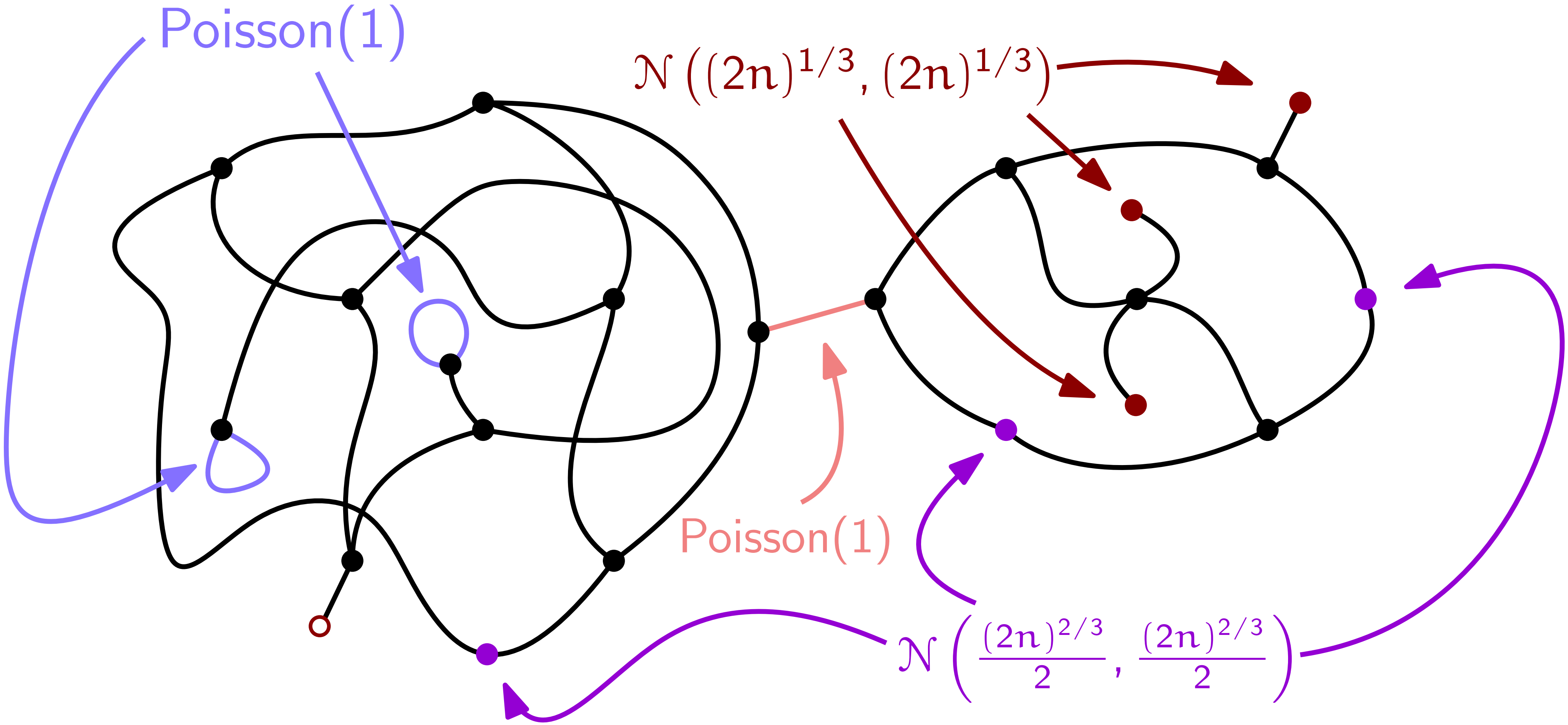
Distributions of parameters in restricted classes of maps and λ-terms



Combinatorics and Arithmetic for Physics, IHES, 1 December 2021

Olivier Bodini (LIPN, Paris 13)

Alexandros Singh (LIPN, Paris 13)

Noam Zeilberger (LIX, Polytechnique)

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not in this talk!

Techniques drawn from combinatorics, logic, and physics may be used in tandem to study them!

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- We're interested in terms up to α-equivalence:

$$(\lambda x.xx)(\lambda x.xx) \stackrel{\alpha}{=} (\lambda y.yy)(\lambda x.xx) \stackrel{\alpha}{\neq} (\lambda y.ya)(\lambda x.xx)$$

Subfamilies of λ-terms

General terms: no restrictions on variable use

$\lambda x.\lambda y.x$

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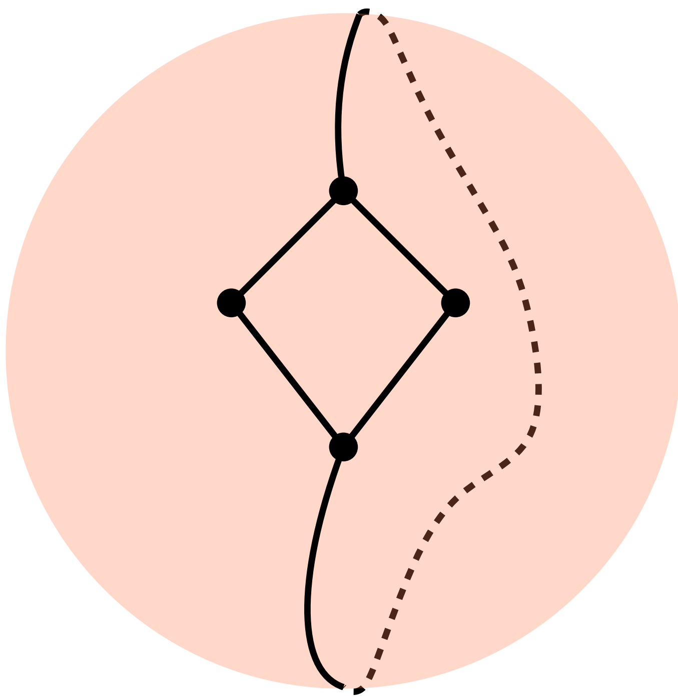
Linear Terms: bound variables occur **exactly** once

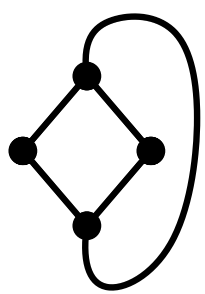
$\lambda x.\lambda y.(y\ x)a$

$\lambda x.\lambda y.(y\ a)(b\ x)$

$\lambda x.a(\lambda z.(\lambda y.y\ (x\ z)))$

What are maps?





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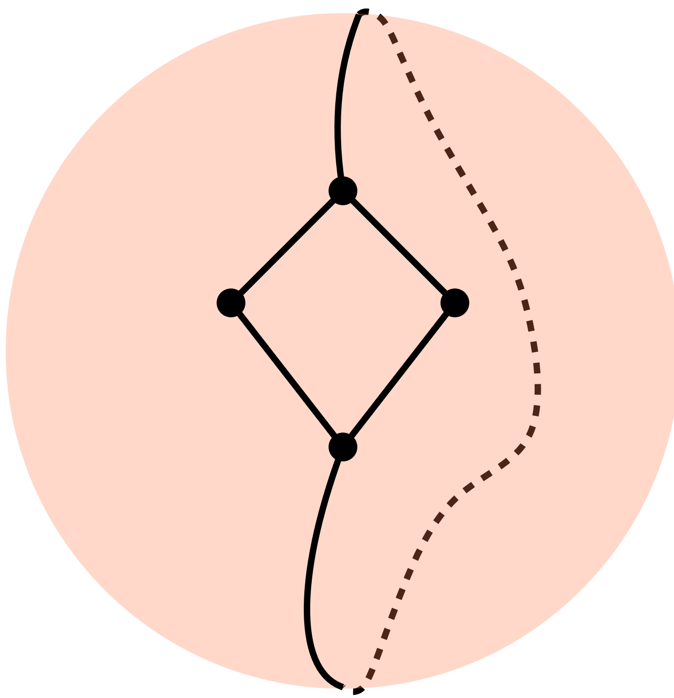

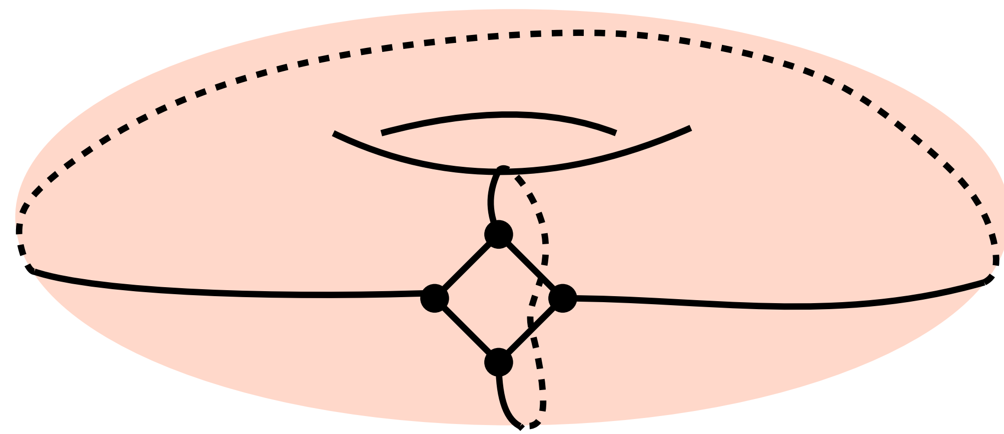

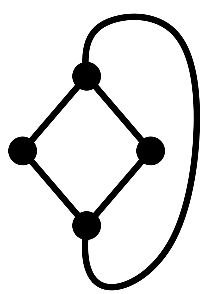

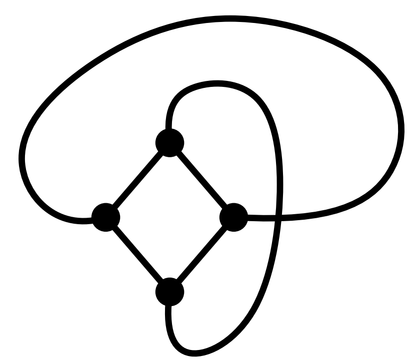

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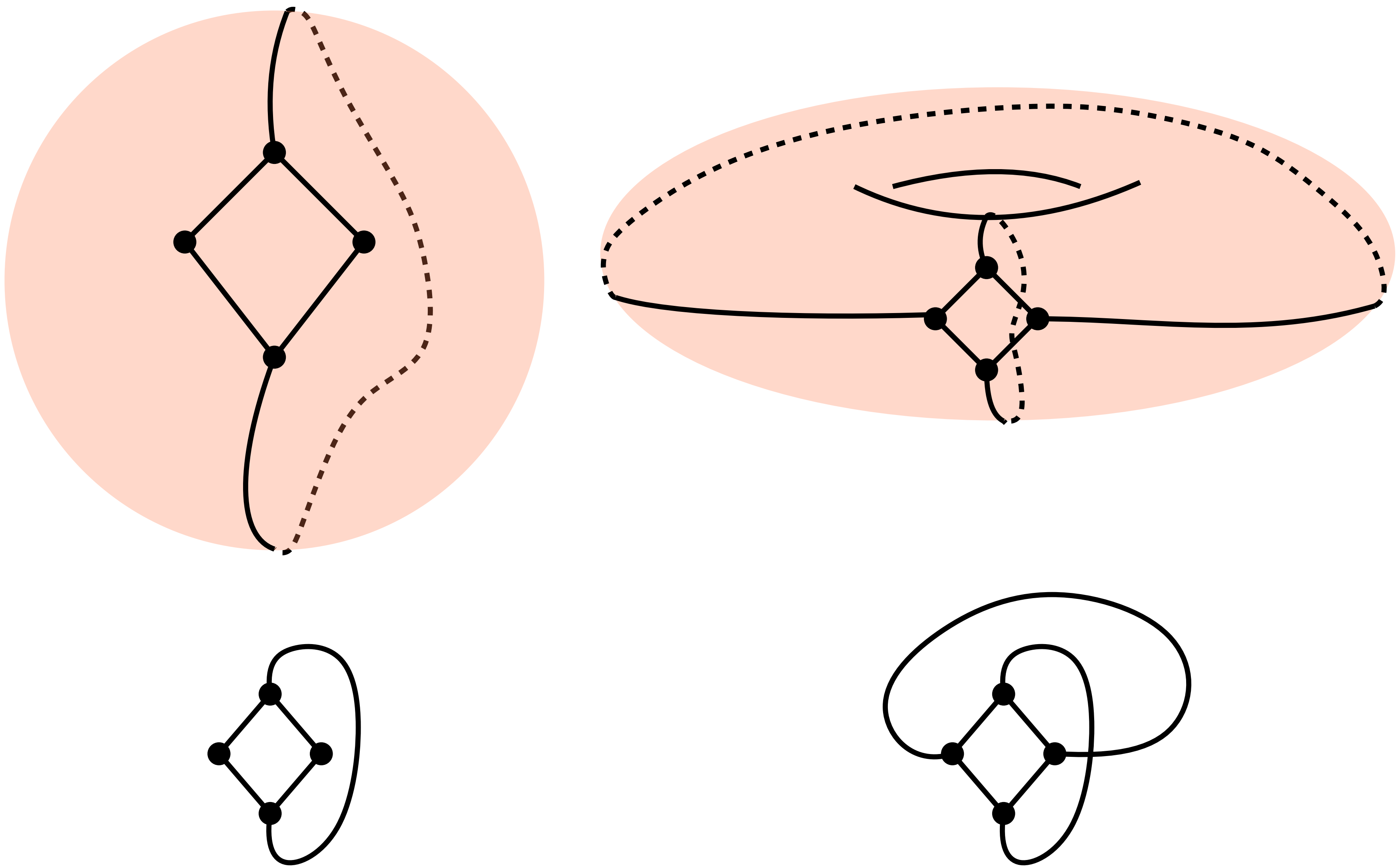
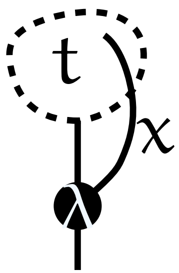


We're interested in unrestricted genus, restricted vertex degrees

Why should you, a logician, be interested in maps?

String diagrams! [BGJ13, Z16]

$\bullet = x$



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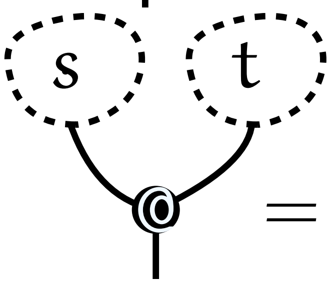

$= (s\ t)$

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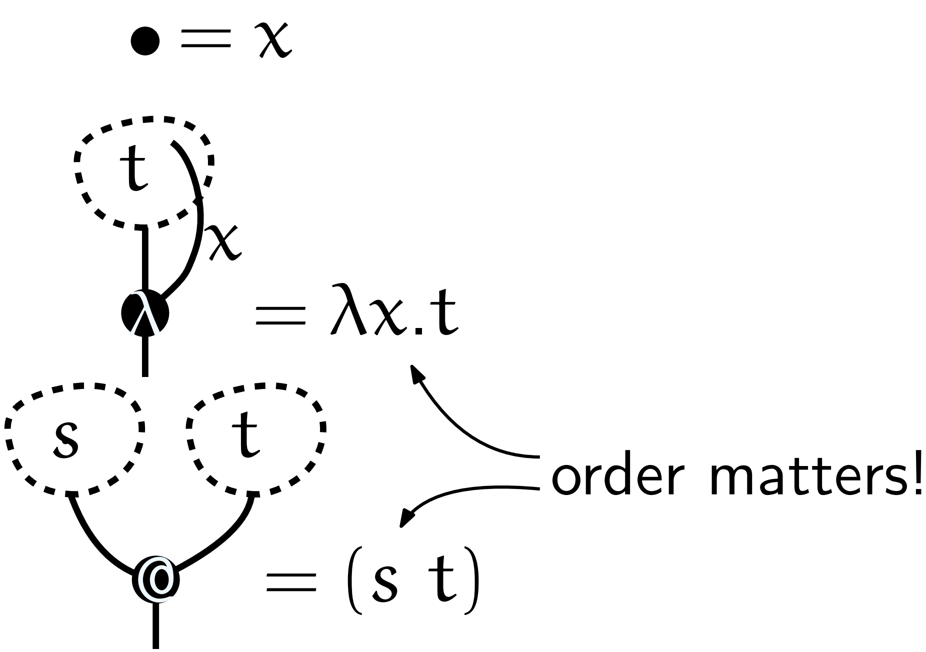

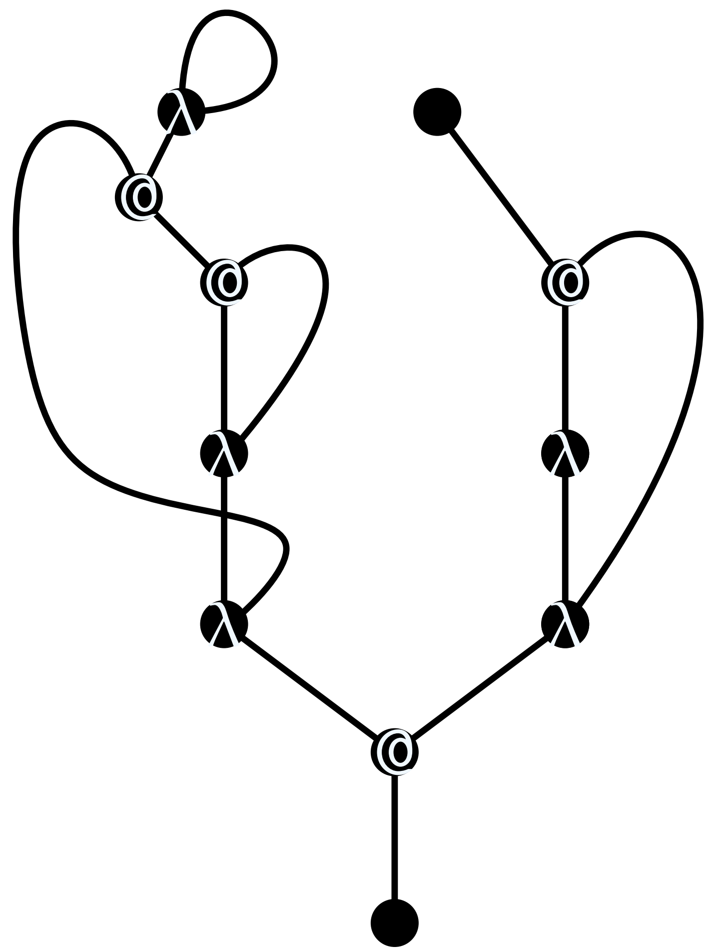

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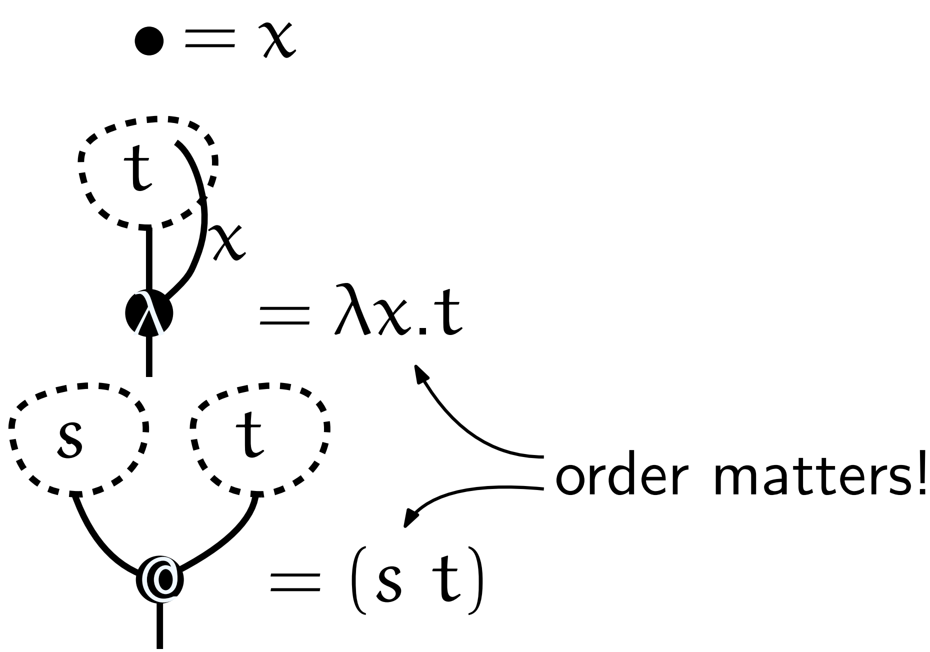

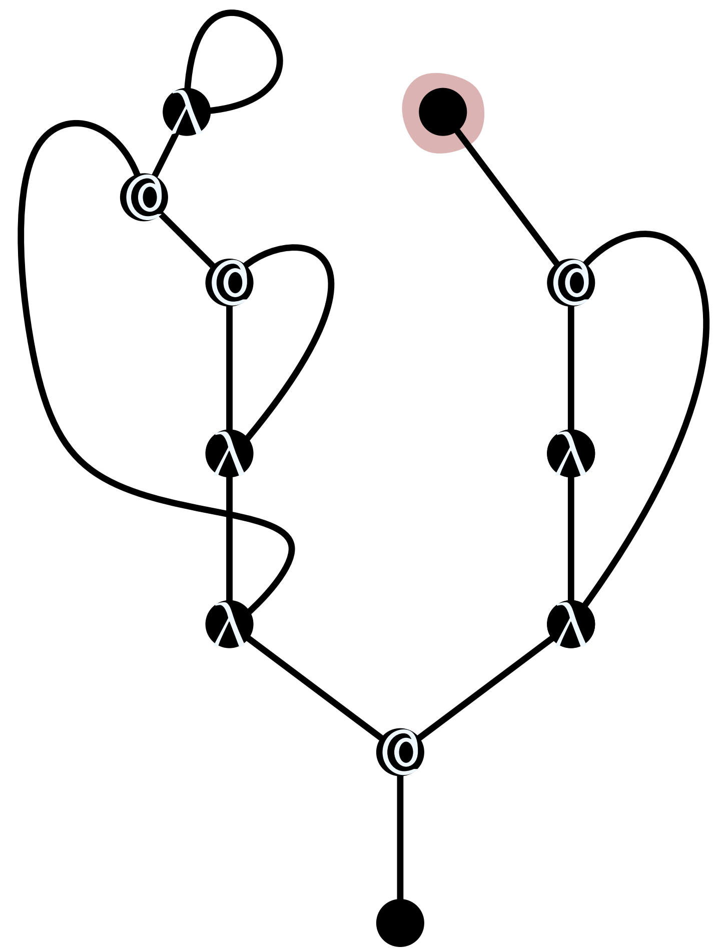

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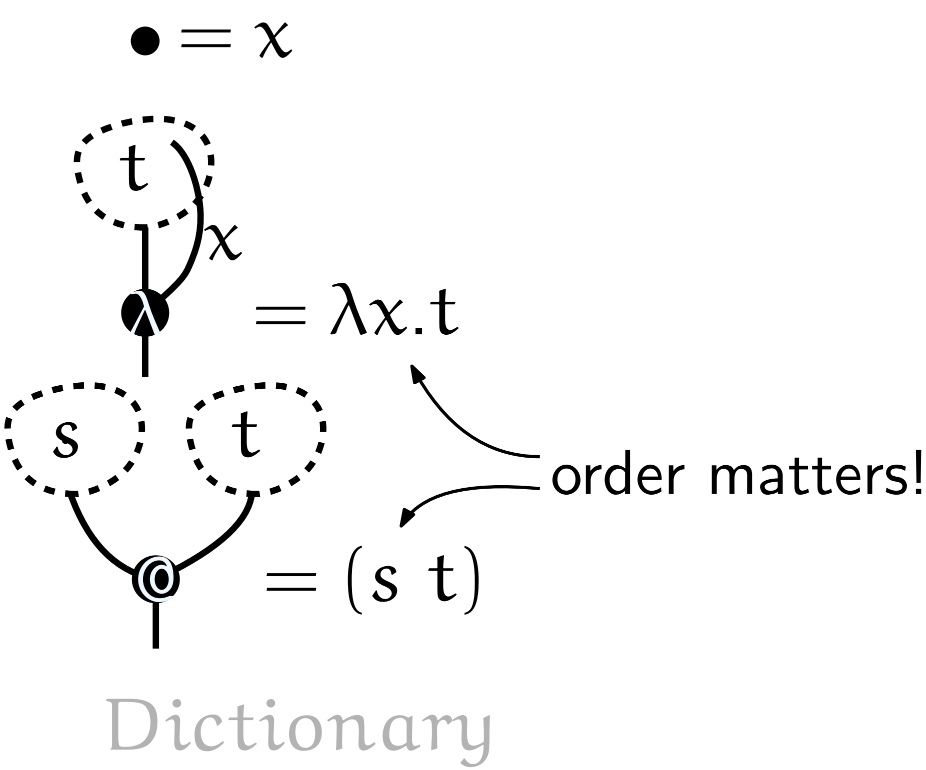

- Free var ↔ unary vertex
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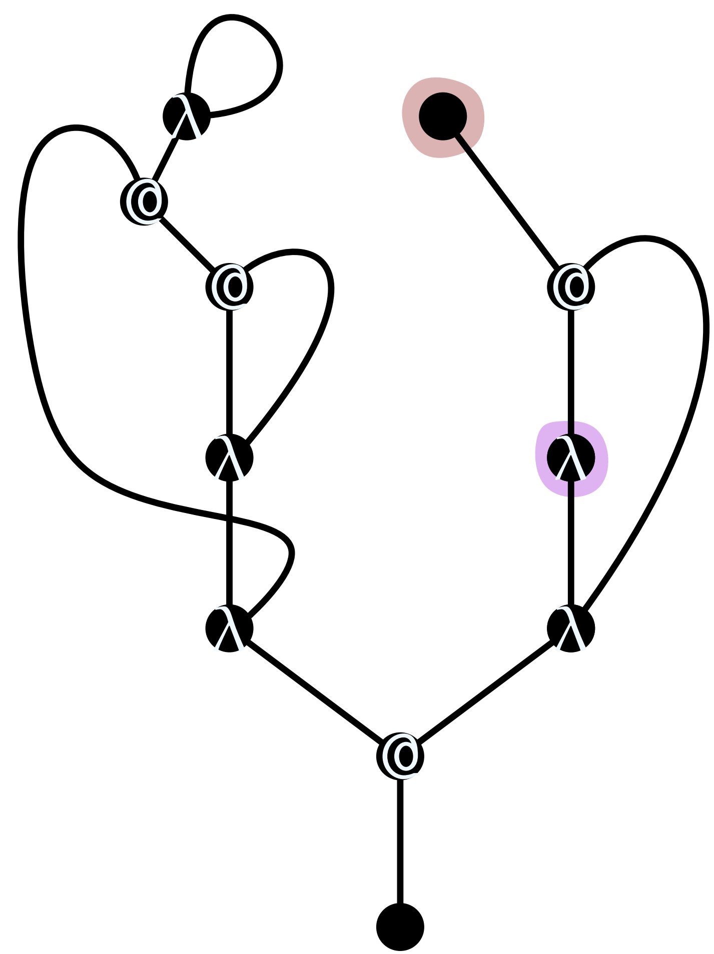

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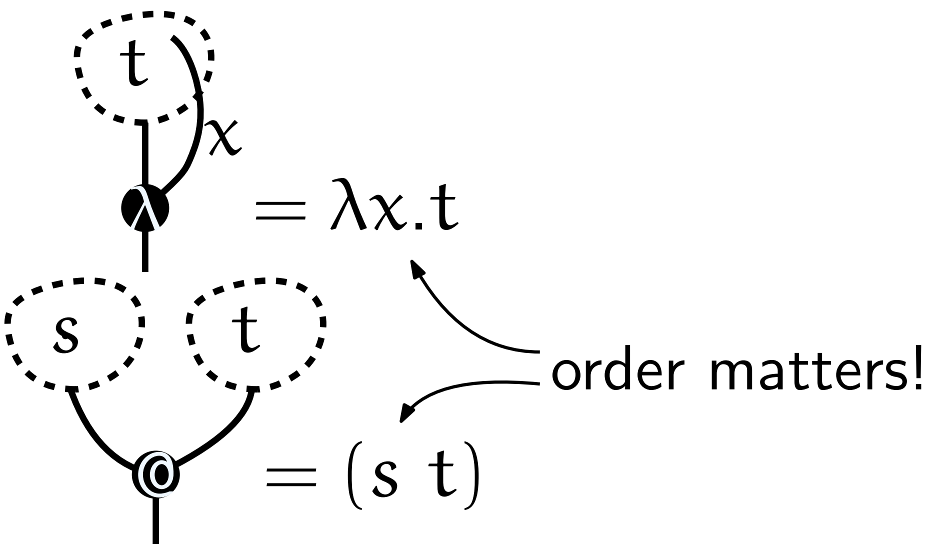

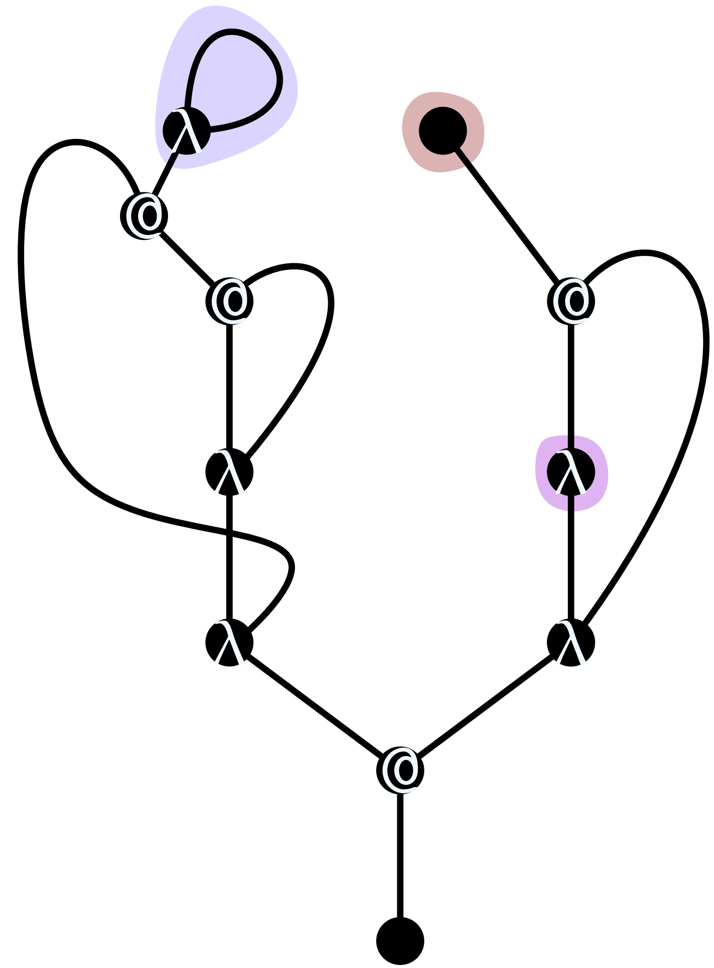

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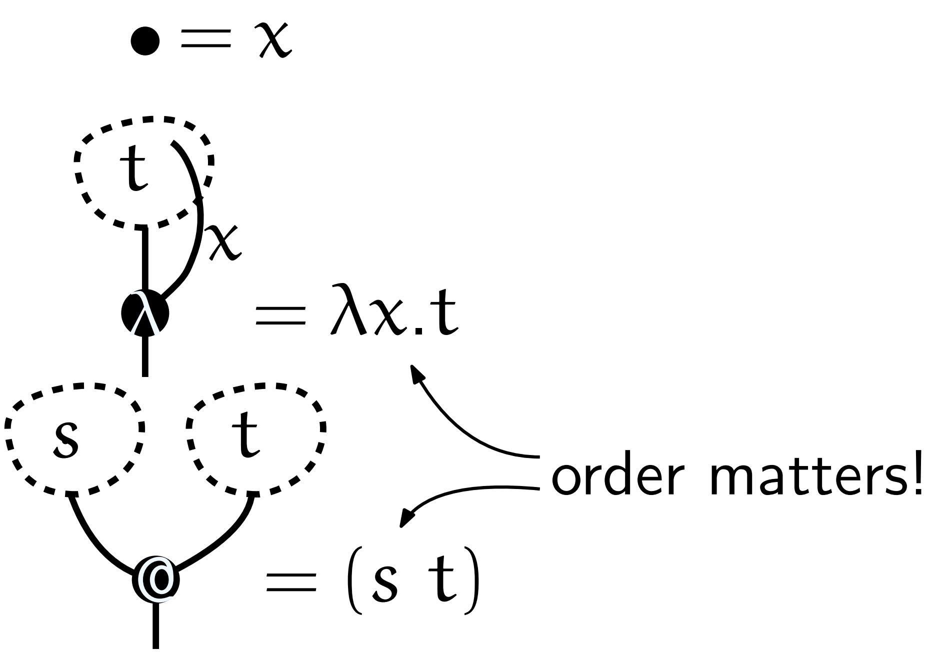

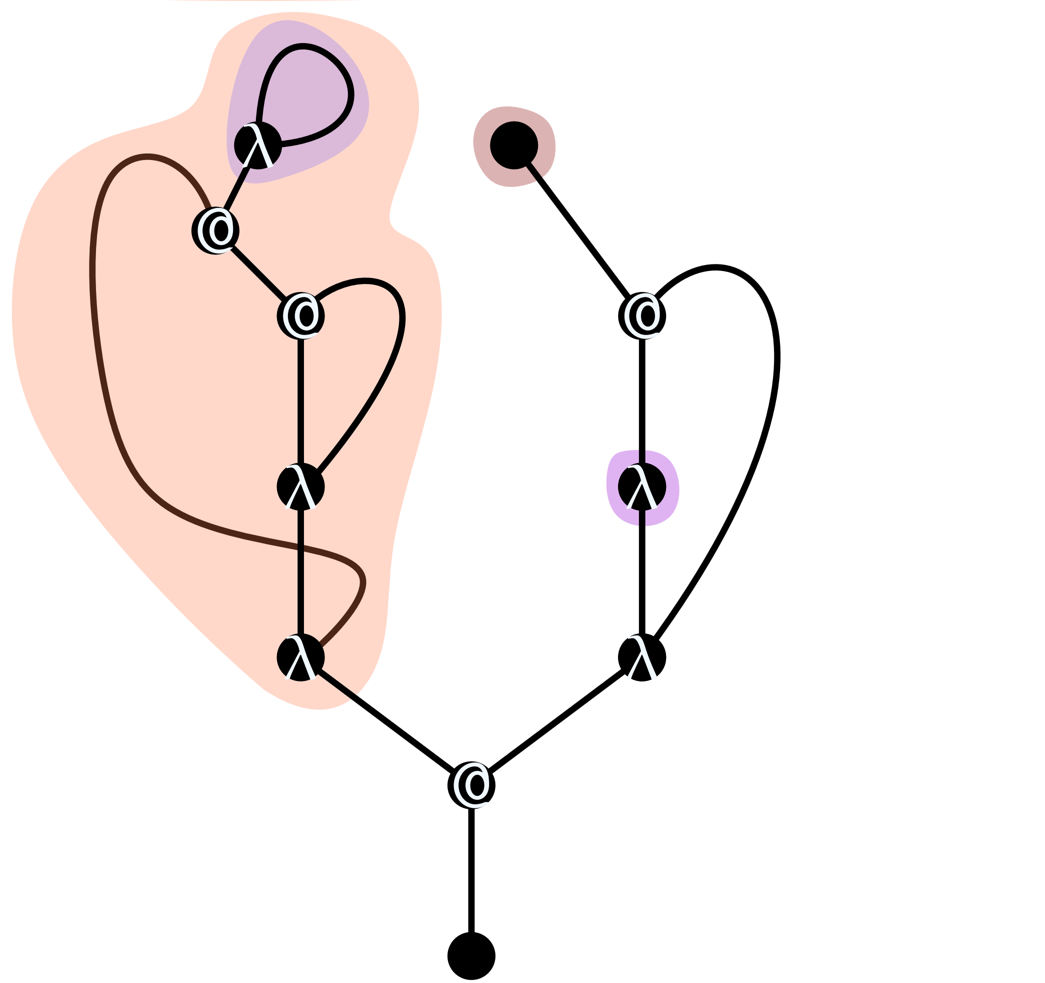

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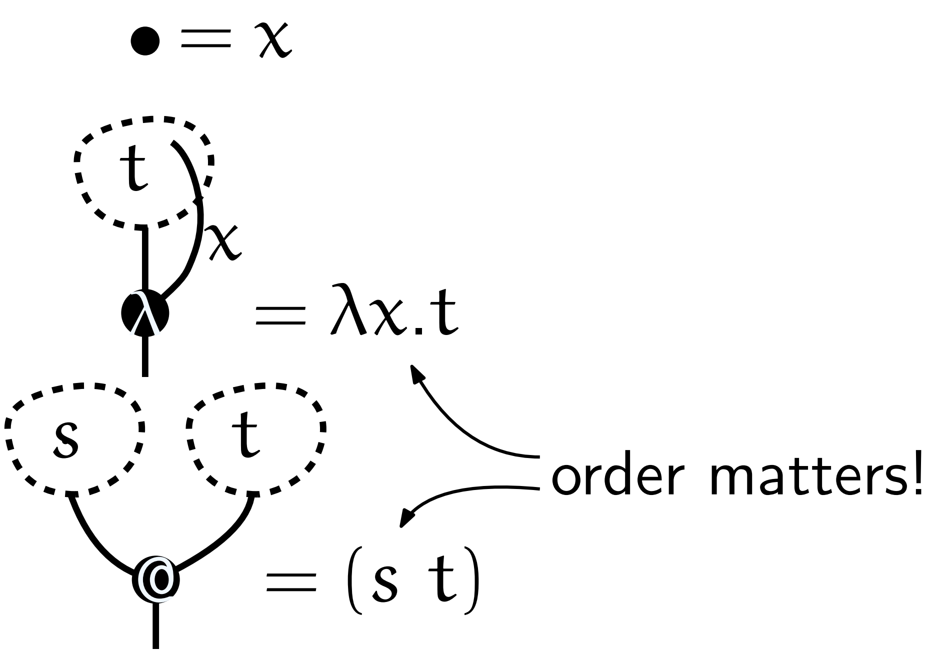

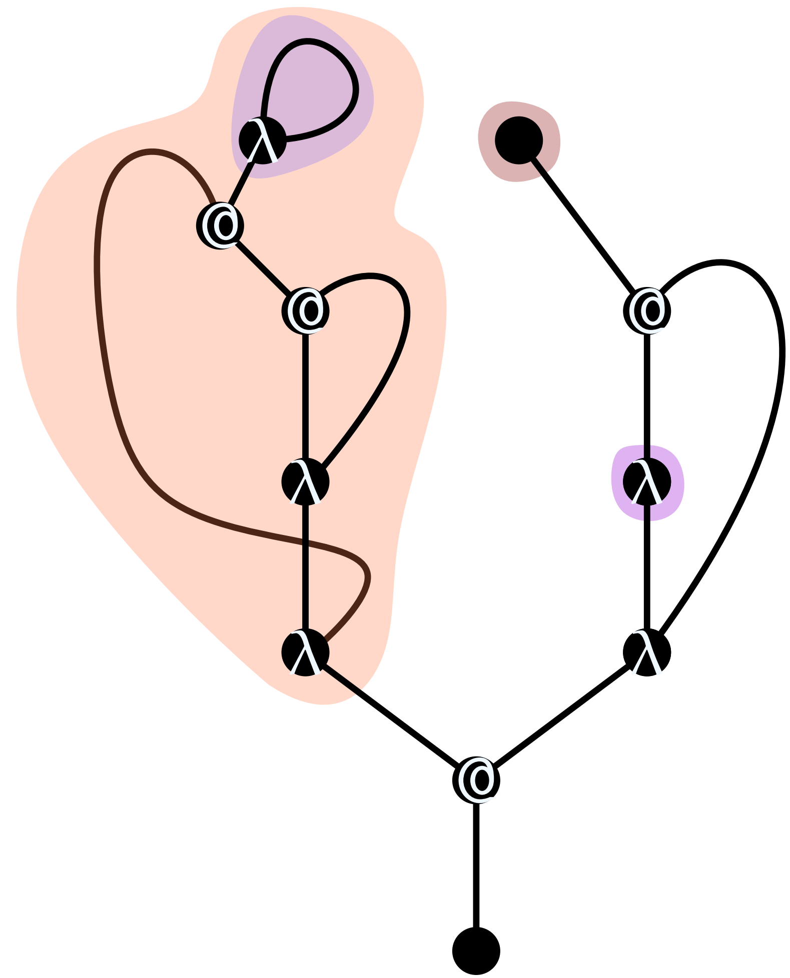

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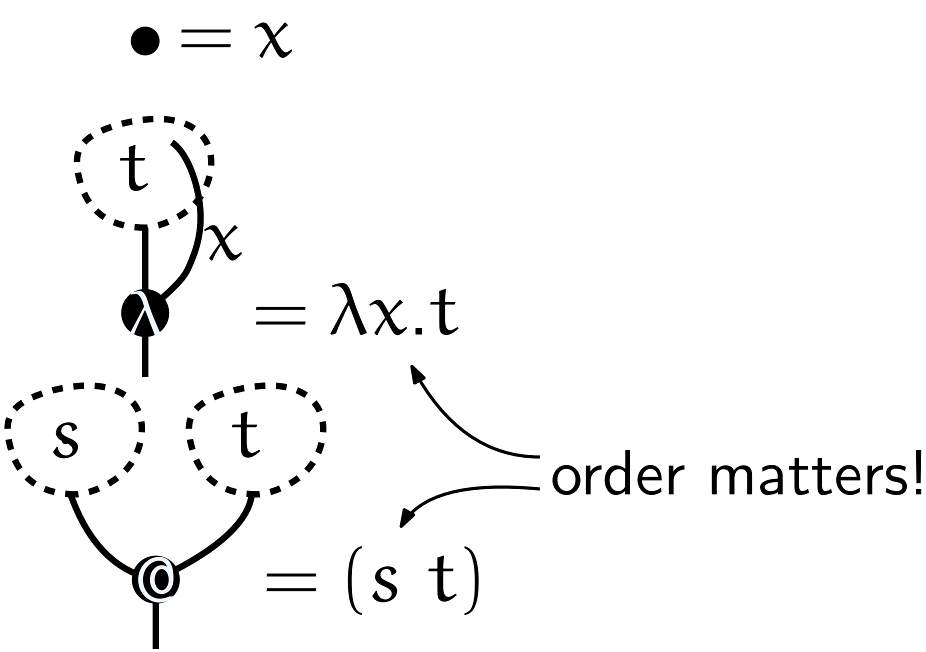

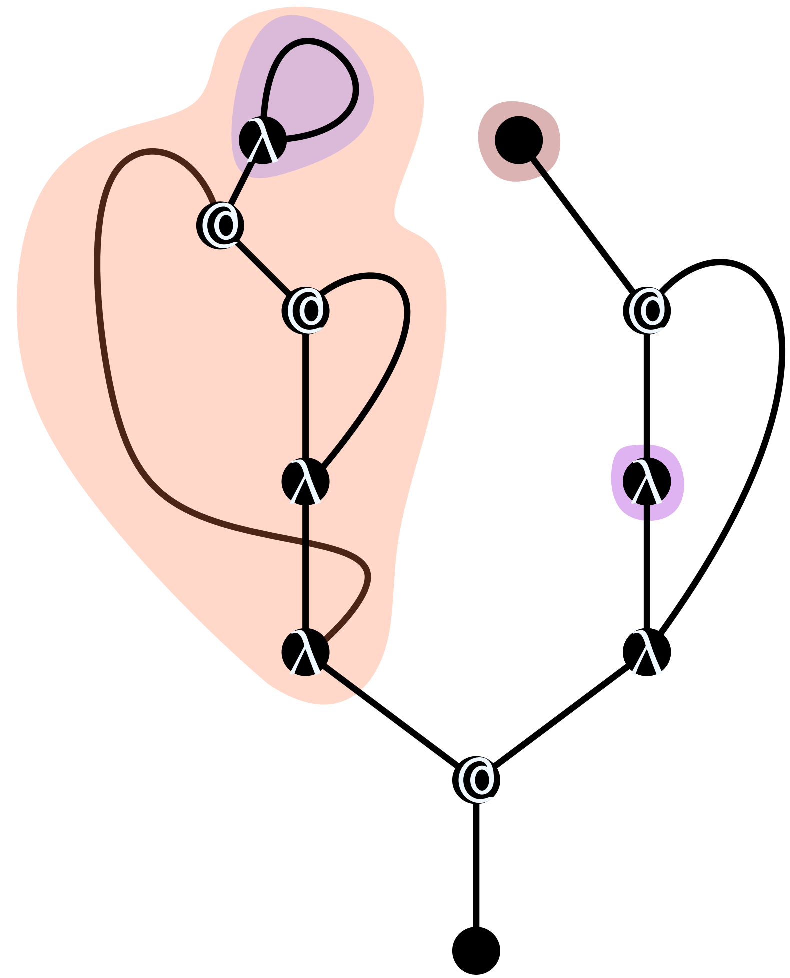

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Closed linear terms ↔ trivalent maps

Closed affine terms ↔ (2,3)-valent maps

Established in [BGJ13, BGGJ13]

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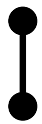

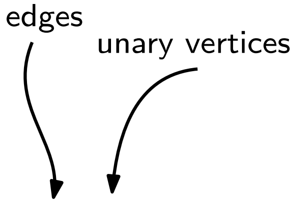

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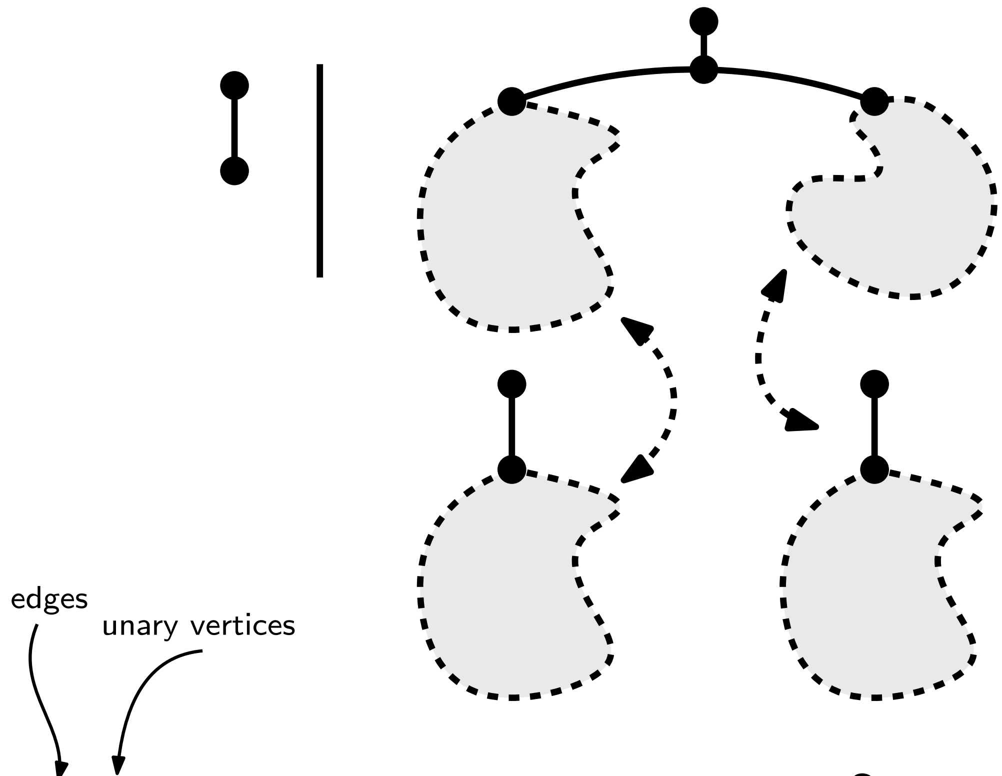

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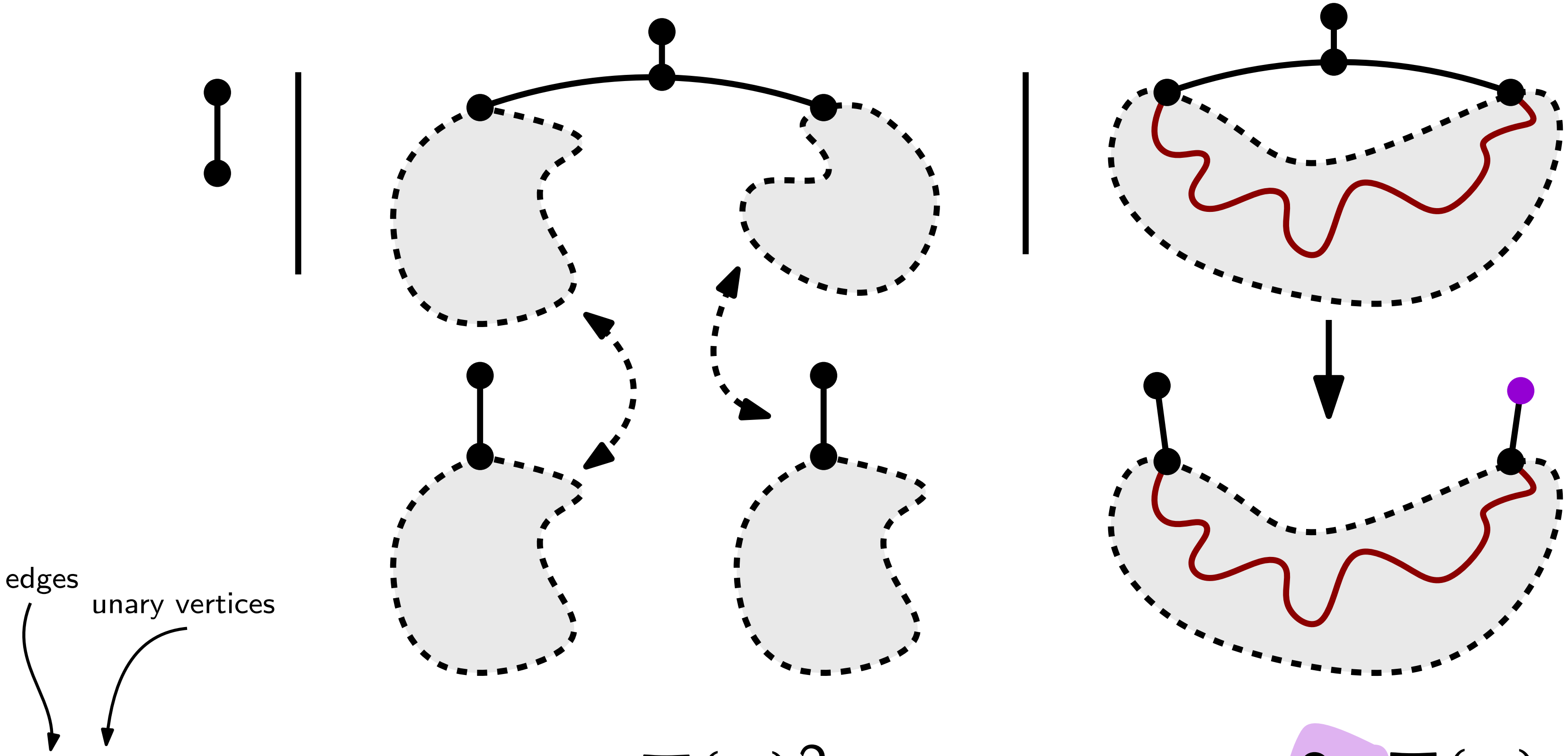

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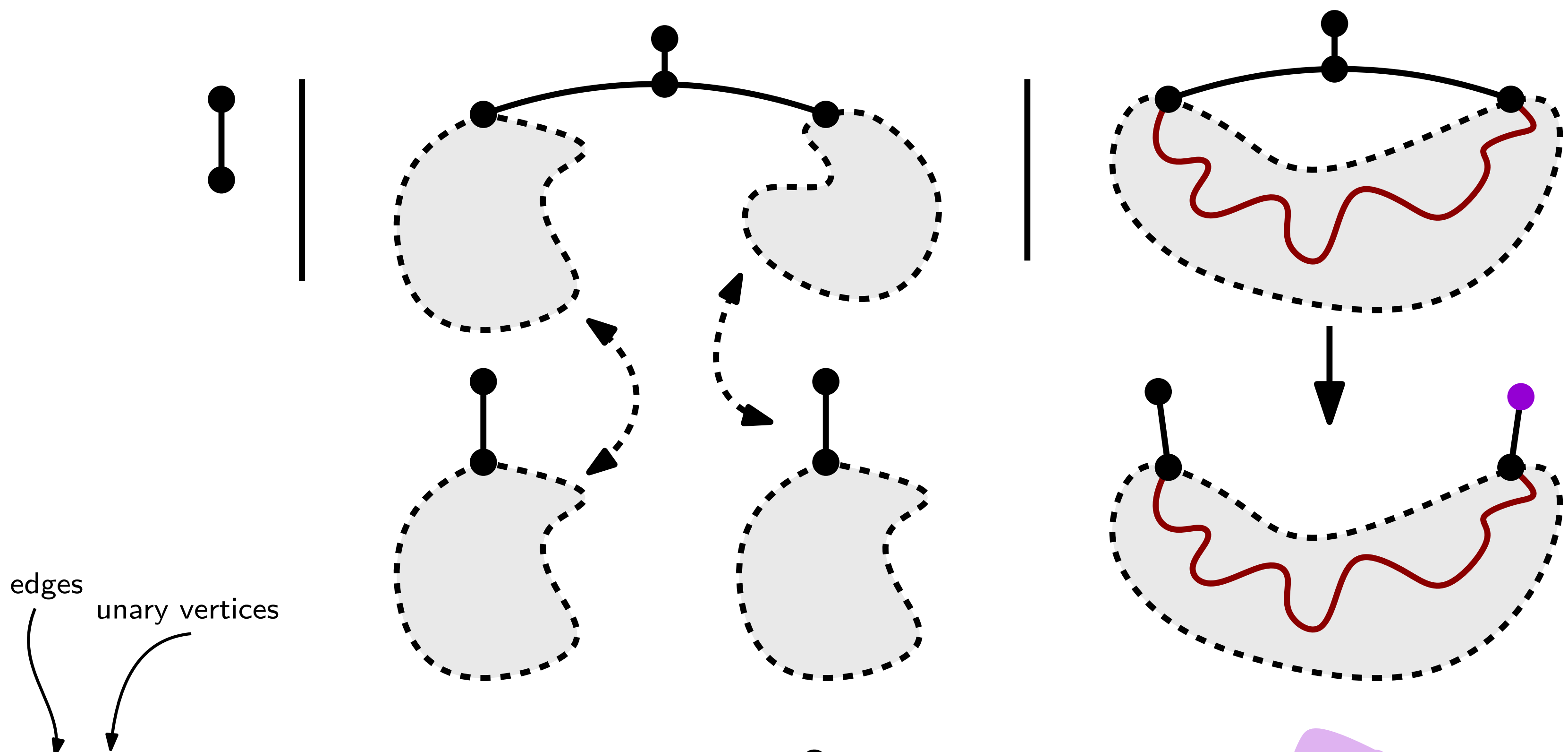

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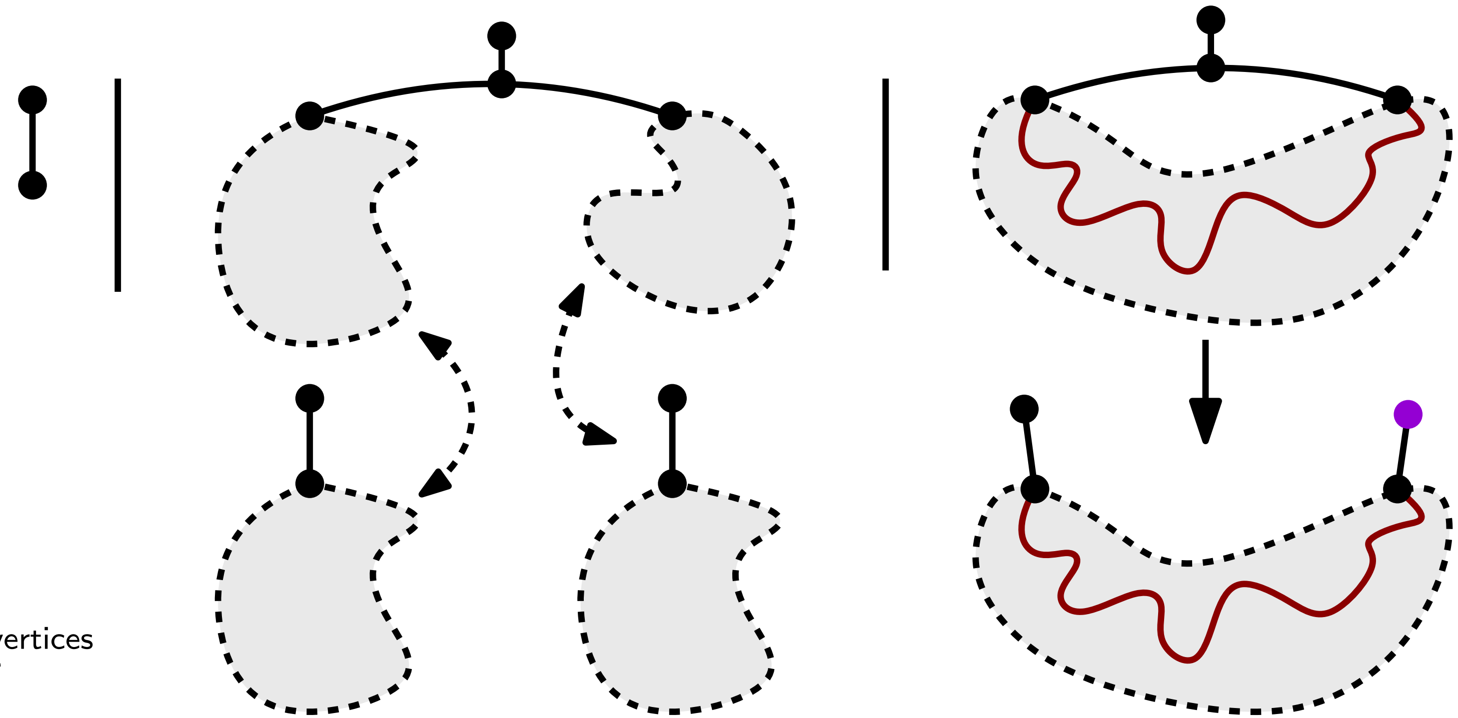

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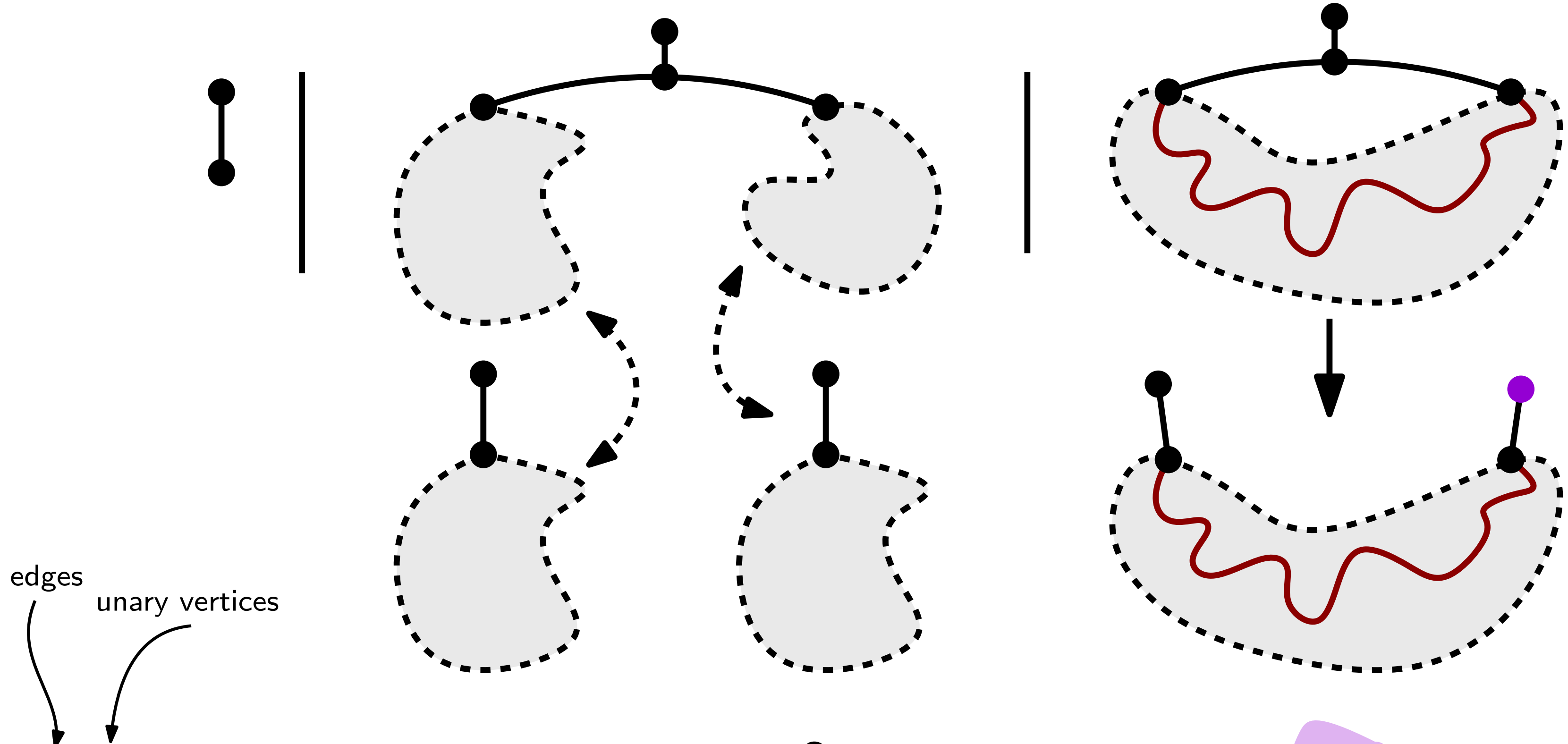

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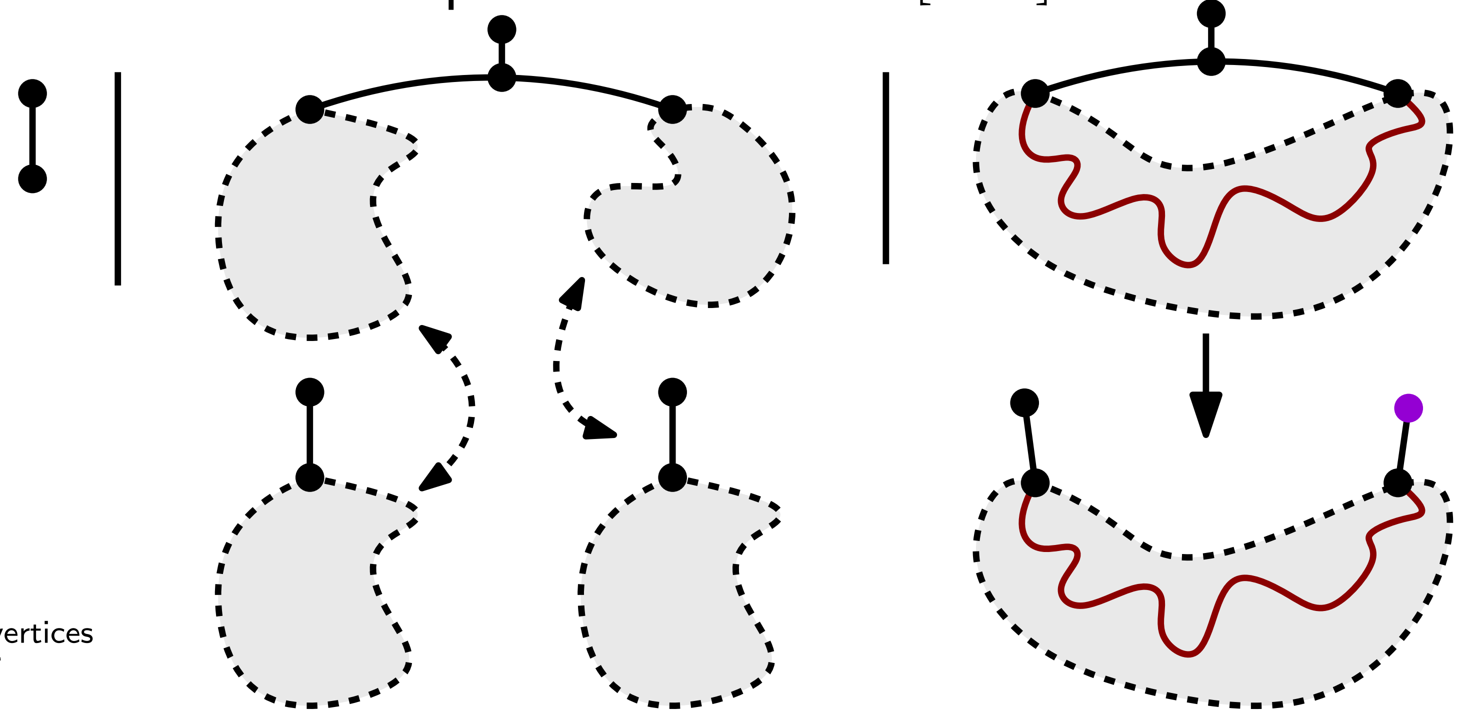
and open linear terms! [Z16]



edges, unary vertices

$$T(z, u) = z + zT(z)^2 + z\partial_u T(z)$$

subterms, free vars

$$\text{lin.term} = x \mid (s\ t) \mid \lambda x.t$$

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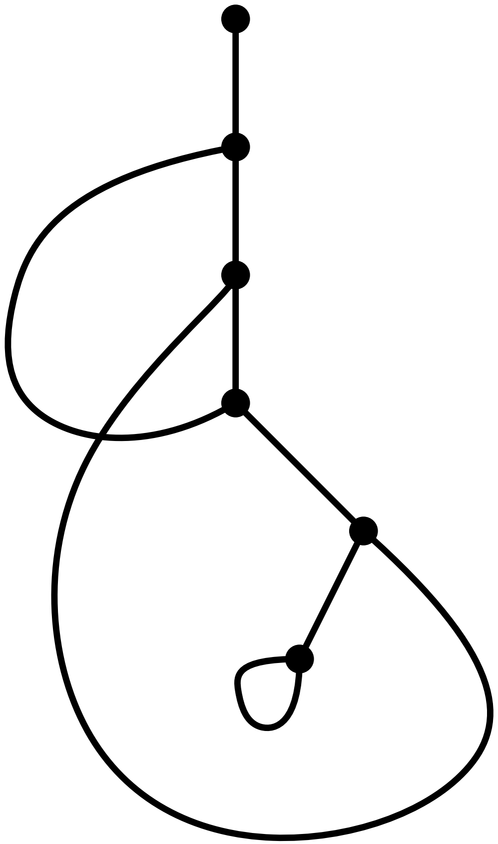
Our results: limit distributions

Closed trivalent maps ↔ closed linear terms



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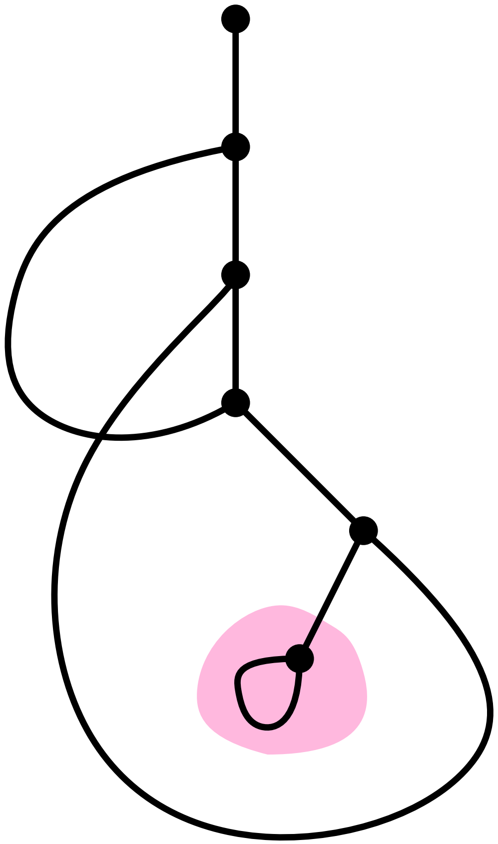

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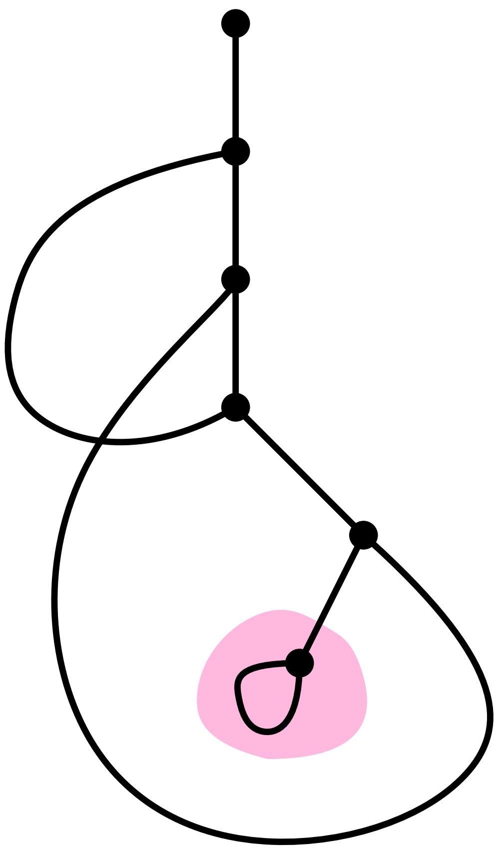

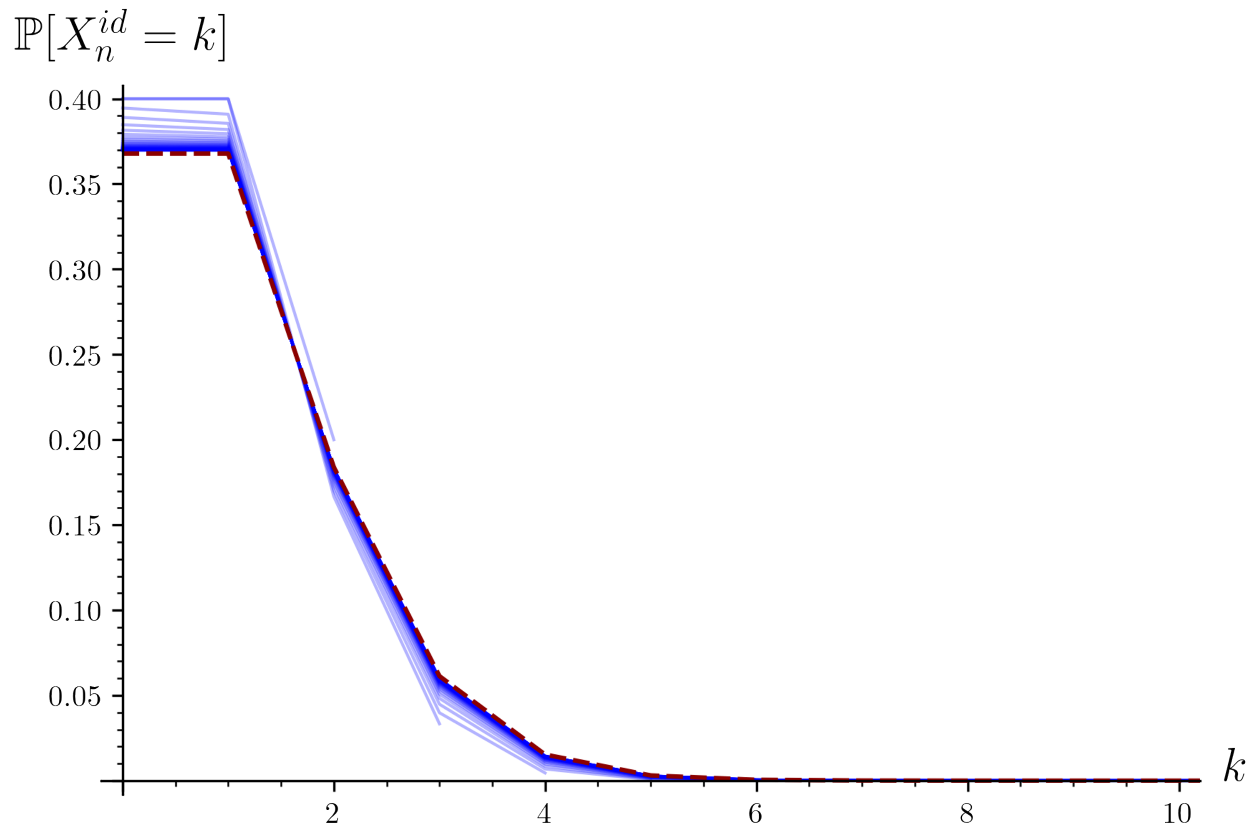

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$X_n^{id} \xrightarrow{D} \mathrm{Poisson}(1)$

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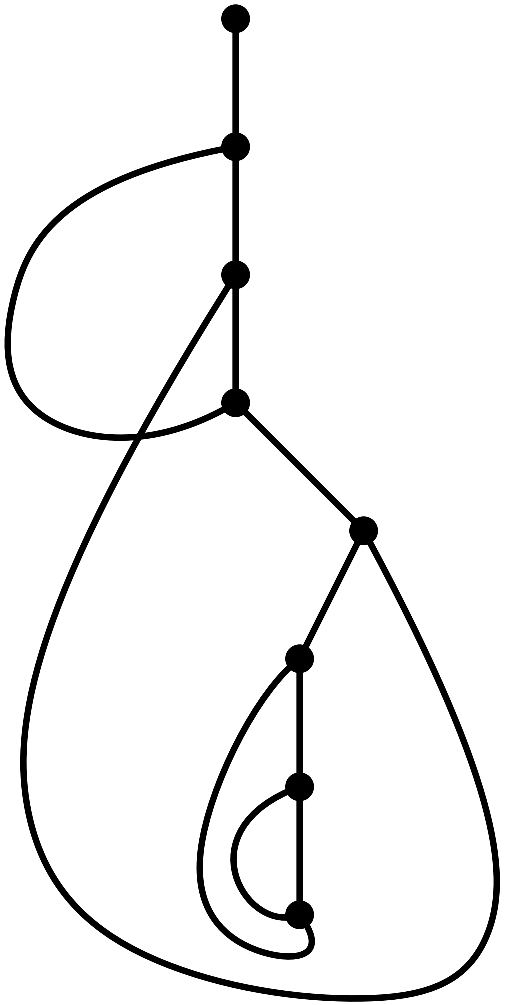
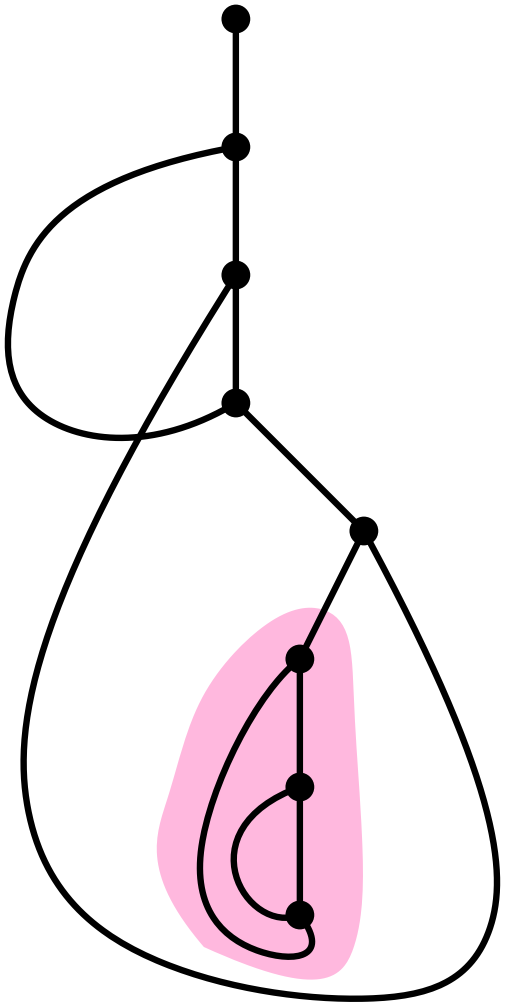


$\lambda x.\lambda y.(y\ \lambda z.\lambda w.zw)x$

Our results: limit distributions

Closed trivalent maps $\leftrightarrow$ closed linear terms

bridges = # closed subterms



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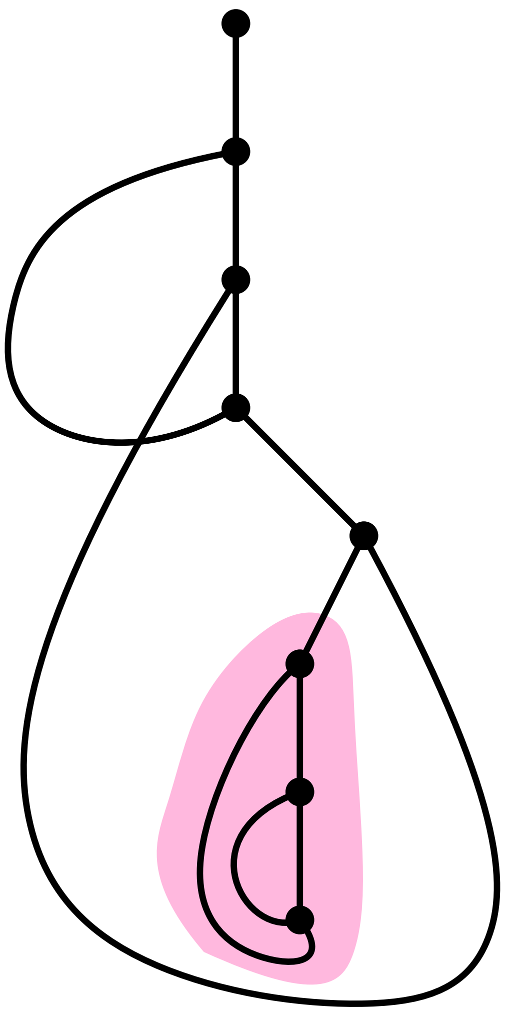

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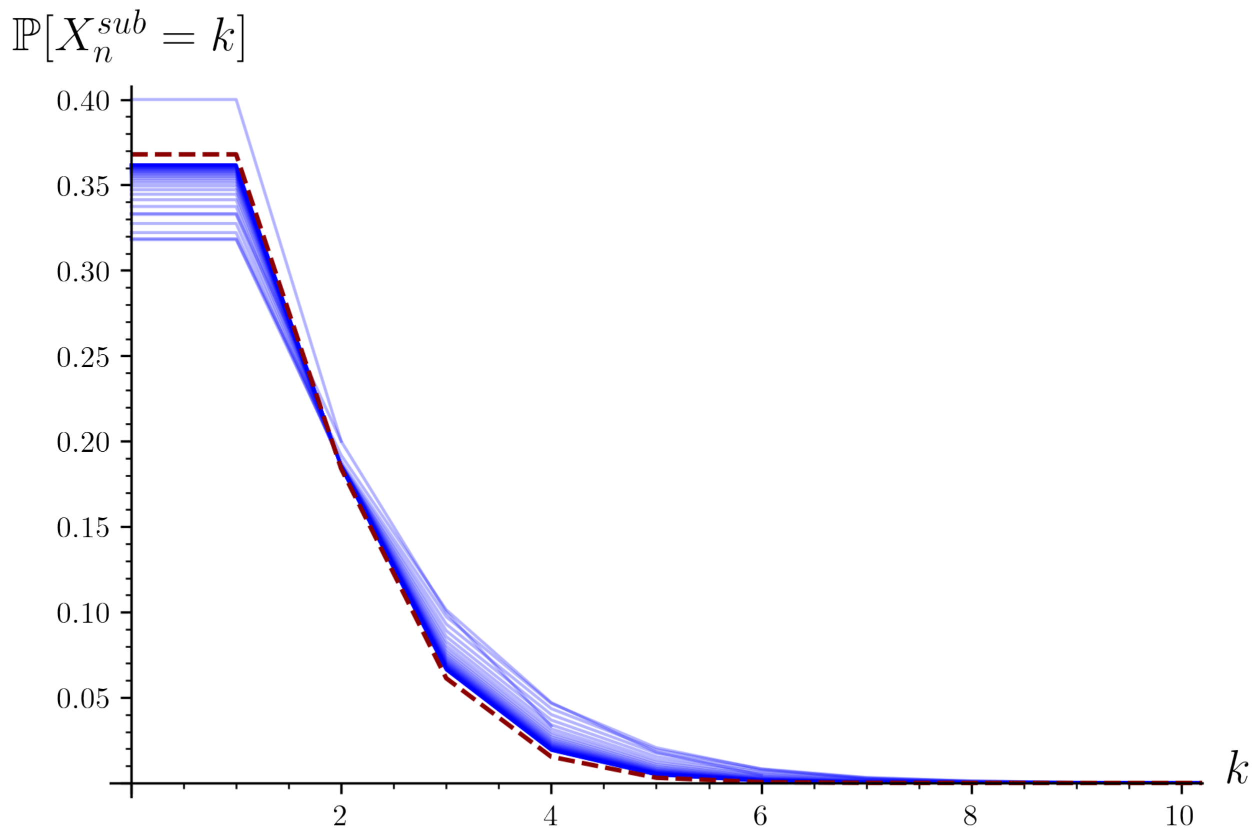

$$X_n^{sub} \xrightarrow{D} \text{Poisson}(1)$$

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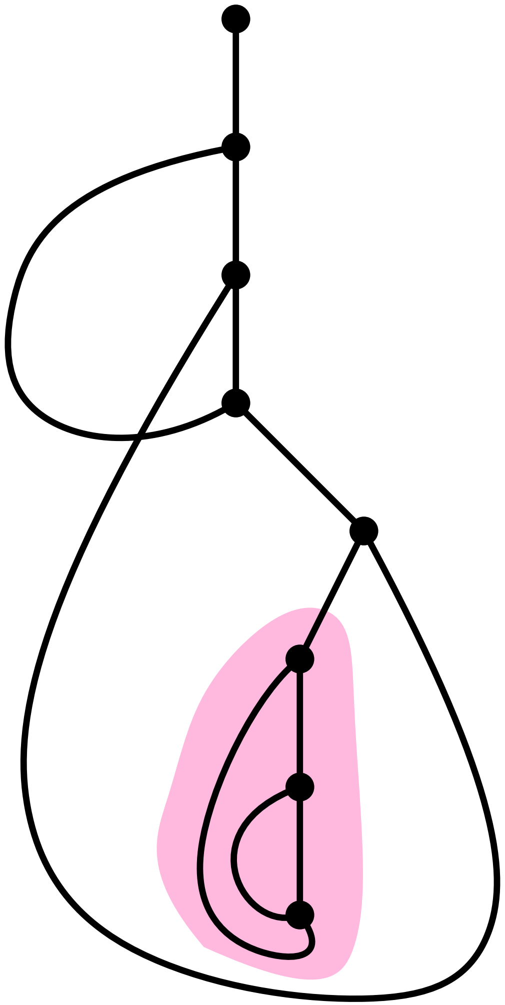
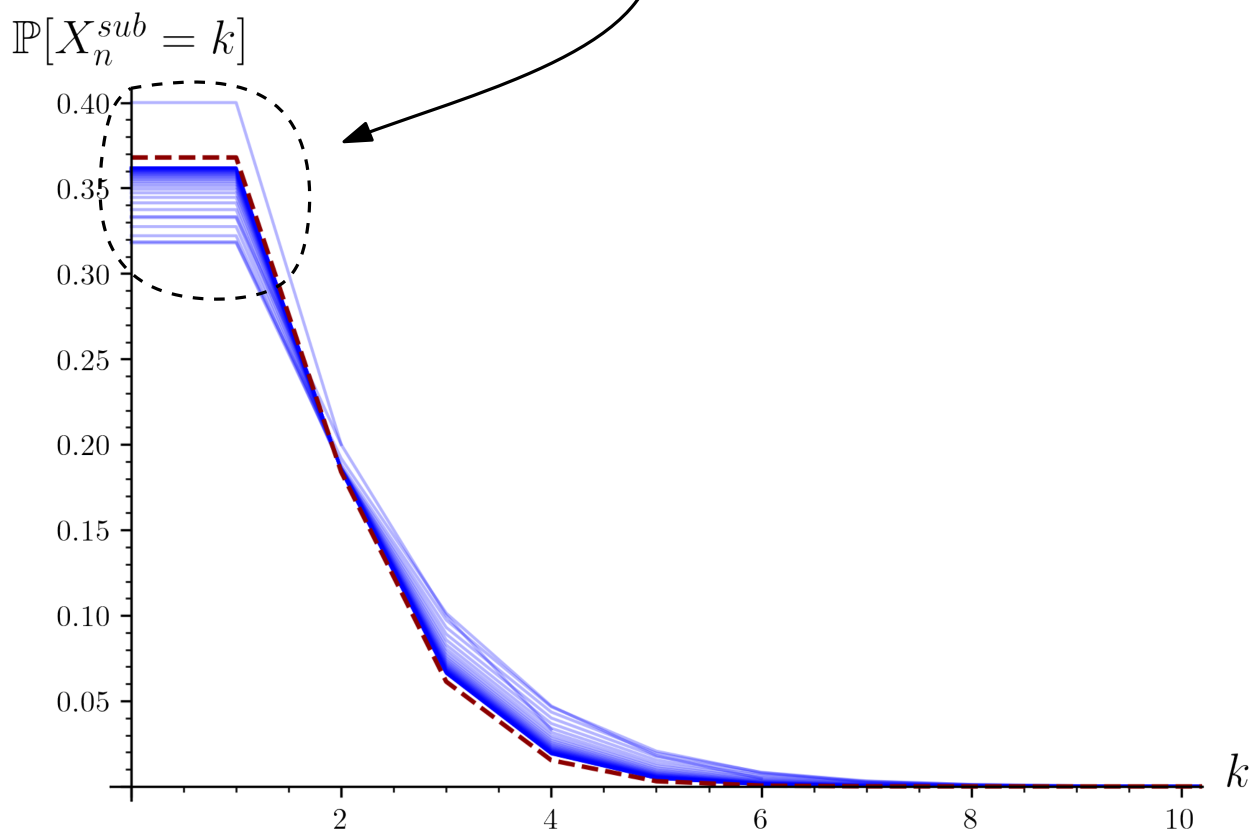
one bridge ↔ no bridge





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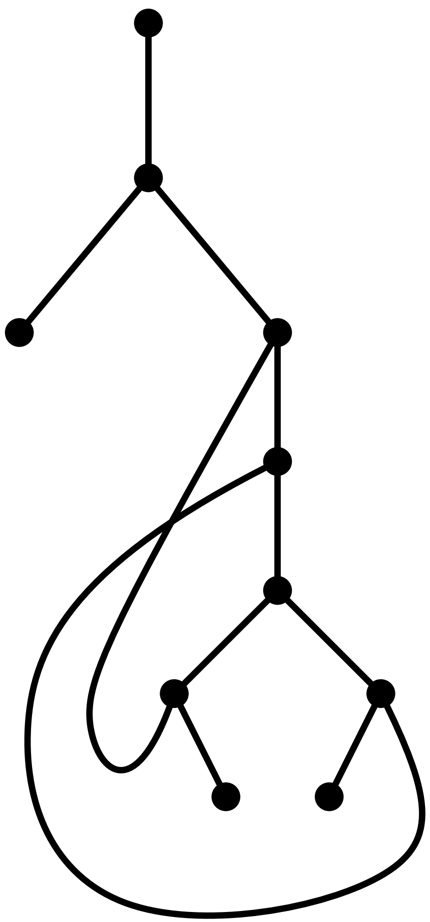
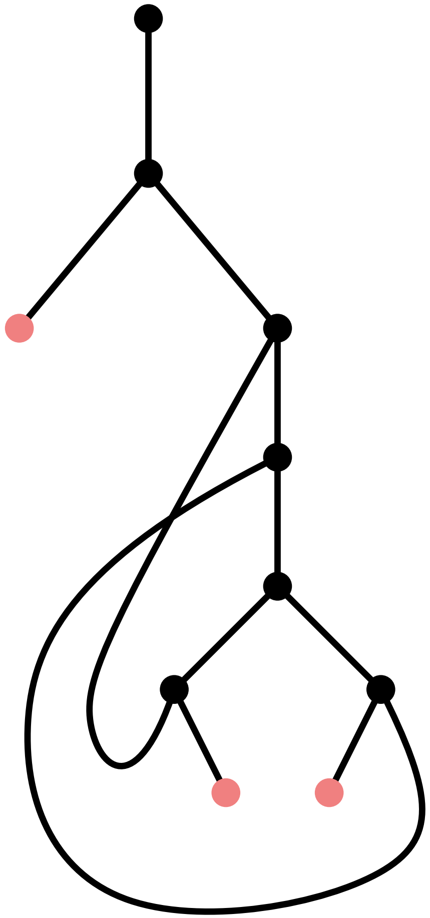


$(a\ (\lambda x.\lambda y.(y\ b)(c\ x)))$

Our results: limit distributions

Open trivalent maps ↔ open linear terms

unary vertices = # free vars



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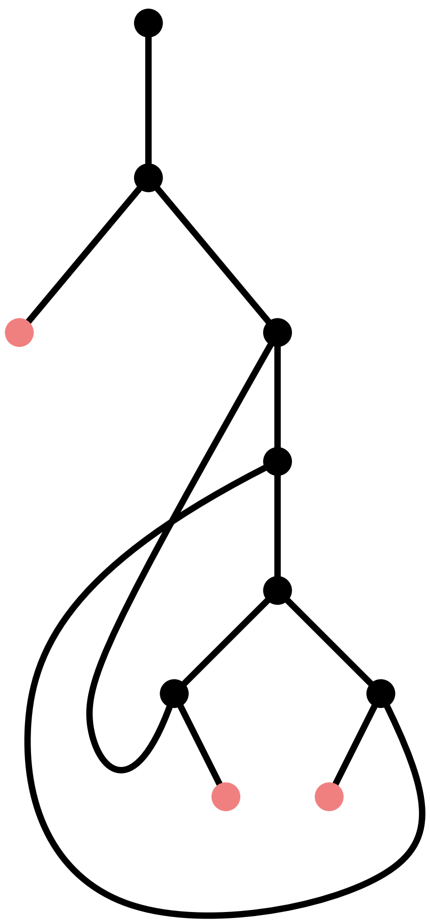

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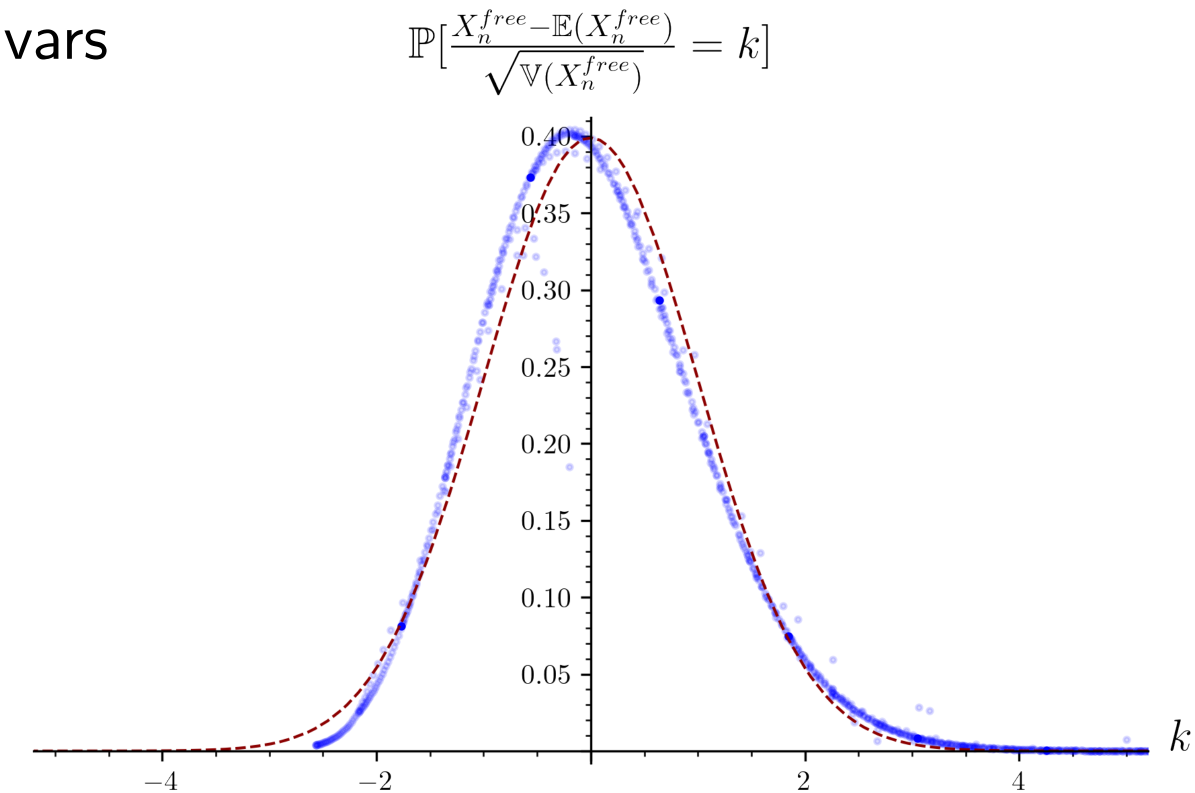
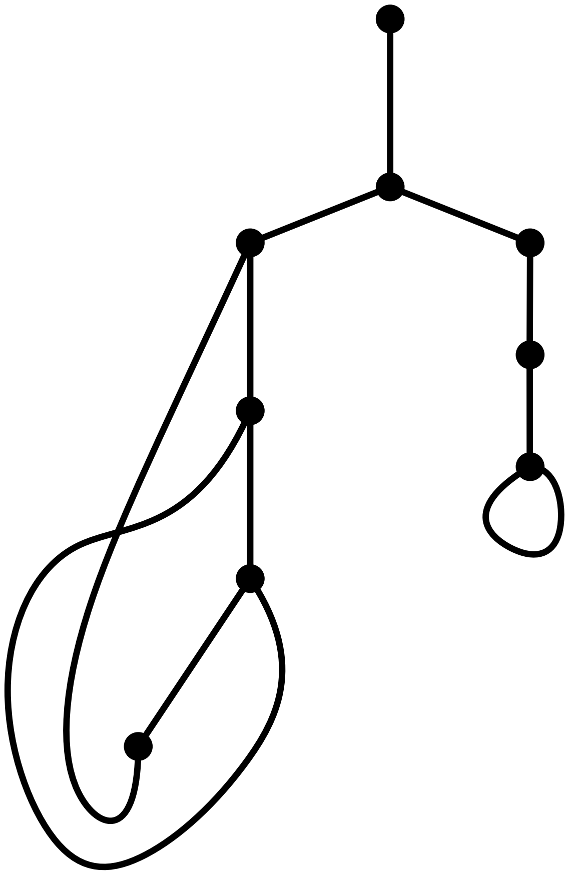


$$\frac{X_n^{free}-\mu_n}{\sqrt{\sigma_n^2}} \xrightarrow{D} \mathcal{N}(0,1)$$

for $\mu = \sigma^2 = (2n)^{1/3}$

Our results: limit distributions

(2,3)-valent maps $\leftrightarrow$ closed affine terms



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Our results: limit distributions

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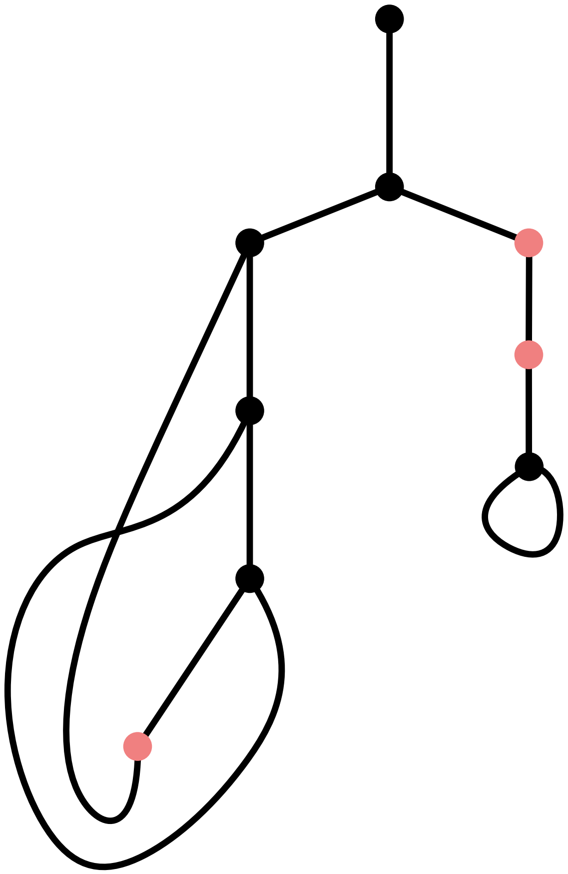

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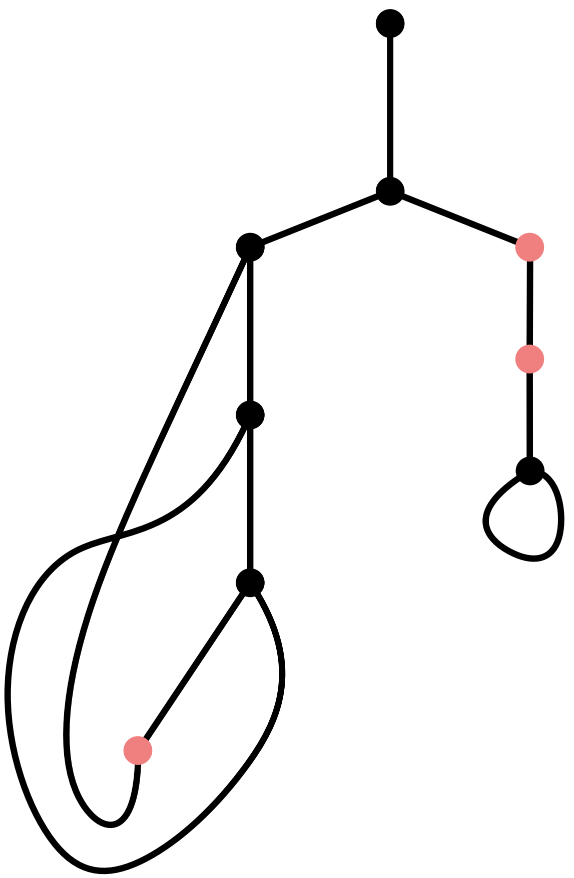

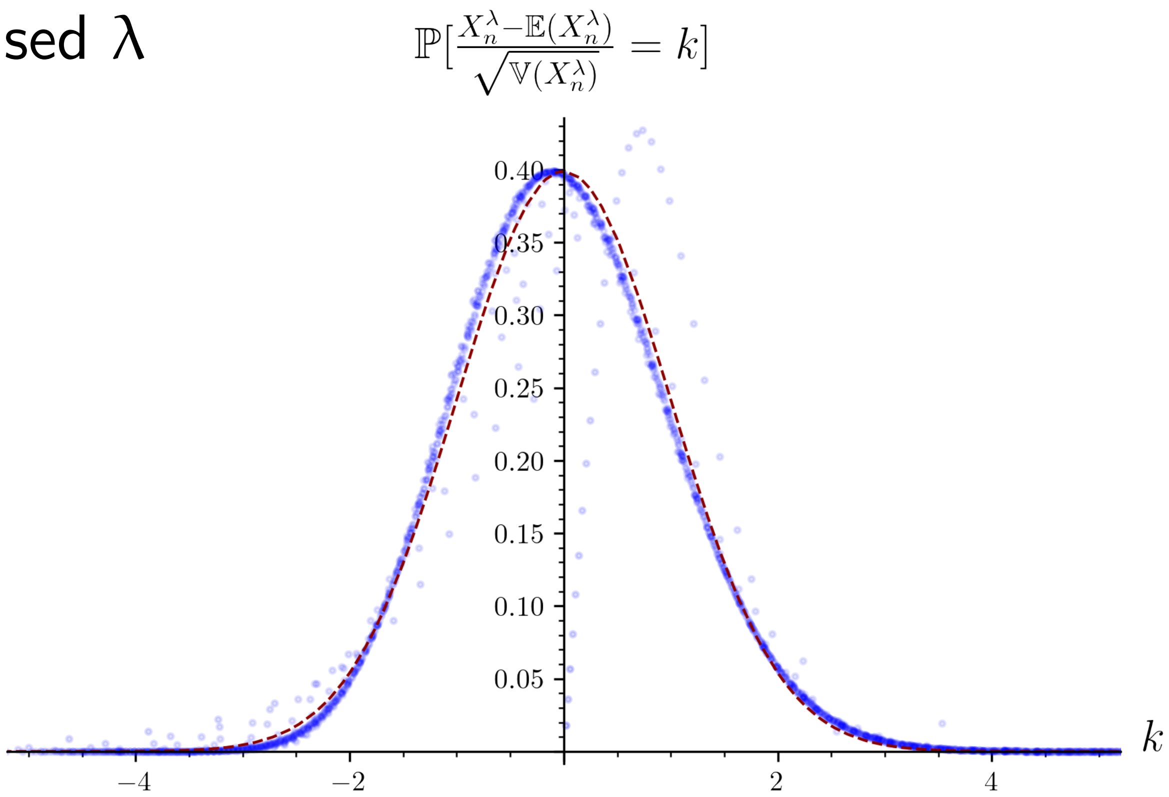

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- Schema based on compositions (see also [B75,FS93,B18,P19,BKW21]):

$F(z, u, G(z, u))$ — inherits the limit law of → $G(z, u)$

Proof sketch for loops/id-subterms:

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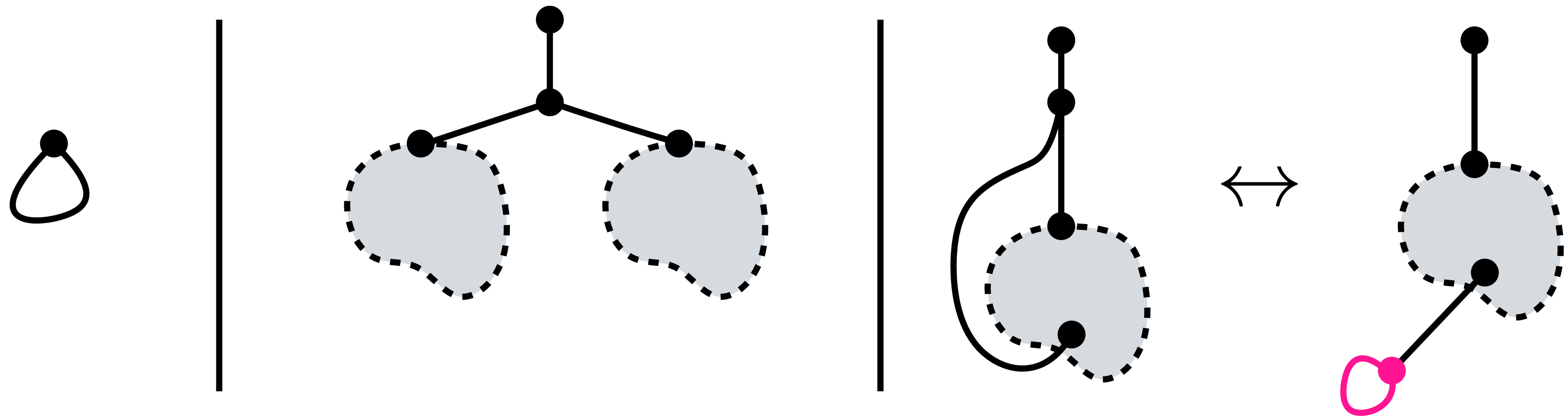
$$T_0^{\mathrm{id}}(z,u) = (u-1)z^2 + zT_0^{\mathrm{id}}(z,u)^2 + \partial_u T_0^{\mathrm{id}}(z,u)$$



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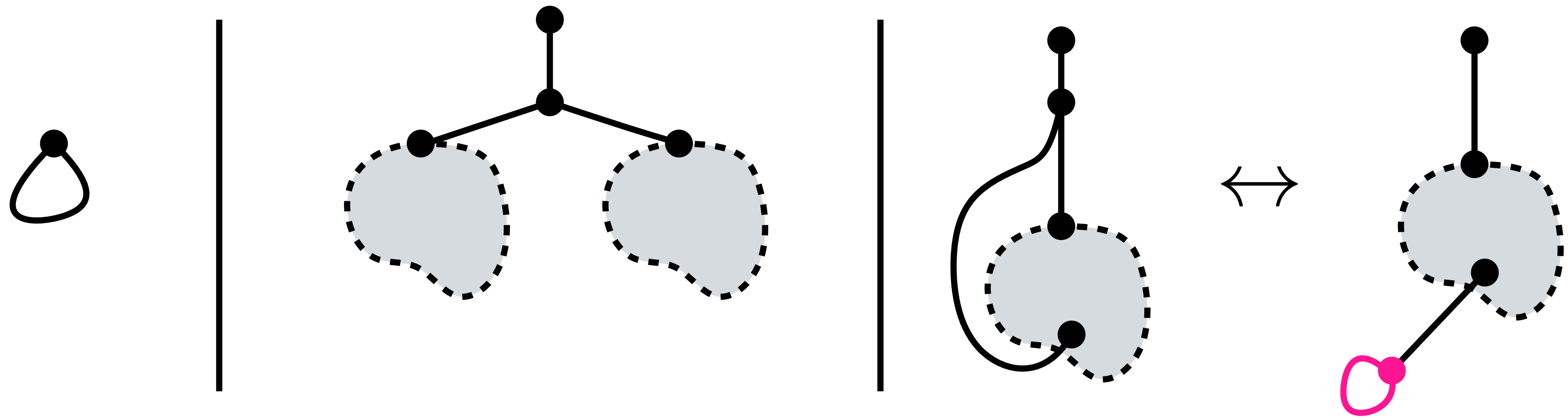

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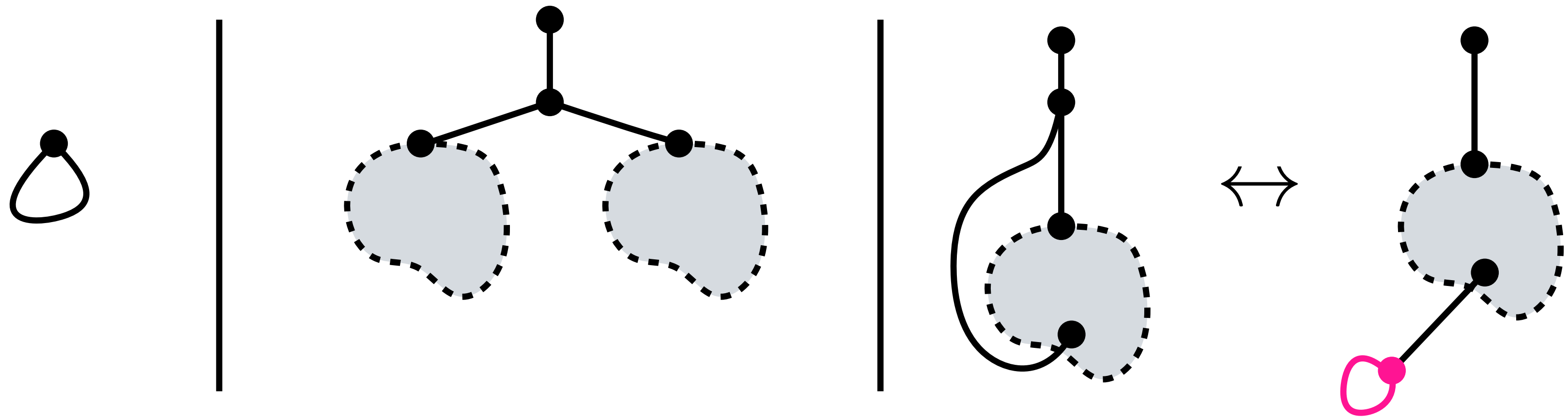

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$$[z^n] \quad \partial_u T_0^{id}\big|_{v=1} = T_0^{id} - (u-1)z^2 - z(T_0^{id})^2 \qquad \sim [z^n]T_0^{id}(z,1)$$

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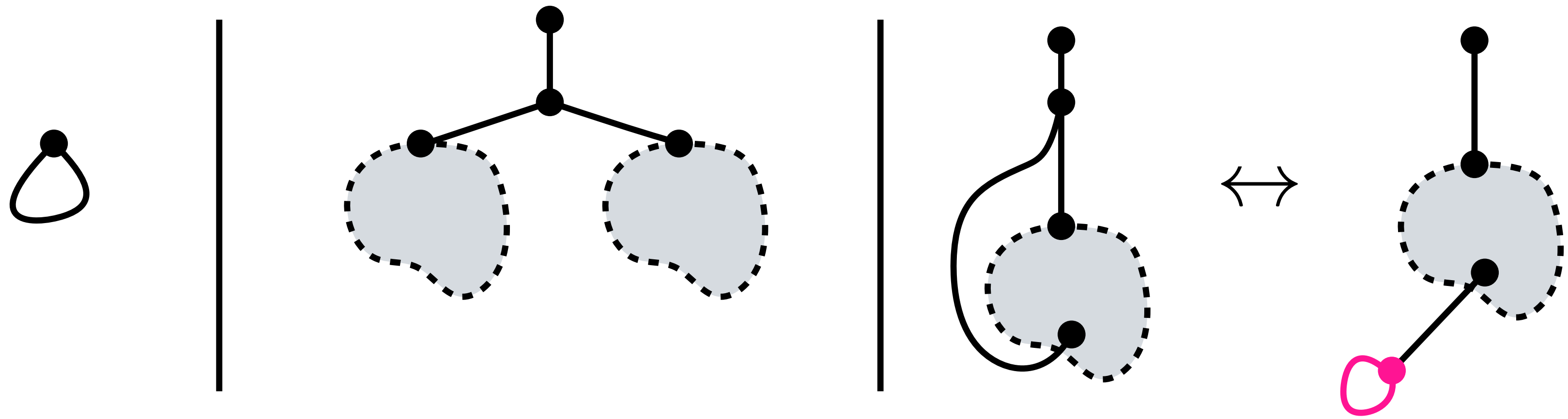

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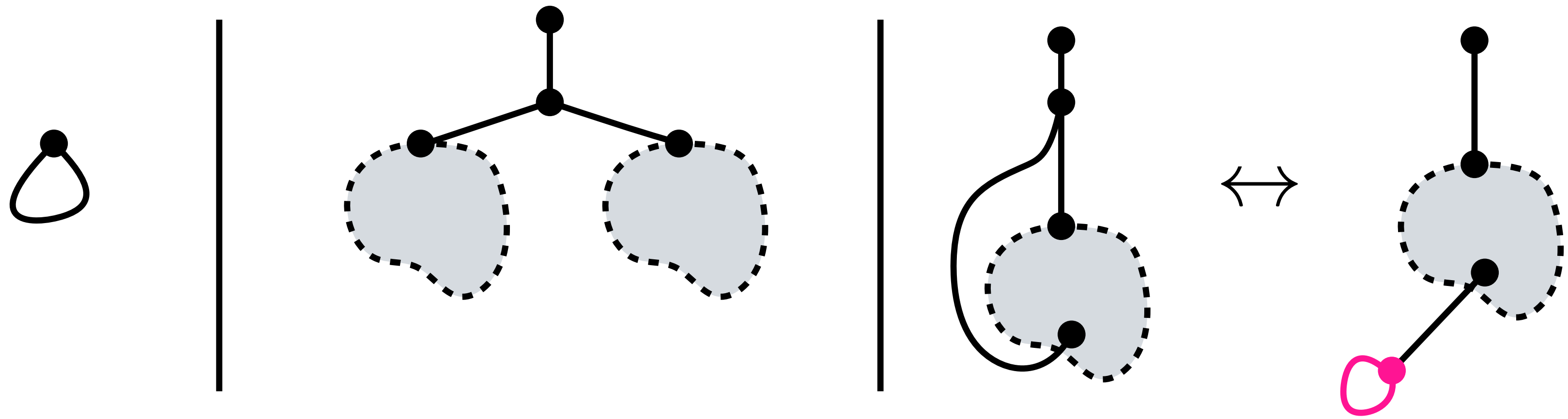

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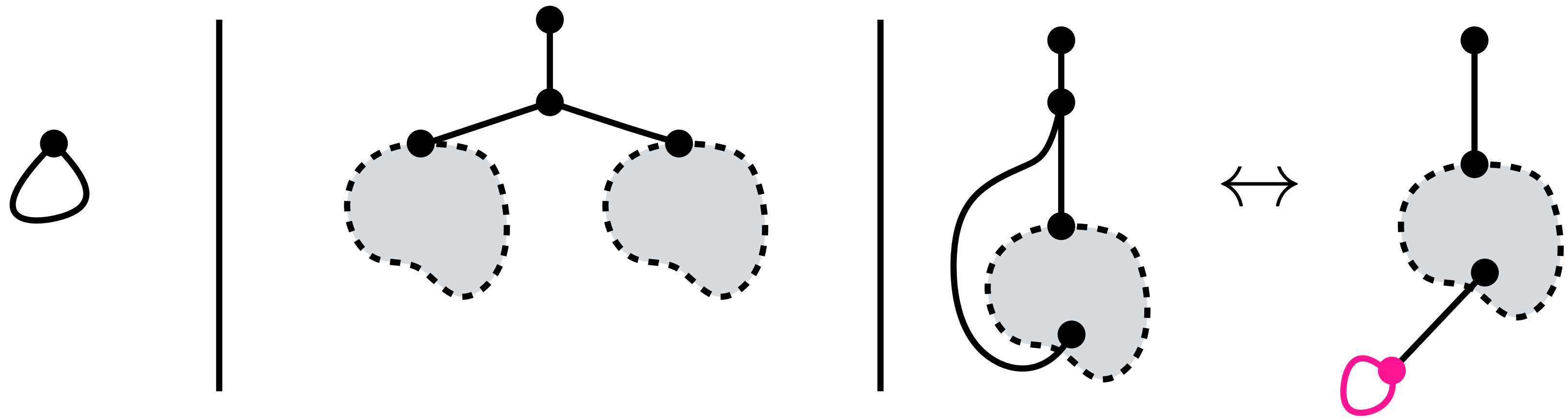

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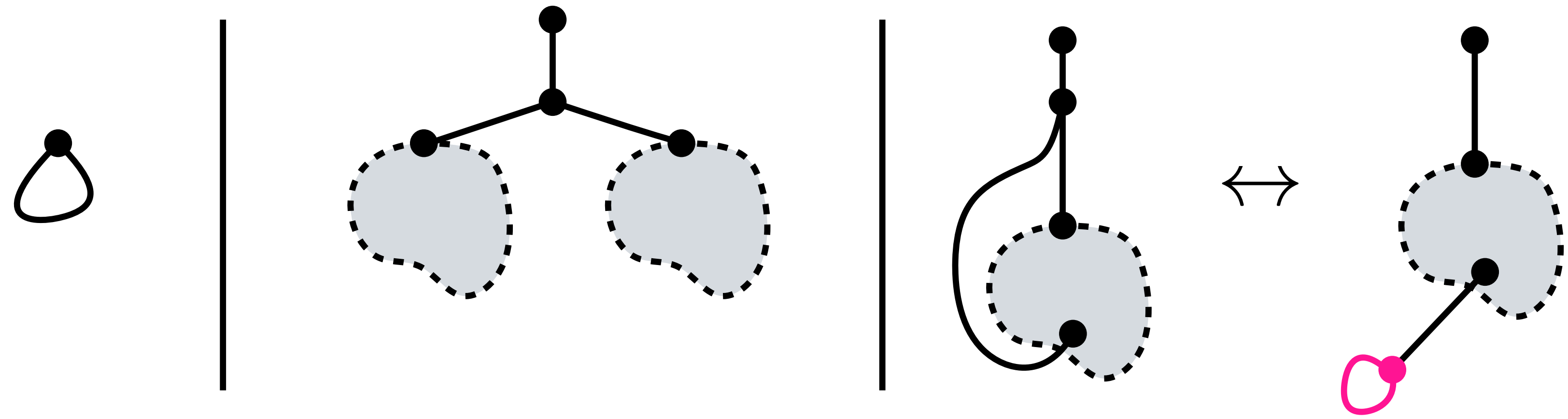

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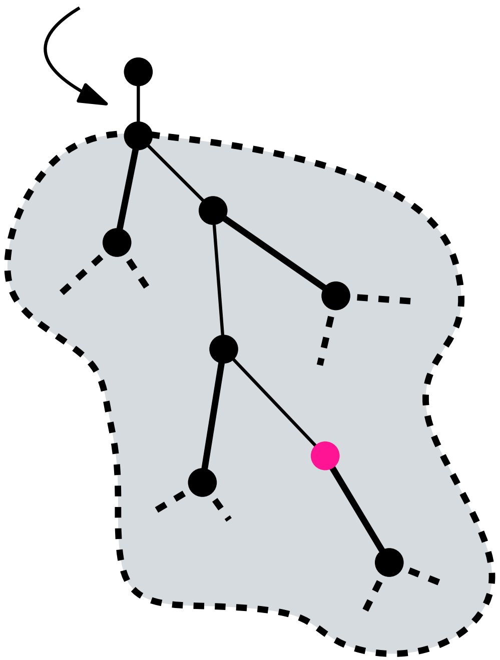
Schema then yields Poisson(1) limit law

Proof sketch for bridges/closed subterms:

spanning tree def'd by term



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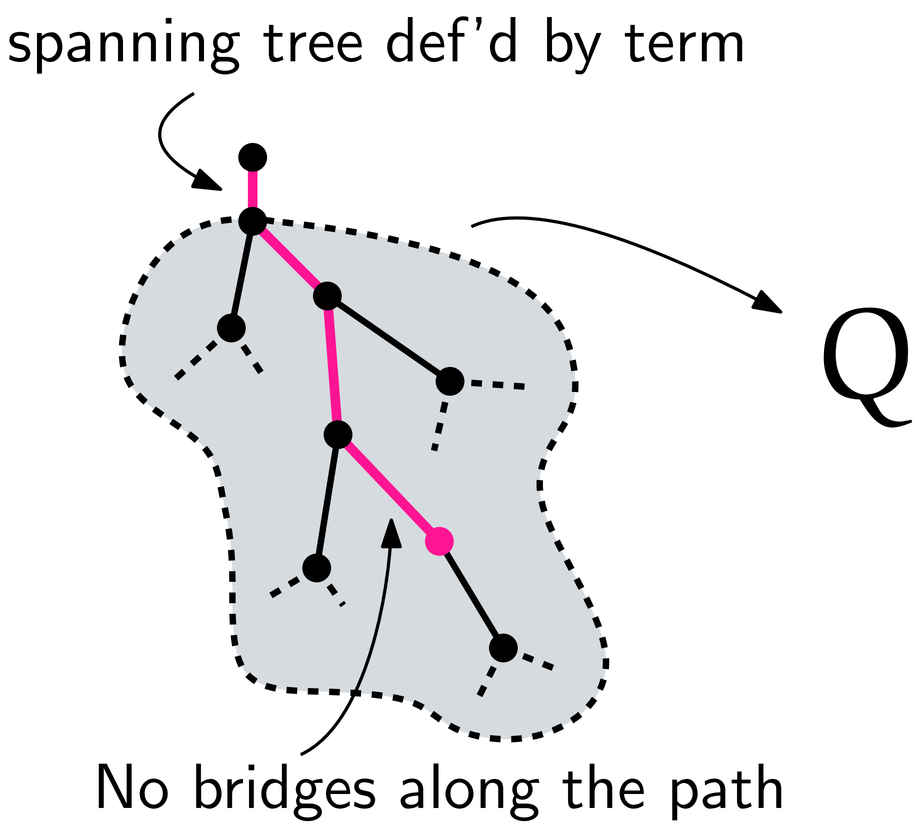

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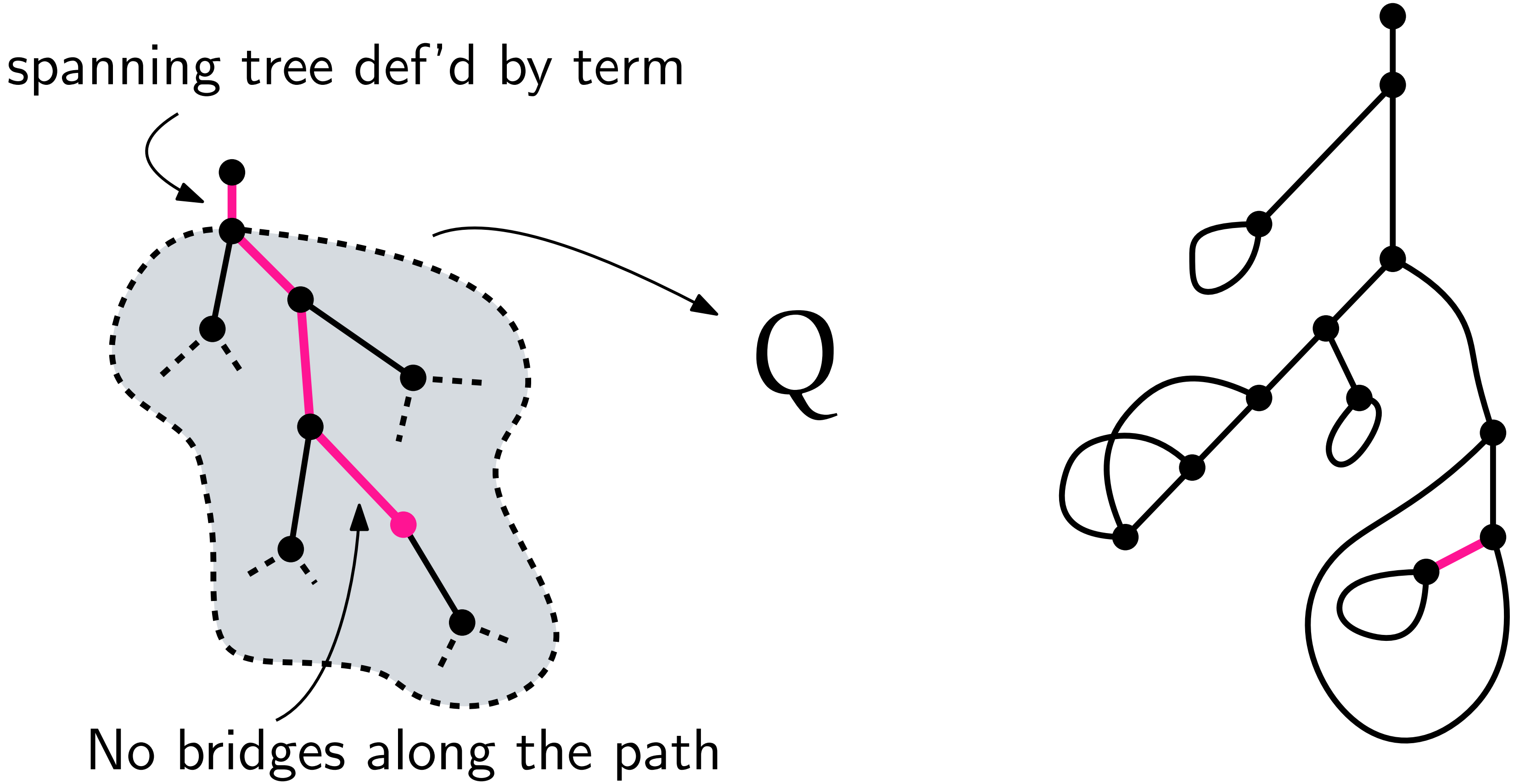

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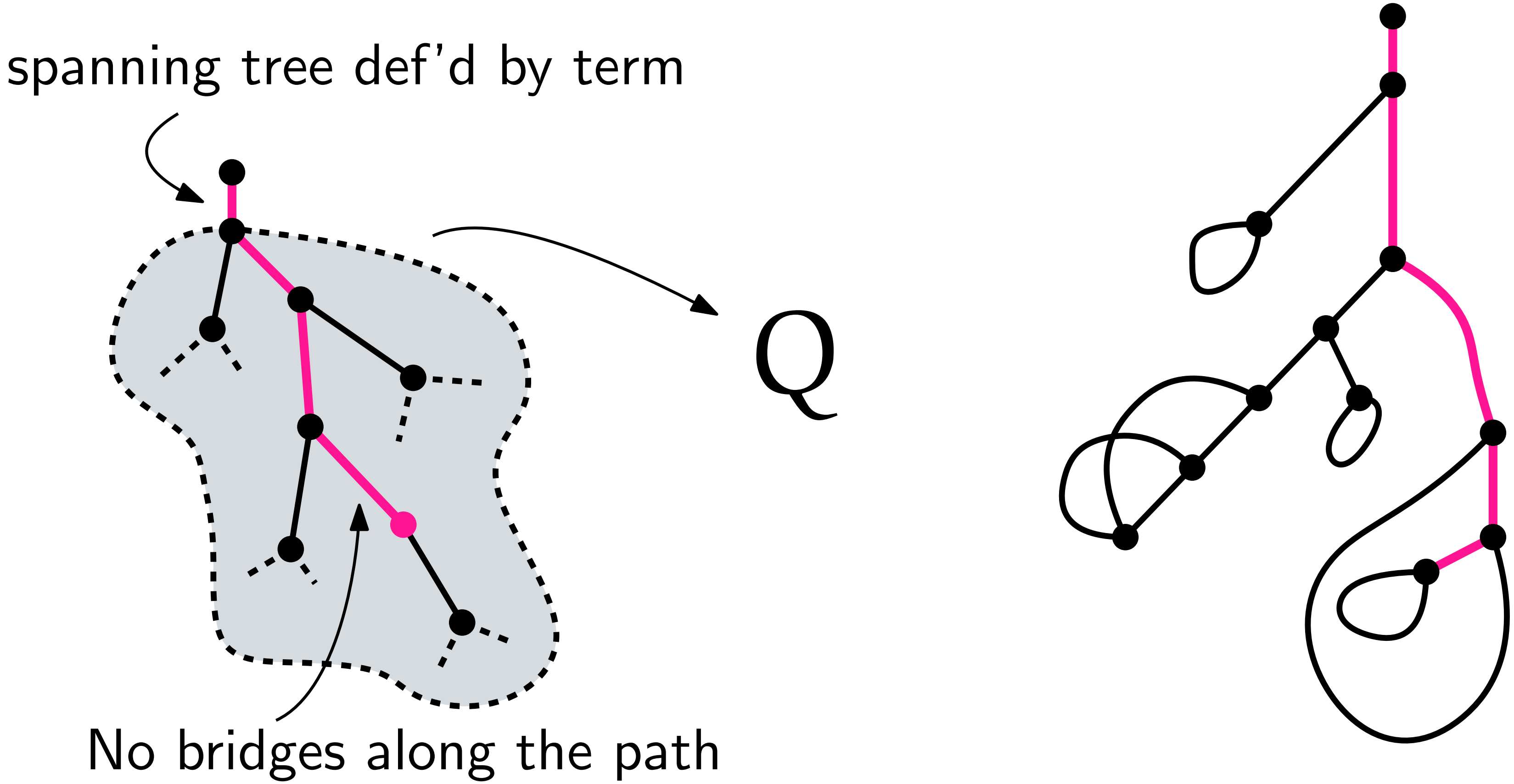

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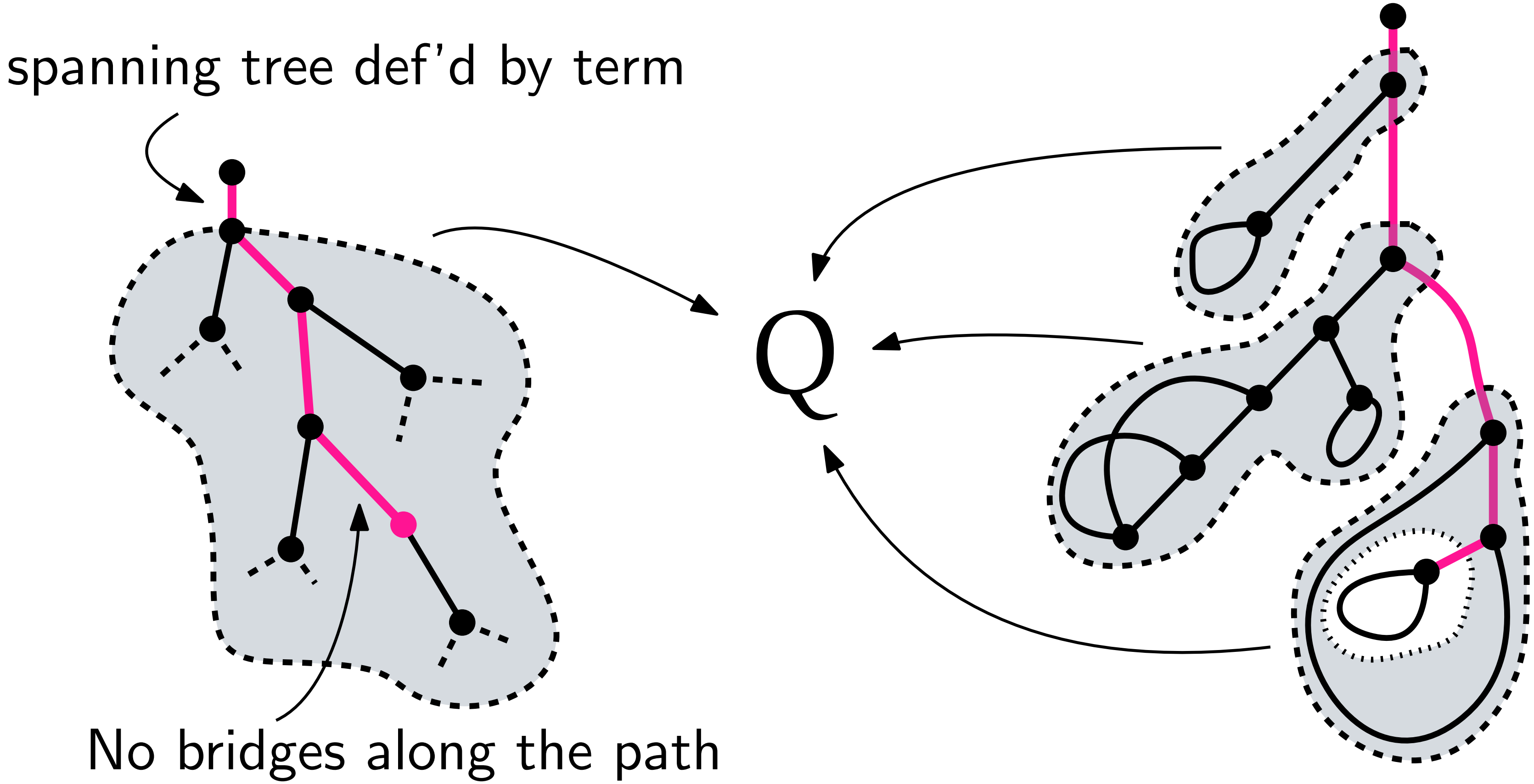

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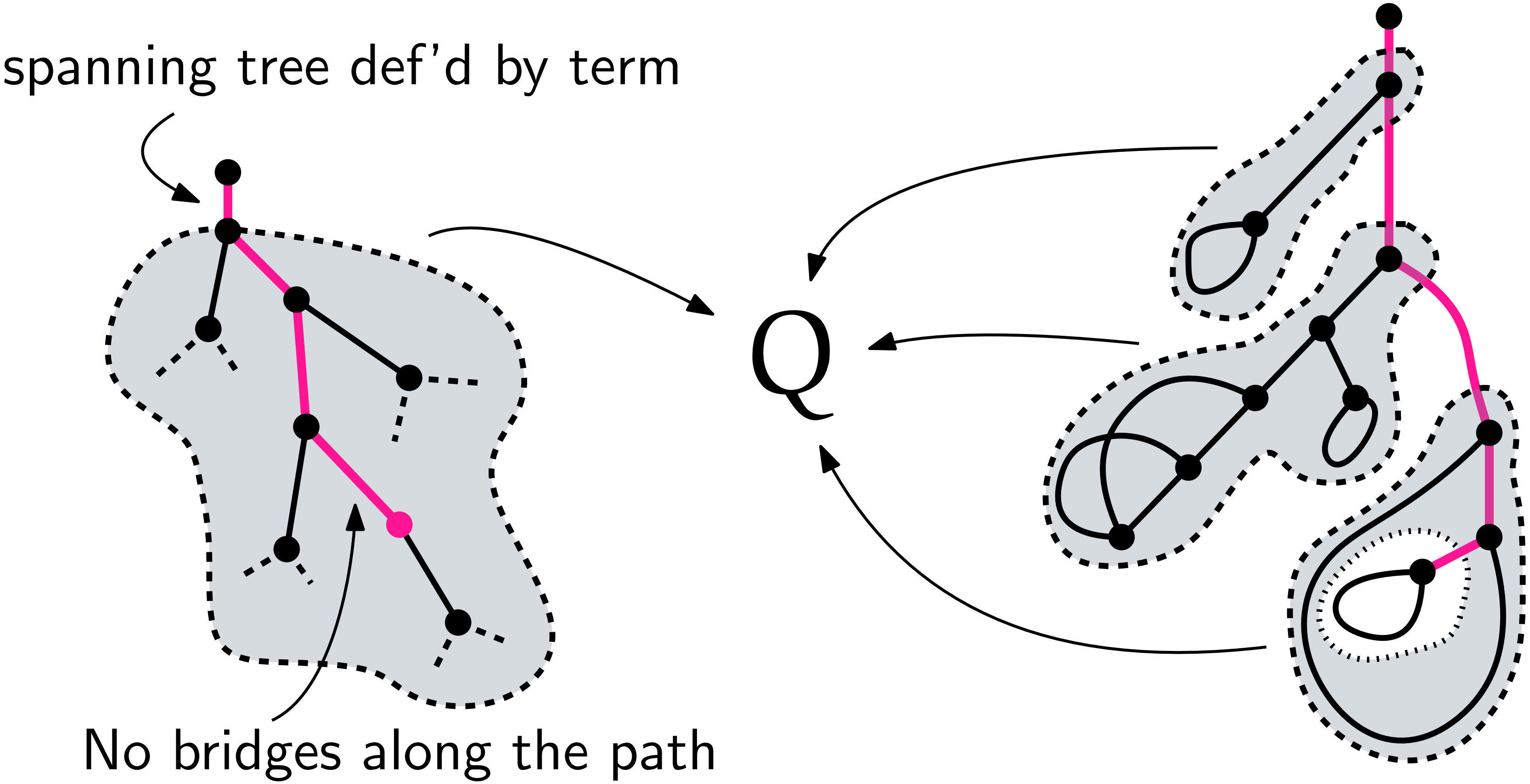

$$\frac{\partial}{\partial v} T_0^{sub}(z,v) = -\frac{v^2 z T_0^{sub}(z,v)^3 + z^2 T_0^{sub}(z,v) - T_0^{sub}(z,v)^2}{(v^3 - v^2) z T_0^{sub}(z,v)^2 + v z^2 - (v-1) T_0^{sub}(z,v)}$$

May be pumped using our schema

Proof sketch for vertices of given degree:

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Specifications based on exponential Hadamard products

$$\mathrm{OT}(z,u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z}\left(\ln\left(\exp(z^2/2) \odot \exp(z^3/3 + uz)\right)\right)$$

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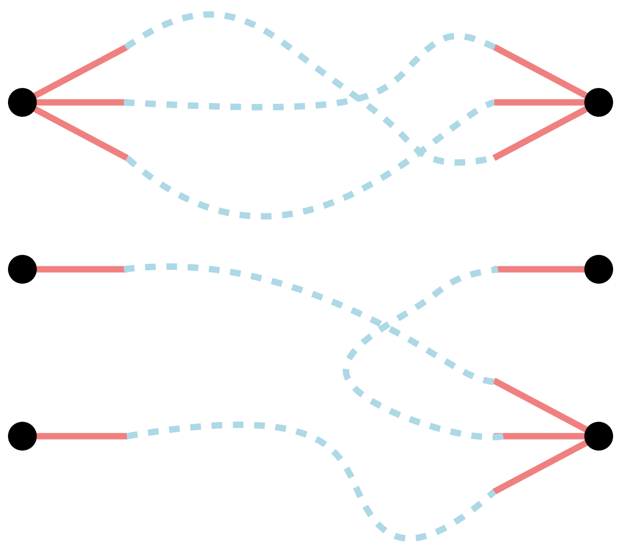

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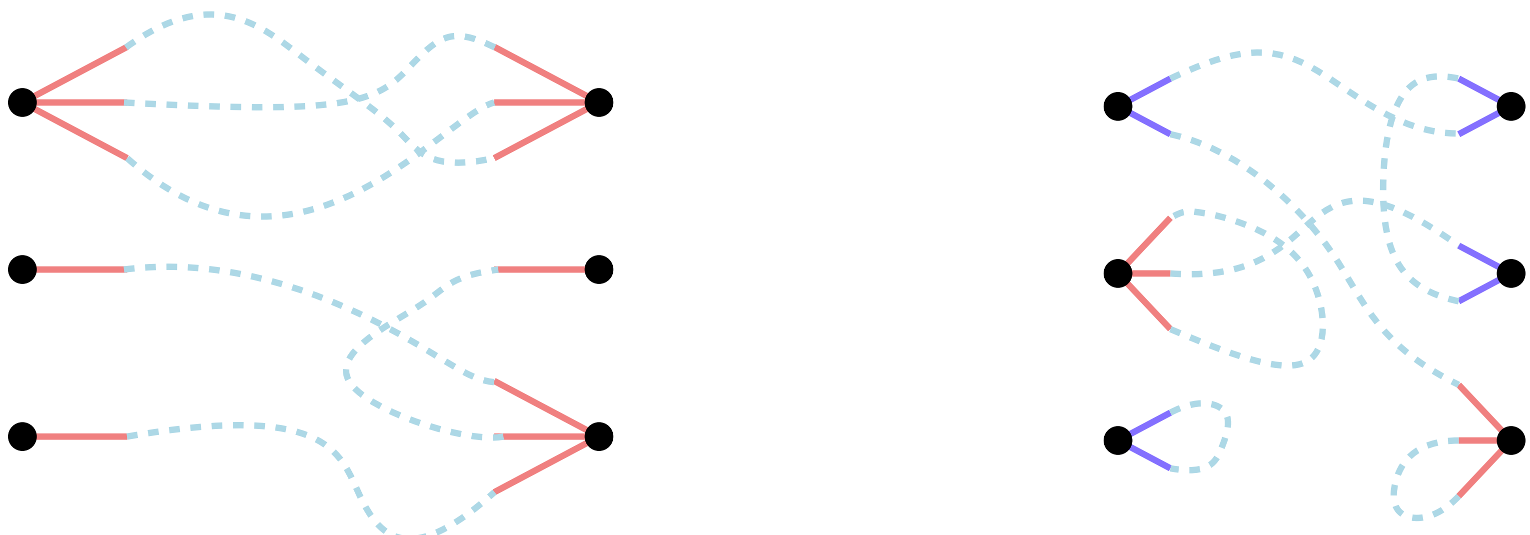

(2,3)-valent maps

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closed affine terms

Compositions for fast-growing series:

$$F(z, u, G(z, u))$$

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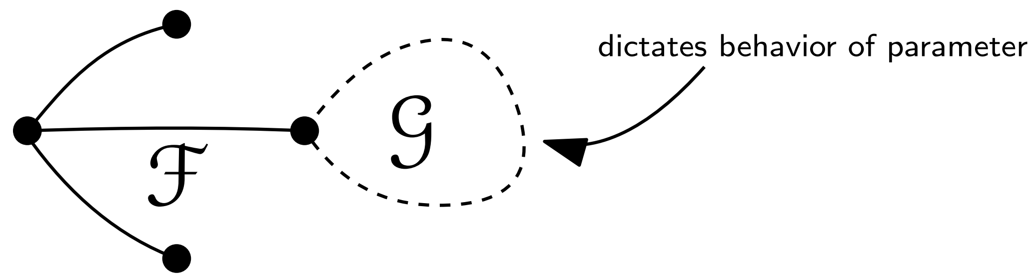
"To build a big $\mathcal{F}(\mathcal{G})$ structure, pick a small $\mathcal{F}$ one and replace one of its atoms with a big $\mathcal{G}$-structure"



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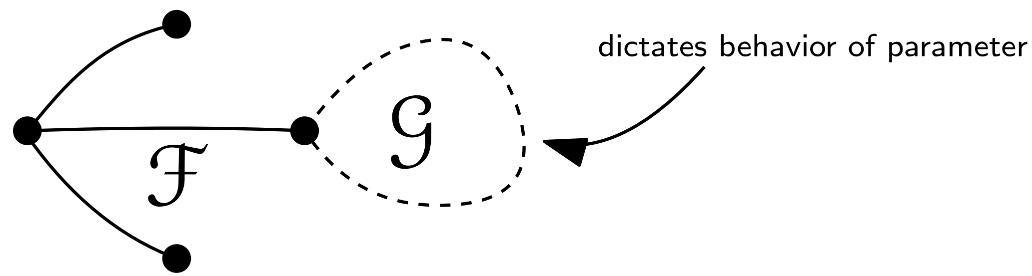

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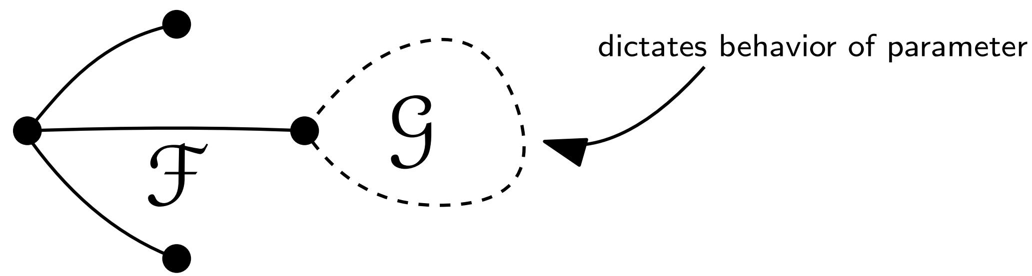

If F is the logarithm:

Asymptotically, almost all not-necessarily-connected $\mathcal{G}$-structures are connected, so the distribution of params. is the same for connected and not-necessarily-so structures!

Proof sketch for bridges/closed subterms (contd.) :

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Ammenable to saddle-point analysis!

Both yield Gaussian limit laws

Proof sketch for bridges/closed subterms (contd.) :

rooted — connected

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Use schema for compositions to show that the results carry over!

Mean number of β-redices in closed terms (WIP)

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- A standard decomposition for closed terms

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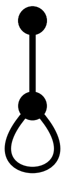
identity



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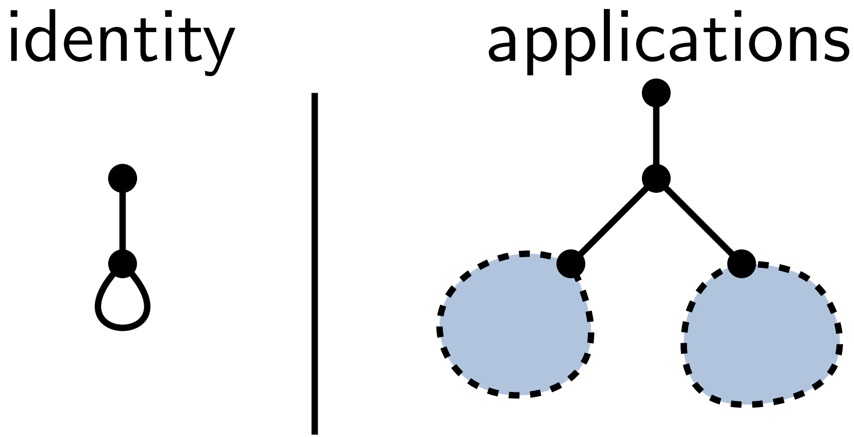
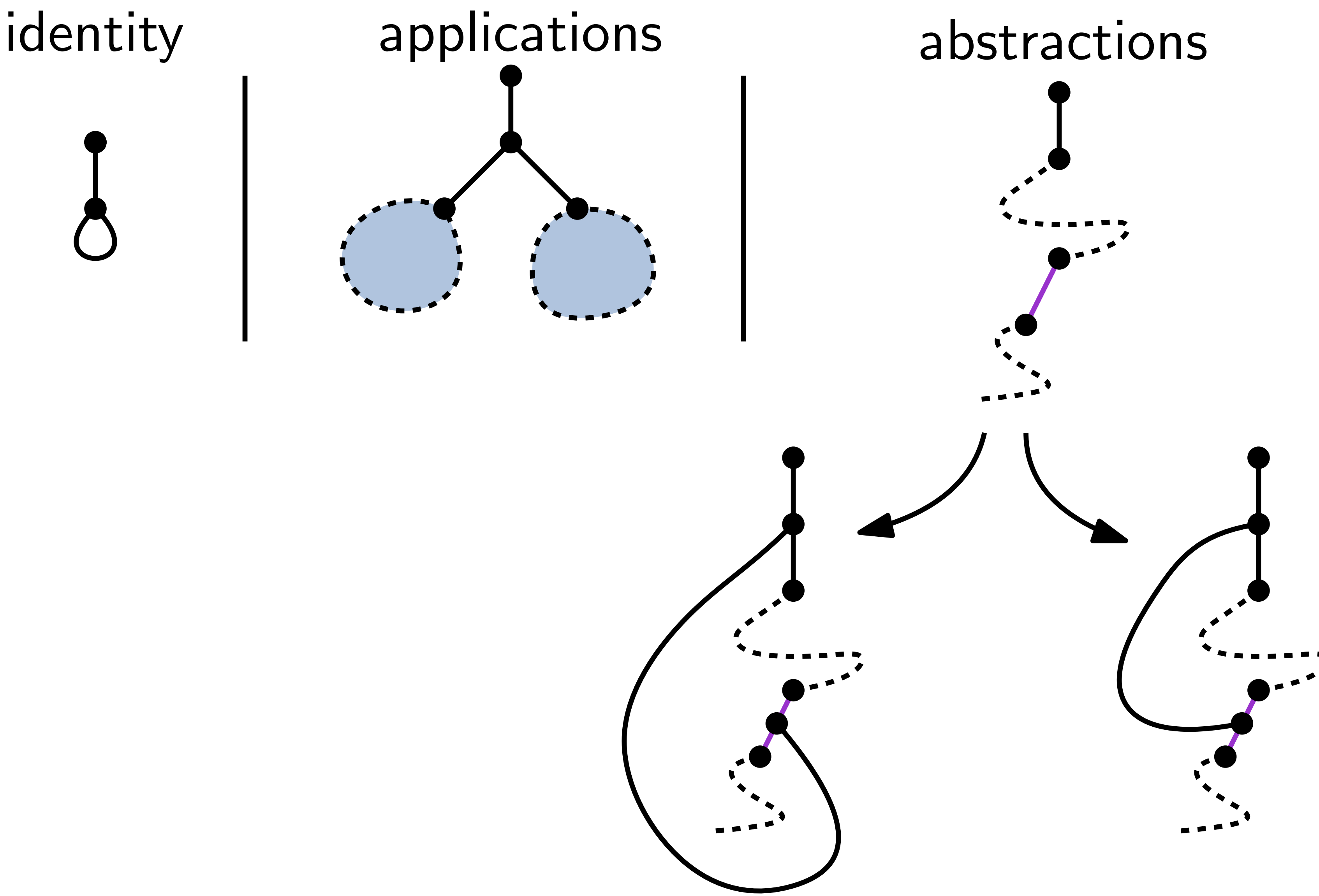
- A standard decomposition for closed terms



Mean number of β-redices in closed terms (WIP)

- A standard decomposition for closed terms



Mean number of β-redices in closed terms (WIP)

Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

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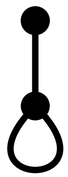
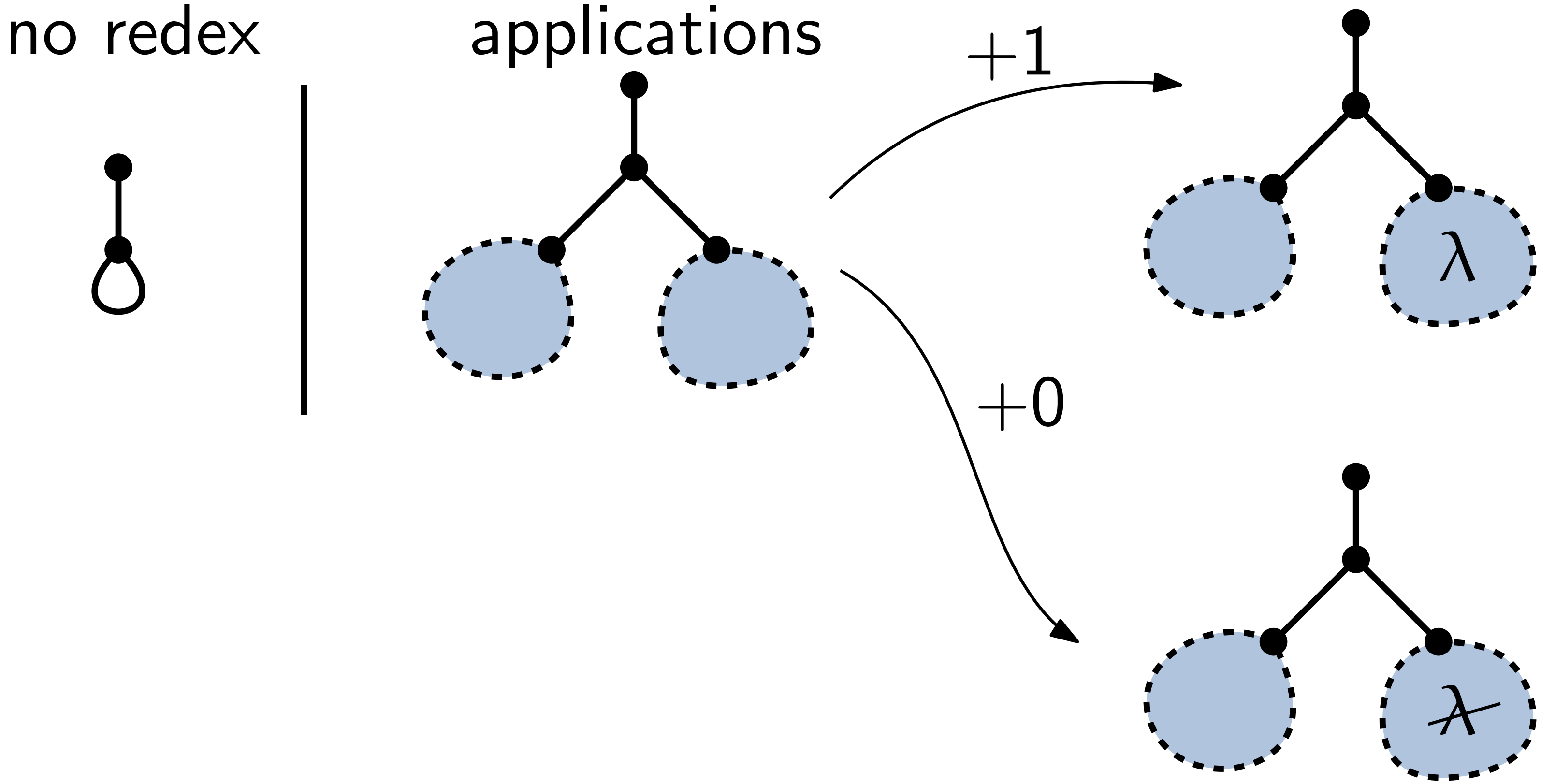


Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition



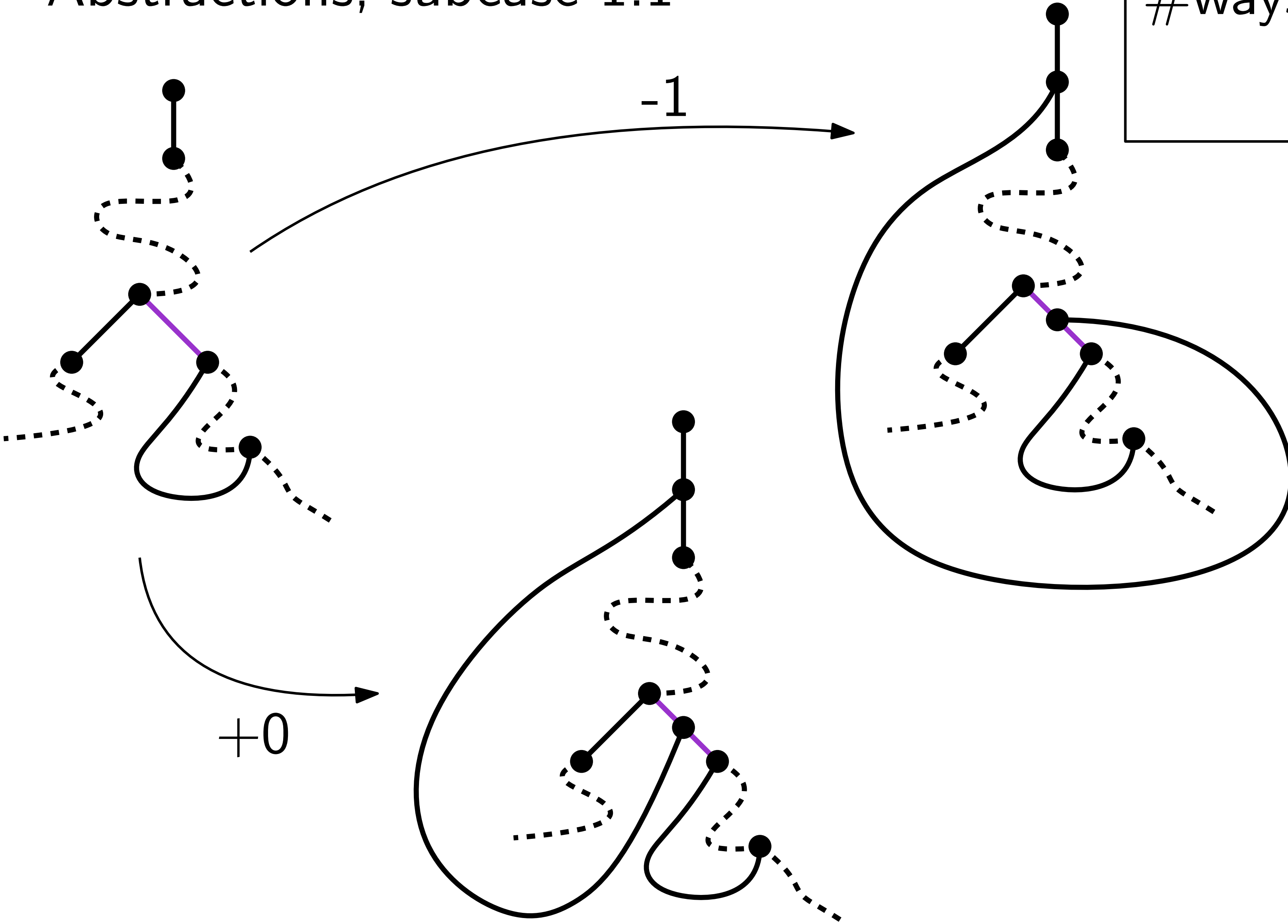
Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

Abstractions, subcase 1.1

#ways to do this

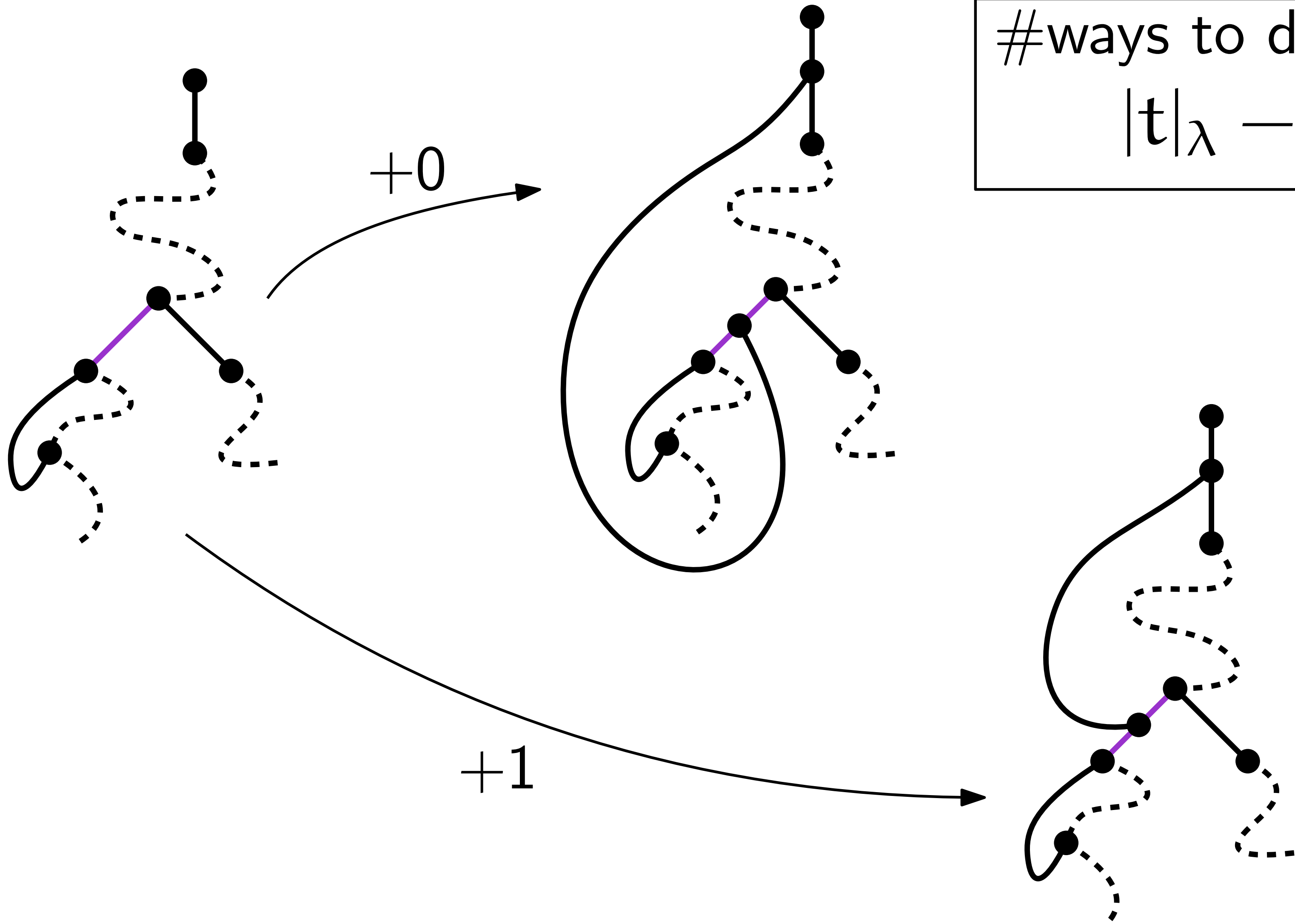
$|t|_\beta$



Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

Abstractions, subcase 1.2



#ways to do this

$$|t|_\lambda - |t|_\beta$$

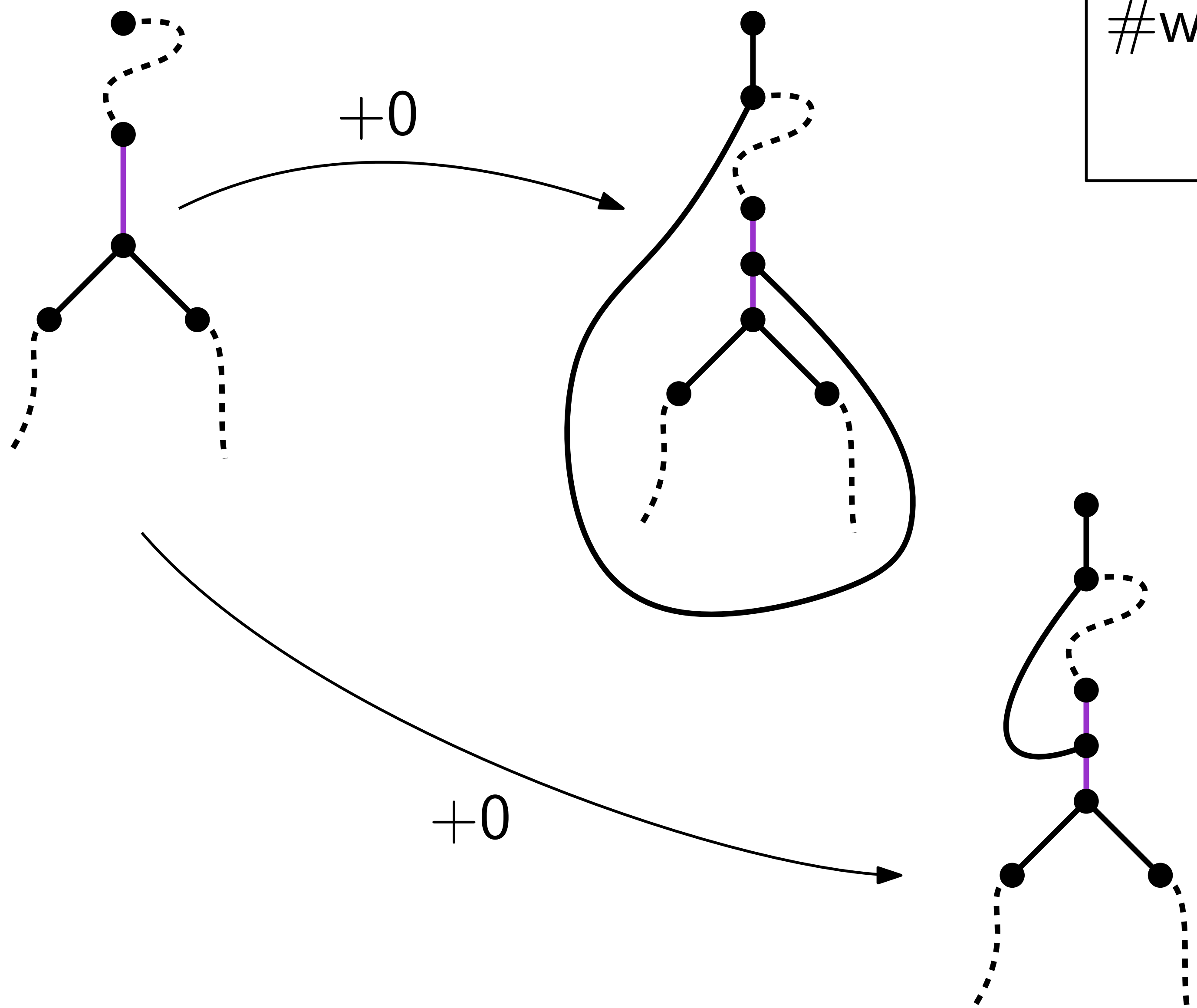
Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition

Abstractions, subcase 1.3

#ways to do this

$$|t| - |t|_\lambda$$



Mean number of β-redices in closed terms (WIP)

- Tracking redices during the decomposition
  - Using the following facts:
    - $|t|_\lambda = \frac{|t|+1}{3}, |t| - |t|_\lambda = \frac{2|t|-1}{3}$
    - $r\partial_r T_0 = \sum_{t \in T_0} |t|_\beta z^{|t|} r^{|t|_\beta}$
    - $\frac{z\partial_z T_0 + T_0}{3} = \sum_{t \in T_0} \frac{|t|+1}{3} z^{|t|} v^{|t|_\beta}$
    - $\frac{2z\partial_z T_0 - T_0}{3} = \sum_{t \in T_0} \frac{2|t|-1}{3} z^{|t|} v^{|t|_\beta}$

Mean number of β-redices in closed terms (WIP)

- Translating to a diff-eq and pumping

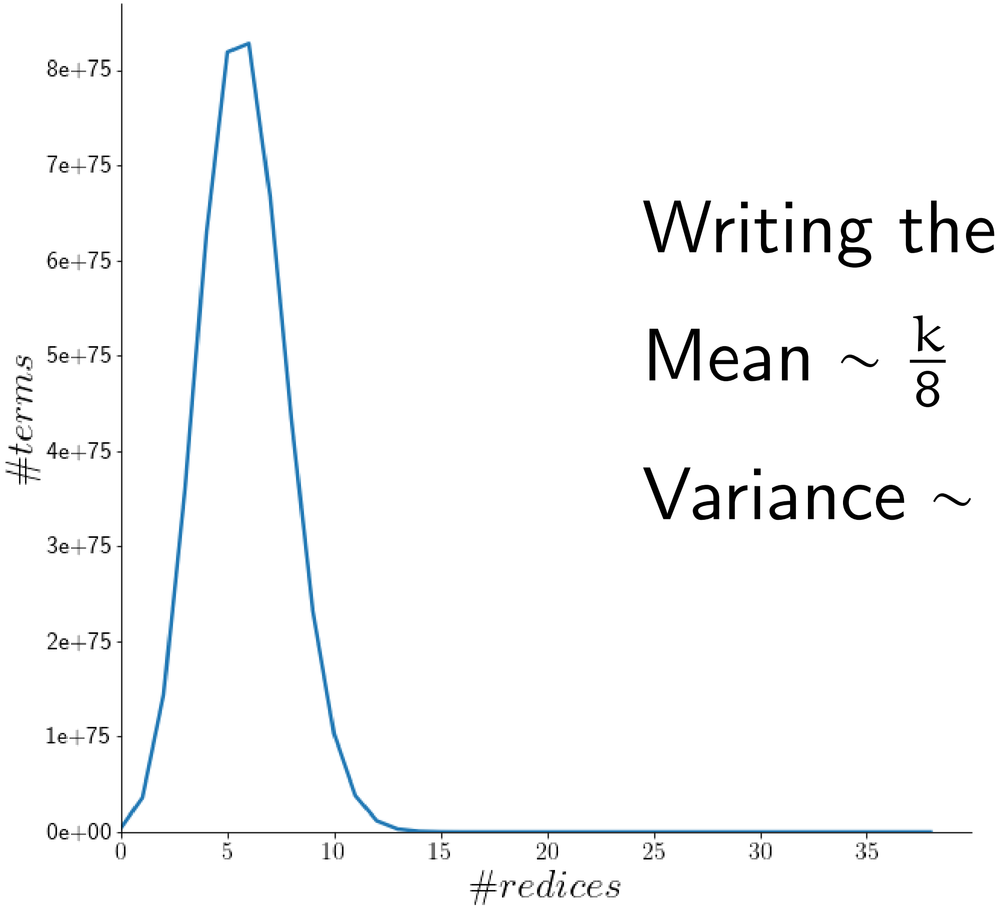
$$T_0 = -z\left(z^2(r+1)(1+(r-1)zT)(r-1)\partial_r T_0 - \frac{(1+z(r-1)T)z^3(r+5)\partial_z T_0}{3} - \frac{z^3(r-1)^2T_0^2}{3} - \frac{4z^2(r-1)T_0}{3} - z - T_0^2\right)$$

Mean number of β-redices in closed terms (WIP)

- Translating to a diff-eq and pumping

$$T_0 = -z\left(z^2(r+1)(1+(r-1)zT)(r-1)\partial_r T_0 - \frac{(1+z(r-1)T)z^3(r+5)\partial_z T_0}{3} - \frac{z^3(r-1)^2T_0^2}{3} - \frac{4z^2(r-1)T_0}{3} - z - T_0^2\right)$$

A plot of the dist. of redices for $n = 119$



Writing the size as $n = 3k + 2$, we have:

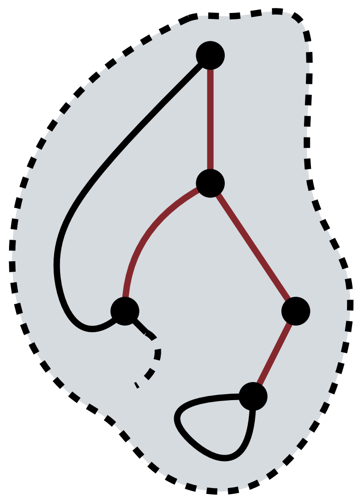
Mean $\sim \frac{k}{8}$

Variance $\sim \frac{29k}{320}$

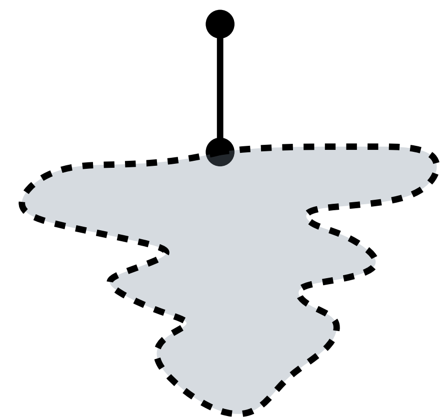
Whats next?

Whats next?

- More parameters:



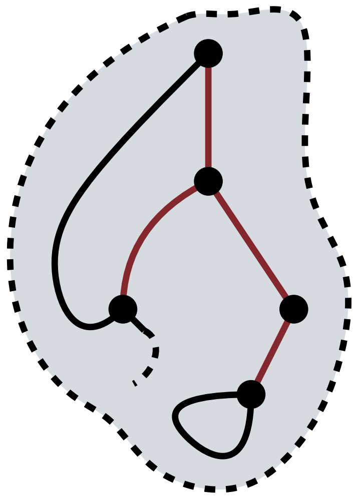
Mean path length



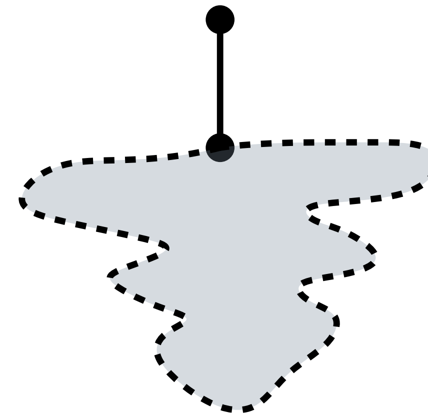
Profile

Whats next?

- More parameters:



Mean path length

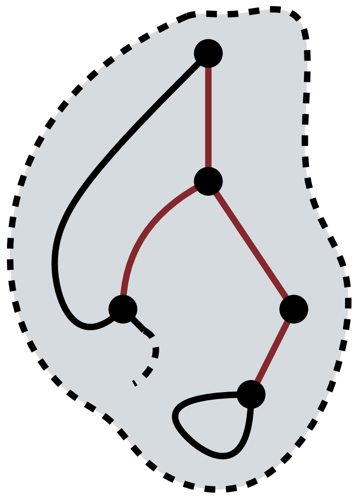


Profile

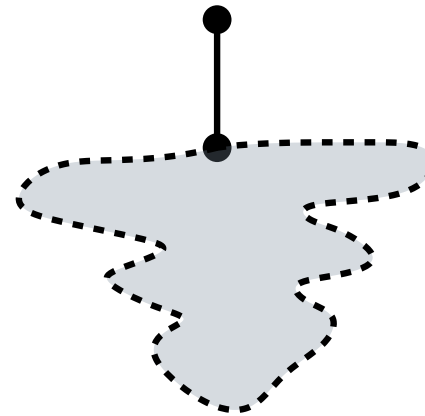
- More map/term families: planar, bridgeless...

Whats next?

- More parameters:



Mean path length



Profile

- More map/term families: planar, bridgeless...

Thank you!

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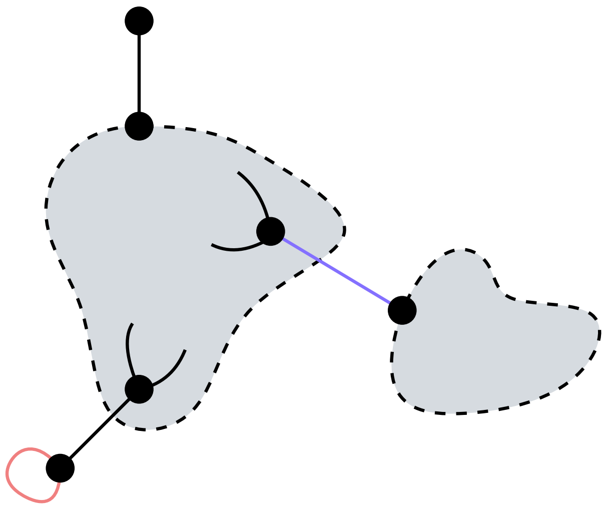
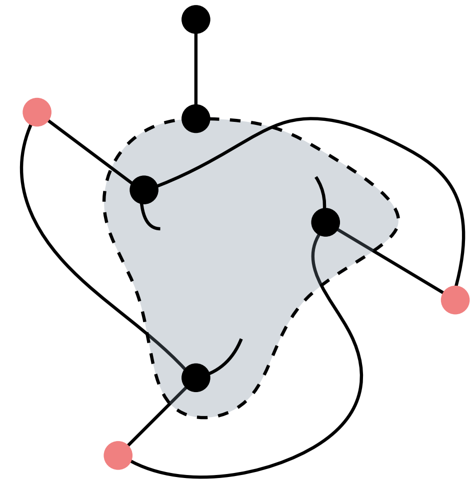
Our results: limit distributions

Trivalent maps ↔ closed linear terms



loops = #id-subterms
bridges = # closed subt.
} Poisson(1)

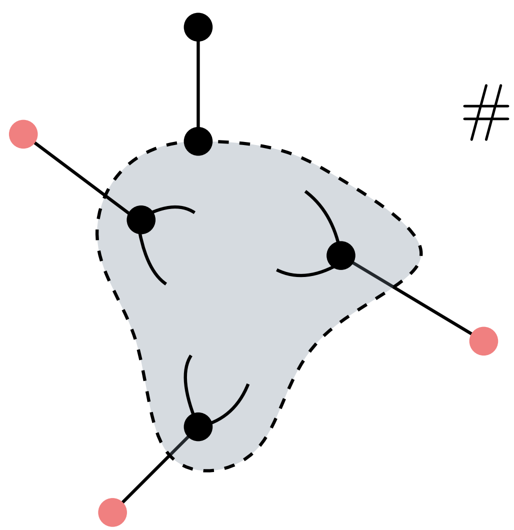
(2,3)-maps ↔ closed affine terms



unary vertices = # free vars

$\mathcal{N}(mu, \sigma^2)$ with $\mu = \sigma^2 = (2n)^{2/3}$

(1,3)-maps ↔ open linear terms



unary vertices = # free vars

$\mathcal{N}(\mu, \sigma^2)$ with

$\mu = \sigma^2 = (2n)^{1/3}$