) = e^{\pi i z/12}\ \eta(z) \qquad \eta\left(-\frac{1}{z}\right) = \sqrt{-i}\ \eta(z)$$

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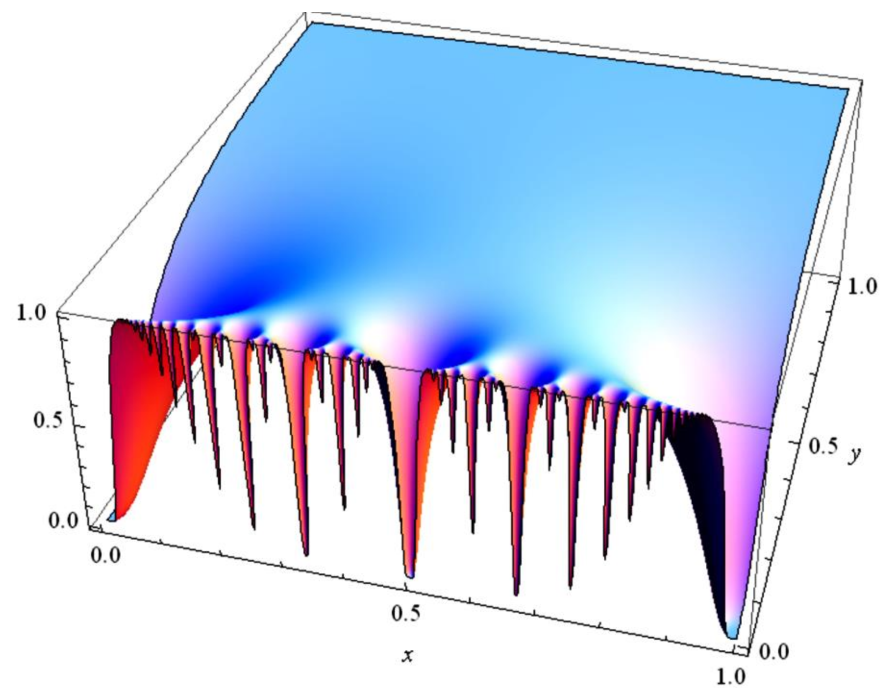


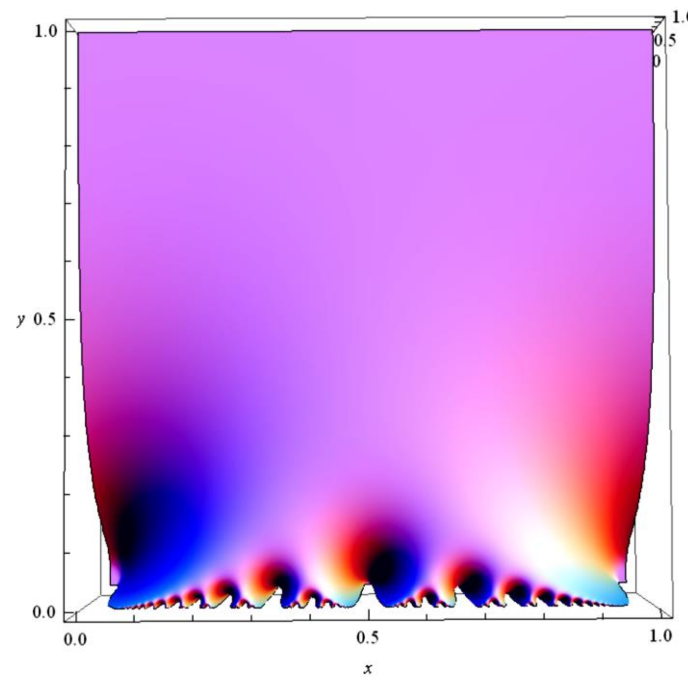

# Discontinuous Riemann "raindrop" function and its regularization

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$$g(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n}, \text{ and } (m, n) \text{ coprime} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

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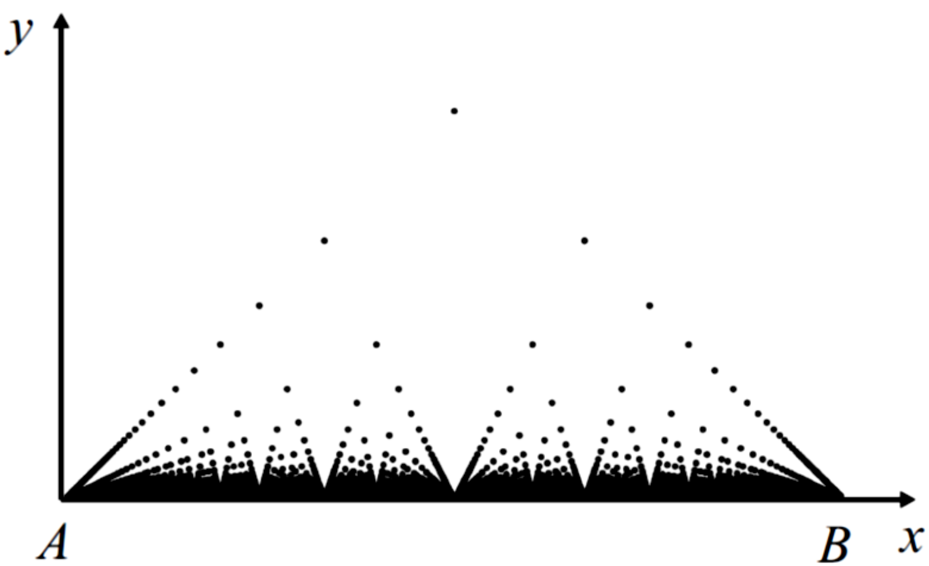
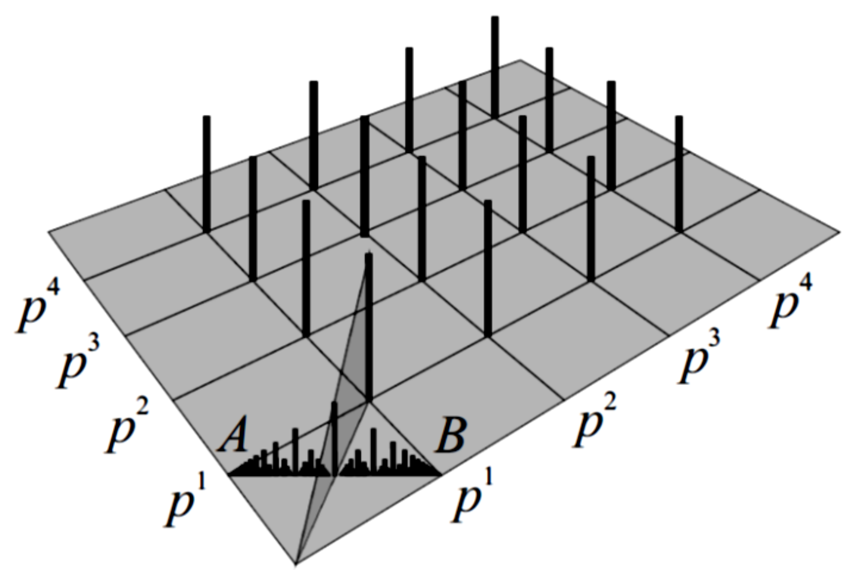
View of the Riemann function

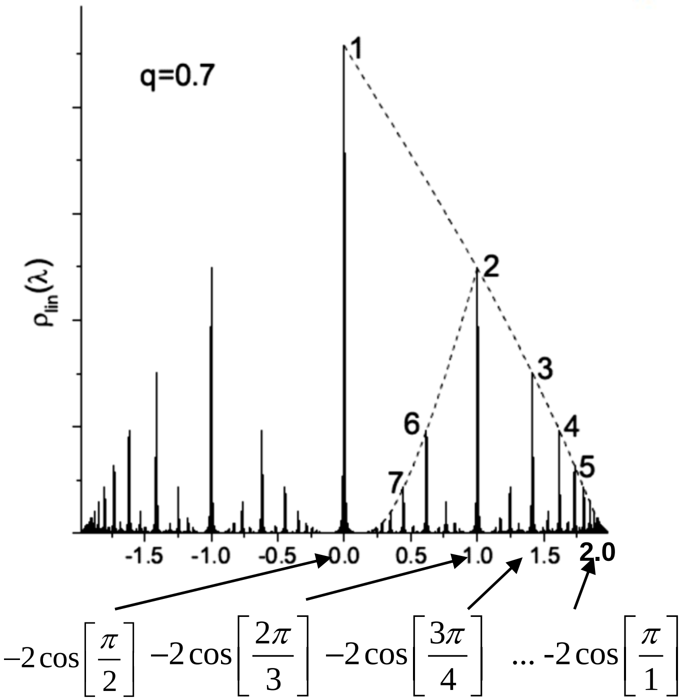



Riemann function as an “Euclid Orchard”



# Expression for the spectral density $\rho(\lambda)$

$$\rho(\lambda) = \frac{p^{1/g(u)}}{1 - p^{1+1/g(u)}}; \qquad u = \frac{1}{\pi}\arccos\frac{\lambda}{2}$$




Spectral tail for $p < 1$

$$S_1:\ \lambda = -2\cos\frac{\pi k}{k+1}\Big|_{k\to\infty} \to 2 - \frac{\pi^2}{k^2},$$

$$\rho^{S_1}(\lambda_k)\Big|_{k\to\infty} \to p^k$$

Lifshitz tail of 1D Anderson localization

$$\rho(\lambda \to 2) \to p^{\pi/\sqrt{2-\lambda}}$$

Laplace transform gives:

$$\rho(N) \sim \int_0^\infty \rho(\lambda) e^{-N\lambda} d\rho \sim \varphi(N) e^{-aN - bN^{-1/3}}$$

# Regularization of a normalized Riemann “raindrop” function

$$f_1(x) = \left(\tfrac{\pi}{12\varepsilon}\right)^{1/2} g(x) \qquad f_2(x) = \sqrt{-\ln|\eta(x+iy)|}$$

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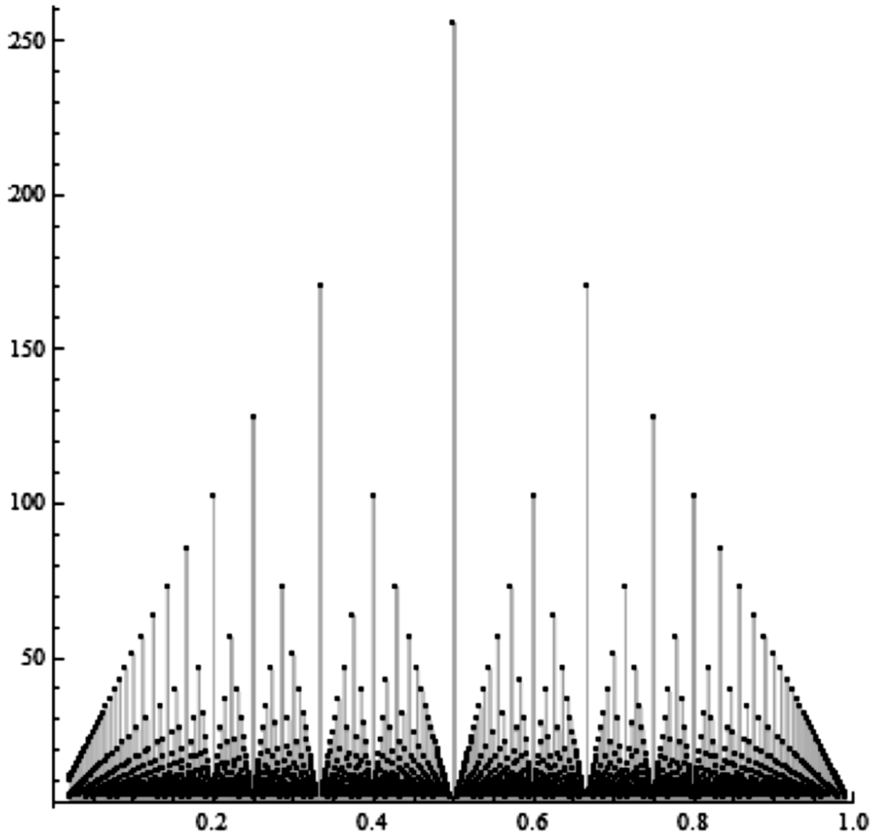

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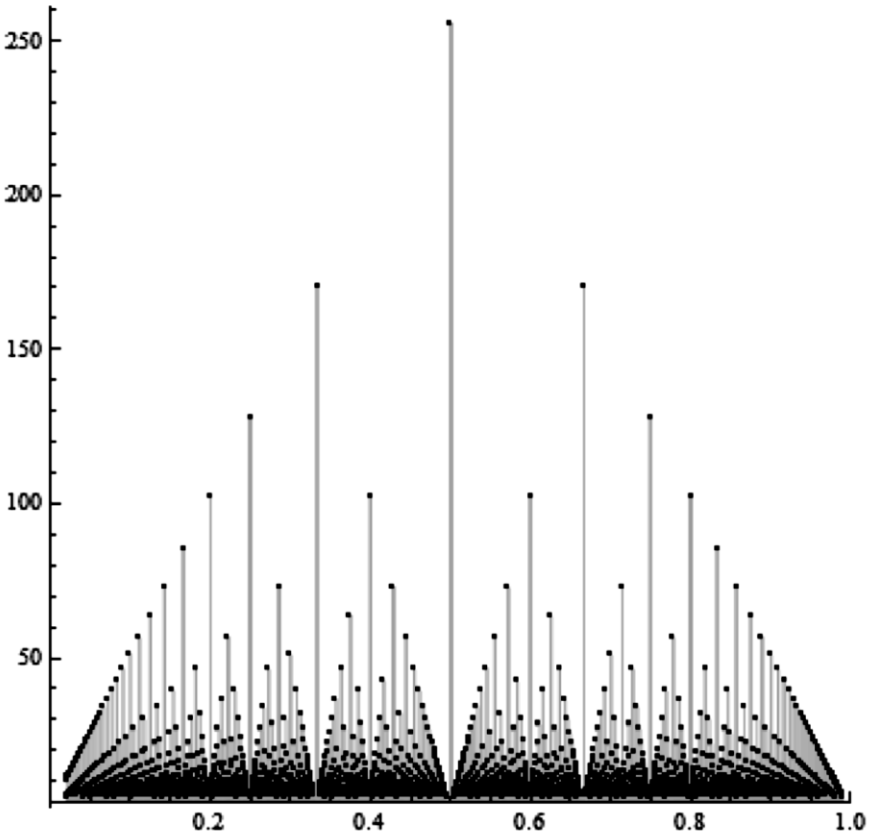
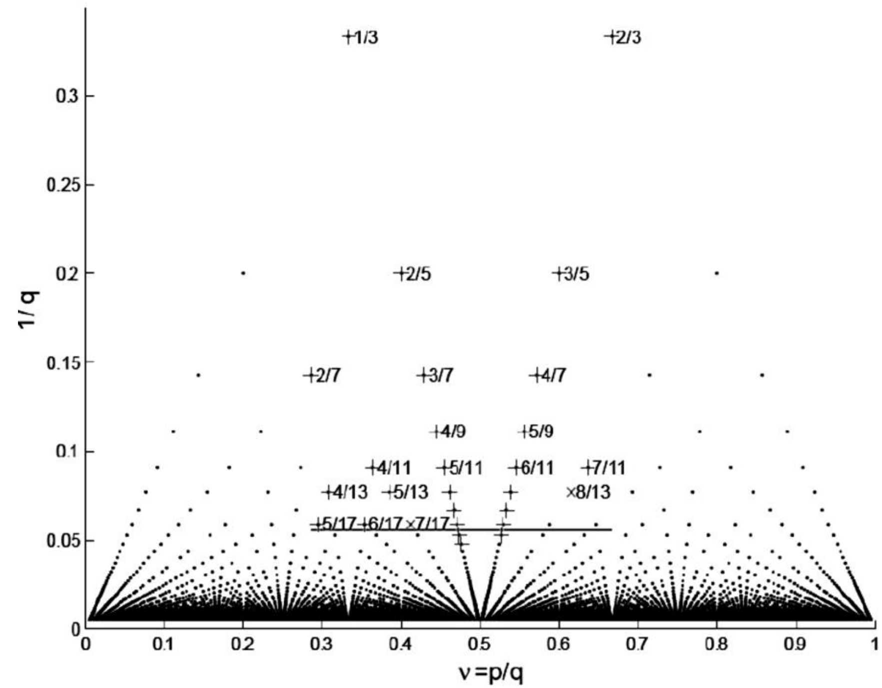
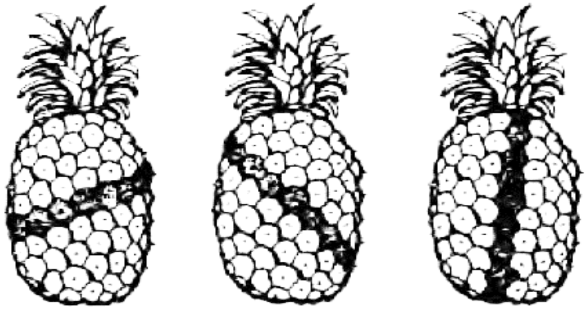



The “stability” diagram of Tao-Thouless Fractional Quantum Hall states [from E.J. Bergholtz et al, 2008]: the lower the disorder, the more fractions are observed

# Phyllotaxis



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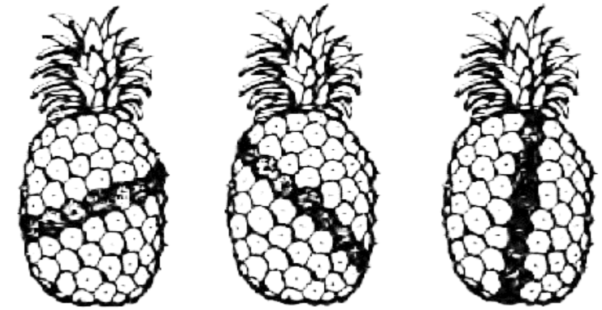

Energetic approach to phyllotaxis, L. Levitov, 1991

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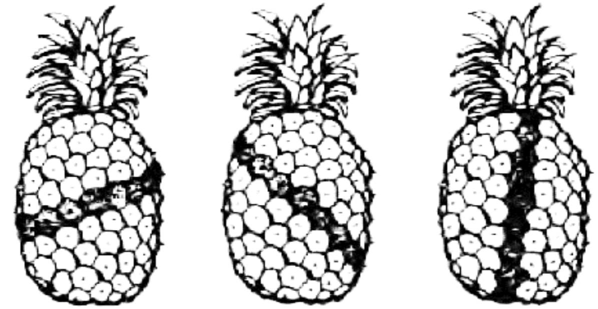
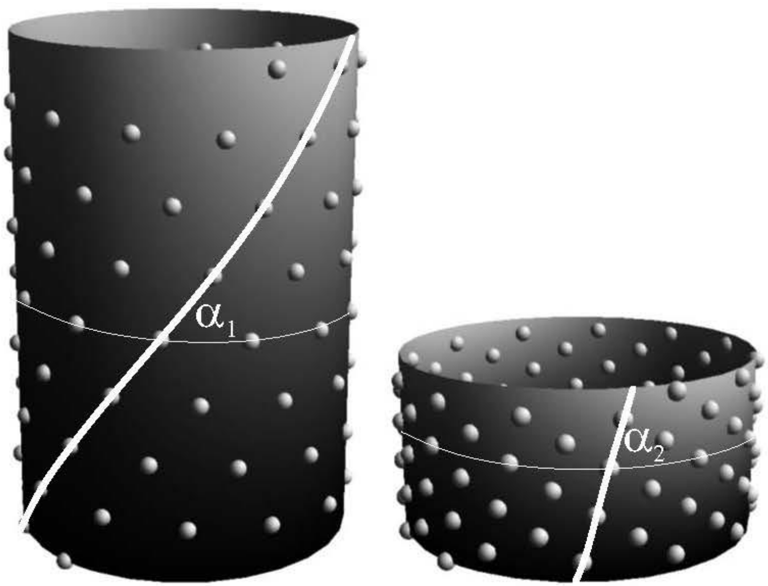
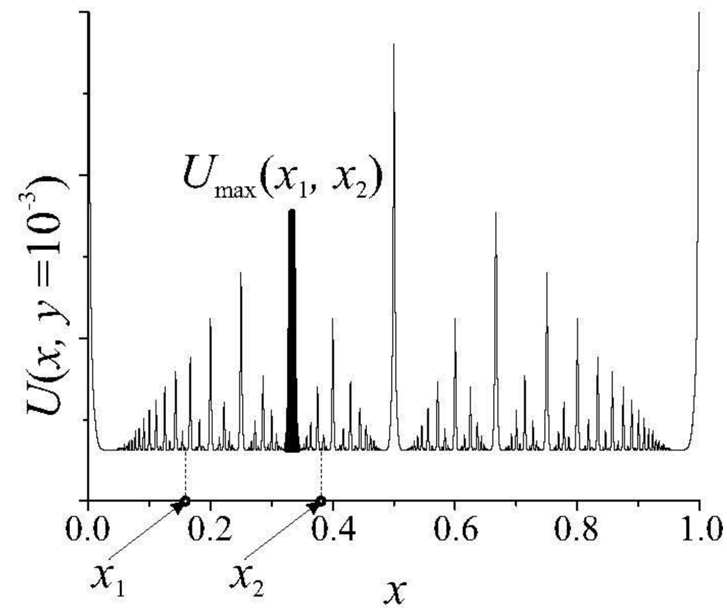


Energetic approach to phyllotaxis, L. Levitov, 1991

$$x = \frac{\alpha}{2\pi},\ y = \frac{h}{2\pi} \quad r_{n,m} = \left( \frac{m + nx}{\sqrt{y}}, n\sqrt{y} \right) \quad U(x,y) \sim \sum_{\{m,n\} \in \mathbb{Z}^2 \setminus \{0,0\}} e^{-\beta \left( \frac{(m+nx)^2}{y} + yn^2 \right)}$$



Static and Dynamical Phyllotaxis in Magnetic Cactus

C. Nisoli et al, ArXiv: cond-mat/0702335

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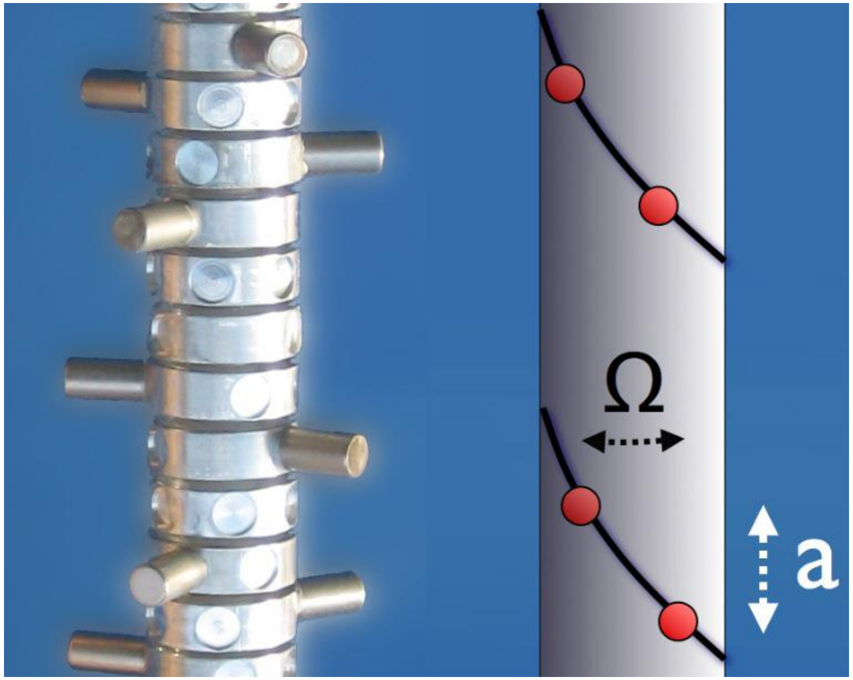


Experimental setting

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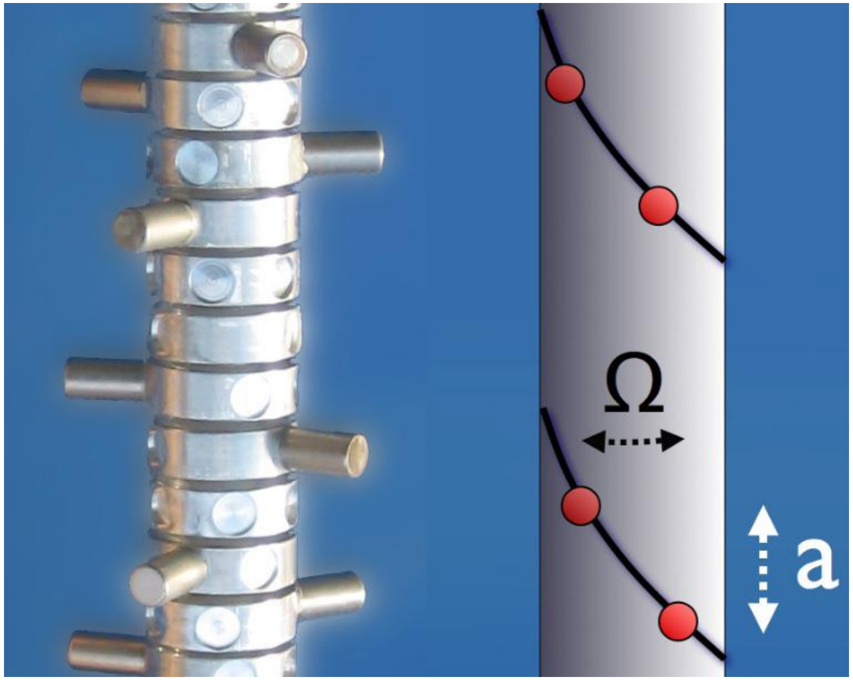
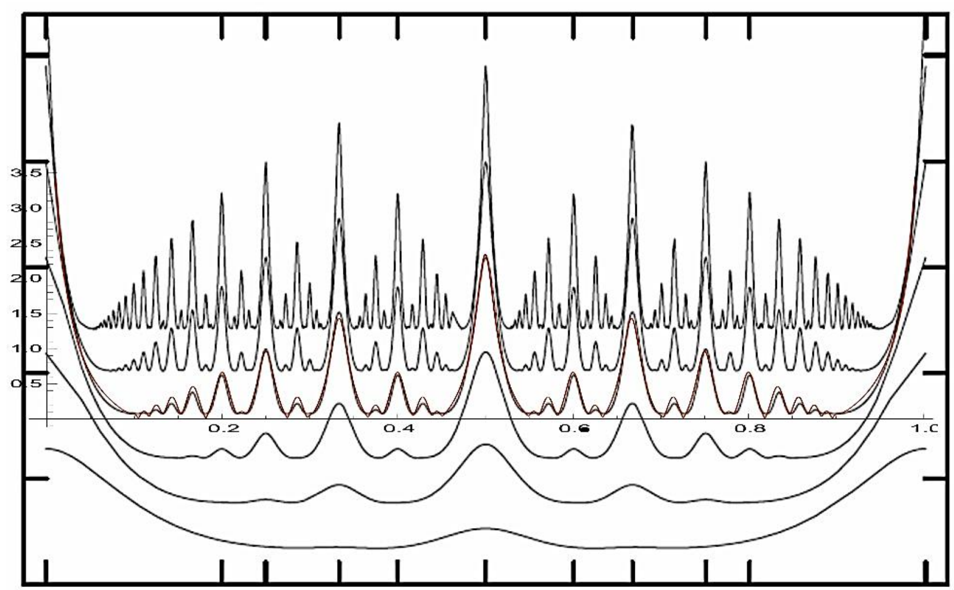



Experimental setting

Hierarchical potential energy relief between states with various Ω



# Phyllotaxis, ultrametric spectra of 1D Schrödinger operators and (maybe) FQHE: what is common?

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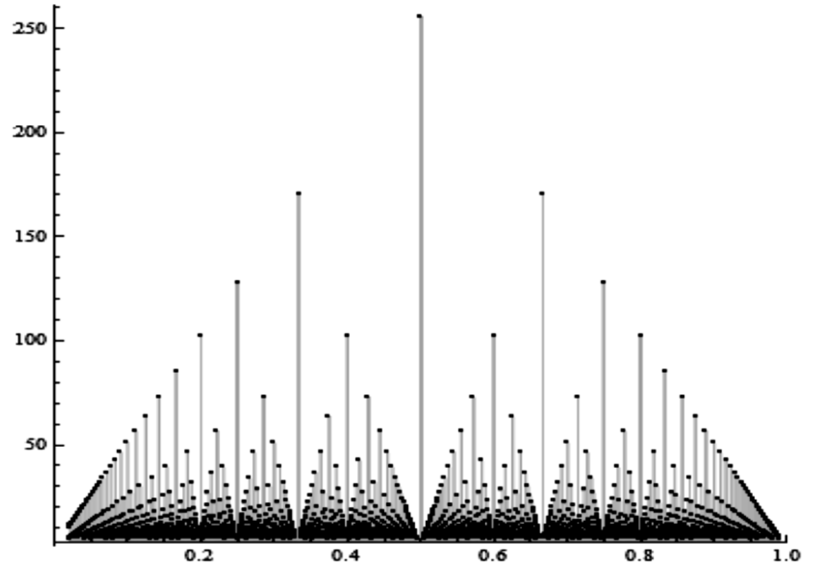
Conjecture:

Discreteness of nature and, in particular, Riemann “raindrop” function lies behind




…however the emergence of modular symmetry in various physical systems is hidden…

We can identify the energy landscape with a metric space

$$U(x,y) \sim \sum_{\{m,n\} \in \mathbb{Z}^2 \setminus \{0,0\}} e^{-\beta\left(\frac{(m+nx)^2}{y} + yn^2\right)}$$

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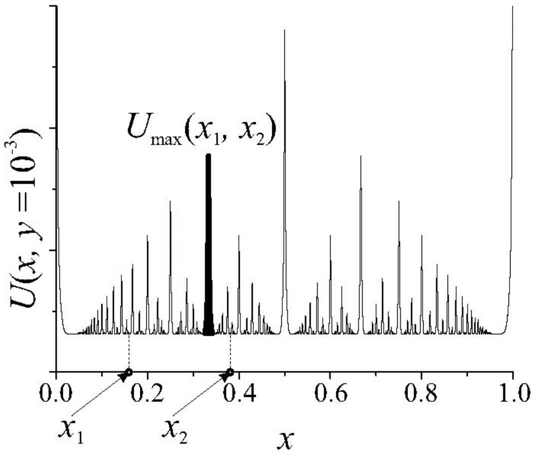


Considering the profile $U(x, y)$ as a function of $x$, we set $d(x_1, x_2) = U_{max}(x_1, x_2)$ where $d(x_1, x_2)$ is the ultrametric pairwise distance

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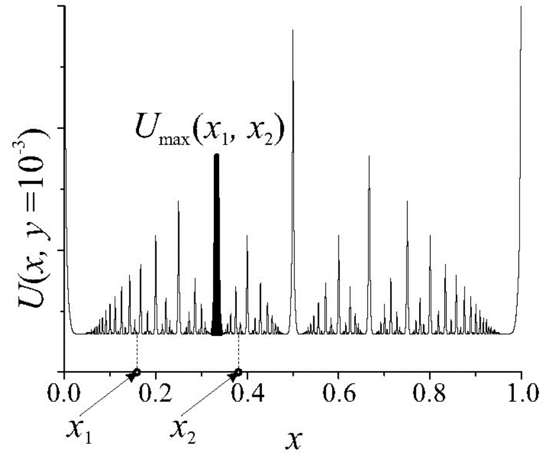


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At $\beta >> 1$ one can approximately rewrite $U(x, y)$ as

$$U(x,y)\Big|_{\beta\gg 1} \approx \sqrt{\frac{\pi}{\beta}} \sum_{\{m,n\}\in\mathbb{Z}^2\setminus\{0,0\}} \delta\left(\frac{(m+nx)^2}{y}+yn^2\right)$$

Making use of regularization of the $\delta$-function, we represent the potential $U(x, y)$ at $\beta >> 1$ as follows:

$$U(x,y)\Big|_{\beta\gg 1} = \lim_{\beta\to\infty}\sqrt{\frac{\beta}{\pi}}\sum_{\{m,n\}\in\mathbb{Z}^2\backslash\{0,0\}}\frac{1}{\beta\left(\frac{(m+nx)^2}{y}+yn^2\right)+\frac{1}{\beta}}$$

$$\approx \frac{1}{\sqrt{\pi\beta}}\sum_{\{m,n\}\in\mathbb{Z}^2\backslash\{0,0\}}\frac{1}{Q^s(m,n)}$$

where $s = 1$ and $Q(m, n)$ is a positive quadratic form

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Recall now the definition of the Eisenstein series

$$E(\tilde{z},s) = \frac{1}{2}\sum_{\{m,n\}\in\mathbb{Z}^2\setminus\{0,0\}} \frac{\tilde{y}^s}{|m\tilde{z}+n|^{2s}}; \qquad \tilde{z} = \tilde{x}+i\tilde{y}$$

# Reminder of Eisenstein series

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$E(z, s)$ as a function of $z$, is a $SL(2,Z)$–invariant auto-morphic solution of the hyperbolic Laplace equation

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$E(z, s)$ is related to the Epstein $\zeta$-function, defined as

$$\zeta(Q, s) = \sum_{\{m,n\} \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{Q^s(m, n, \tilde{z})} = \frac{2}{d^{s/2}} E(\tilde{z}, s)$$

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We can make now an identification

$$\frac{1}{y}=a,\;\;\frac{x}{y}=b,\;\;\frac{x^2}{y}+y=c,\;\;d=ac-b^2=1.$$

$$z=\frac{-b+i\sqrt{d}}{a}\quad\Rightarrow\quad\left(\tilde{x}=-\frac{b}{a},\;\tilde{y}=\frac{\sqrt{d}}{a}=\frac{1}{a}\right)$$

# Kronecker 1st limit formula

The residue of $\zeta(Q, s)$ at $s = 1$ is known as 1st Kronecker limit formula

$$\zeta(Q,s) = \frac{\pi}{\sqrt{d}}\frac{1}{s-1} + \frac{2\pi}{\sqrt{d}}\left(\gamma + \ln\sqrt{\frac{a}{4d}} - 2\ln|\eta(z)|\right) + O(s-1)$$

where $\eta(z) = e^{\pi iz/12}\prod_{,n=0}^{\infty}(1-e^{2\pi inz})$ is the Dedekind eta-function

$$\eta(z+1) = e^{\pi iz/12}\,\eta(z) \quad \eta\left(-\frac{1}{z}\right) = \sqrt{-i}\,\eta(z)$$

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Dropping the divergent at $s\to 1$ term, we get for the ultrametric potential $U(x, y)$:

$$U(x,y) \approx \sqrt{\frac{\pi}{\beta}}\left(\frac{\pi}{s-1} + 2\pi\gamma - \ln 2 - 4\pi\ln h(x,y)\right)$$

where $h(x,y) = y^{1/4}|\eta(x+iy)|$

Relief of the function

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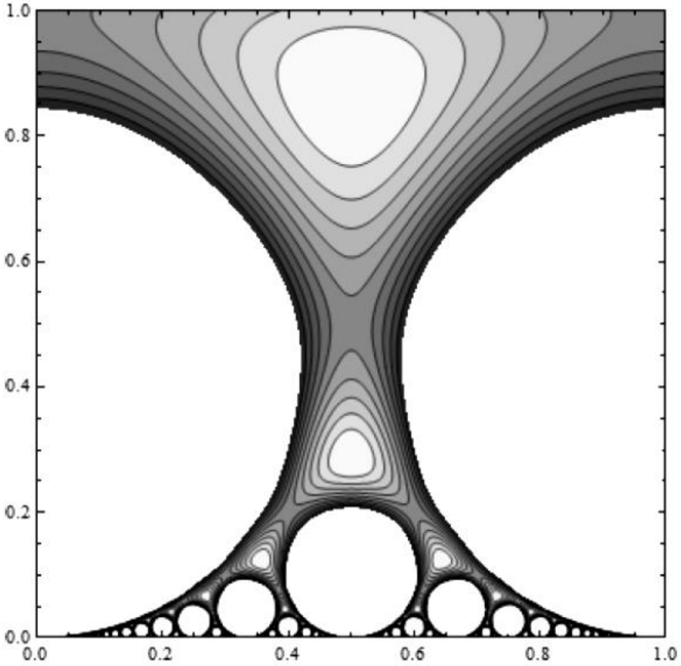
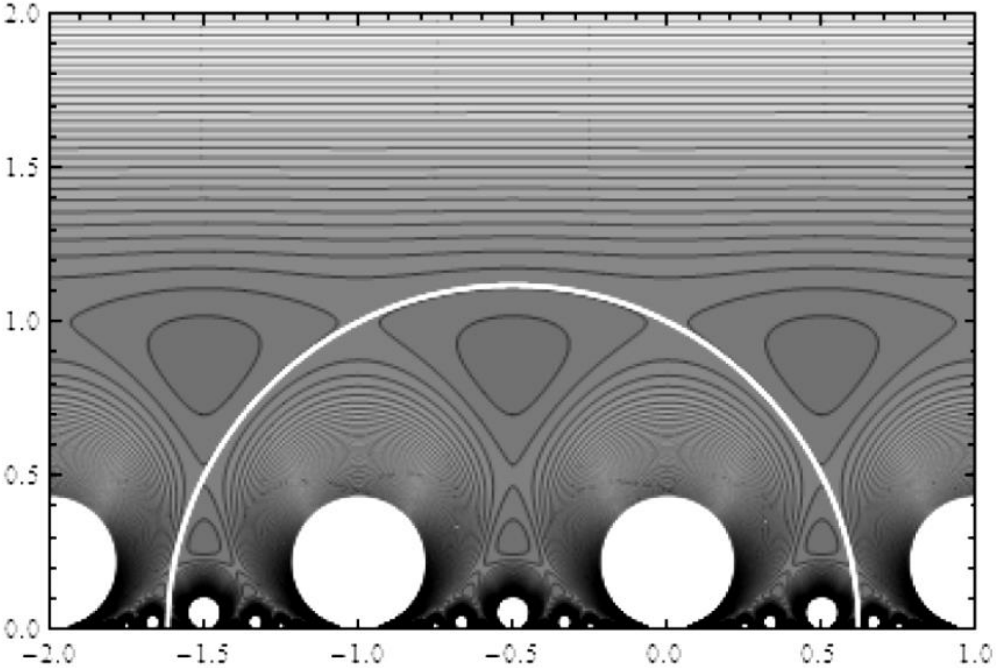



Relief of the potential

$$\tilde{U}(x,y) = -\ln h(x,y)$$



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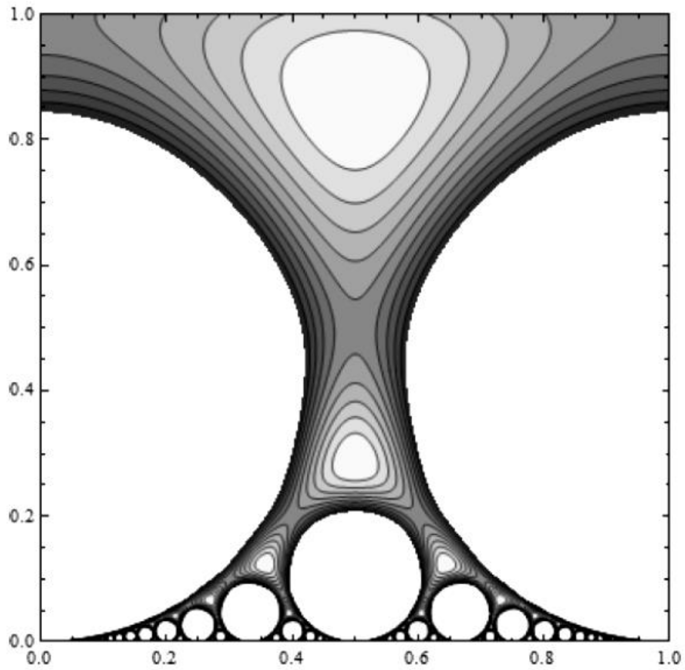


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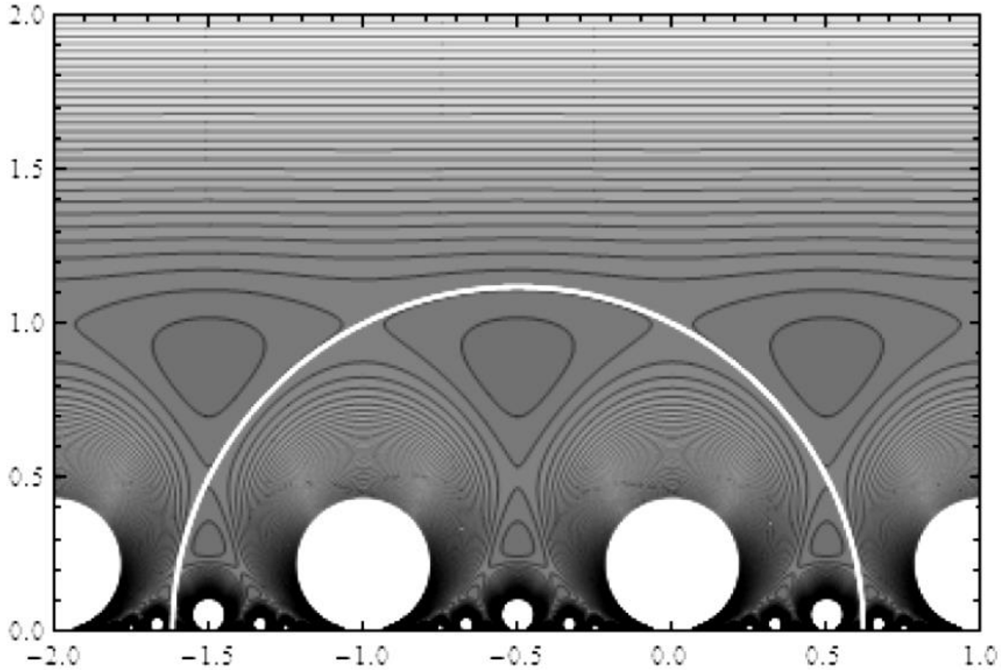


The *longest open geodesic* is the largest horocycle defined by the equation

$$\left(x+\frac{1}{2}\right)^2 + y^2 = \frac{5}{4}$$

To proceed with regularization of a Riemann function, return to the "phyllotaxis potential" and assess $U(x, y)$ at small $y$:

$$U(x, y \to 0)\Big|_{\beta \gg 1} \approx \frac{1}{y\sqrt{\beta\pi}} \sum_{\{m,n\} \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{y^2}{(m+nx)^2 + y^2 n^2}$$

$$\approx \frac{2}{y\sqrt{\beta\pi}} \sum_{m=1} \sum_{n=1} \frac{1}{n^2} \delta\left(x - \frac{m}{n}\right)$$

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Comparing with the Riemann function, we get

$$\frac{\pi}{12y}\, g^2(x) = -\ln|\eta(x+iy)|_{y\to 0} + O(\ln y)$$

and, finally

$$g(x) \to \sqrt{-\frac{12y}{\pi}\ln|\eta(x+iy)|}\Bigg|_{y\to 0}$$