Categorification of Rule Algebras

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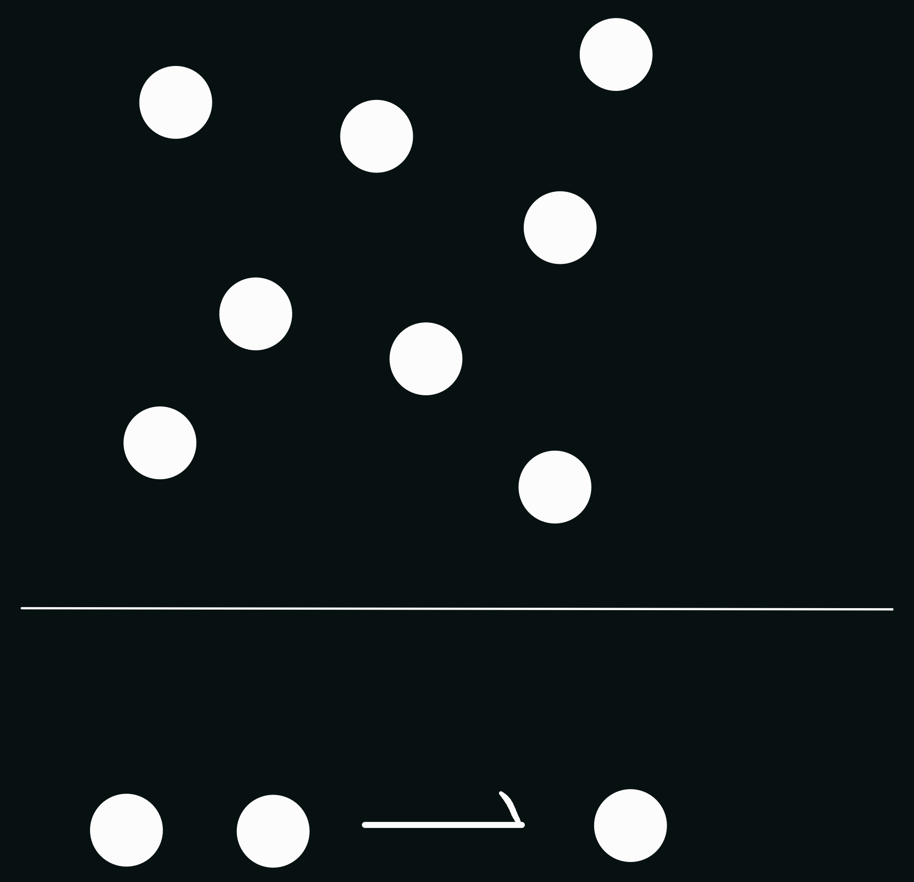
Combinatorics & Arithmetic for Physics: Special Days
IHÉS, November 28, 2022

Based upon joint work with

- P.-A. Melliès & N. Zeilberger
- R. Harmer & J. Krivine (2204.07175)

① MOTIVATION



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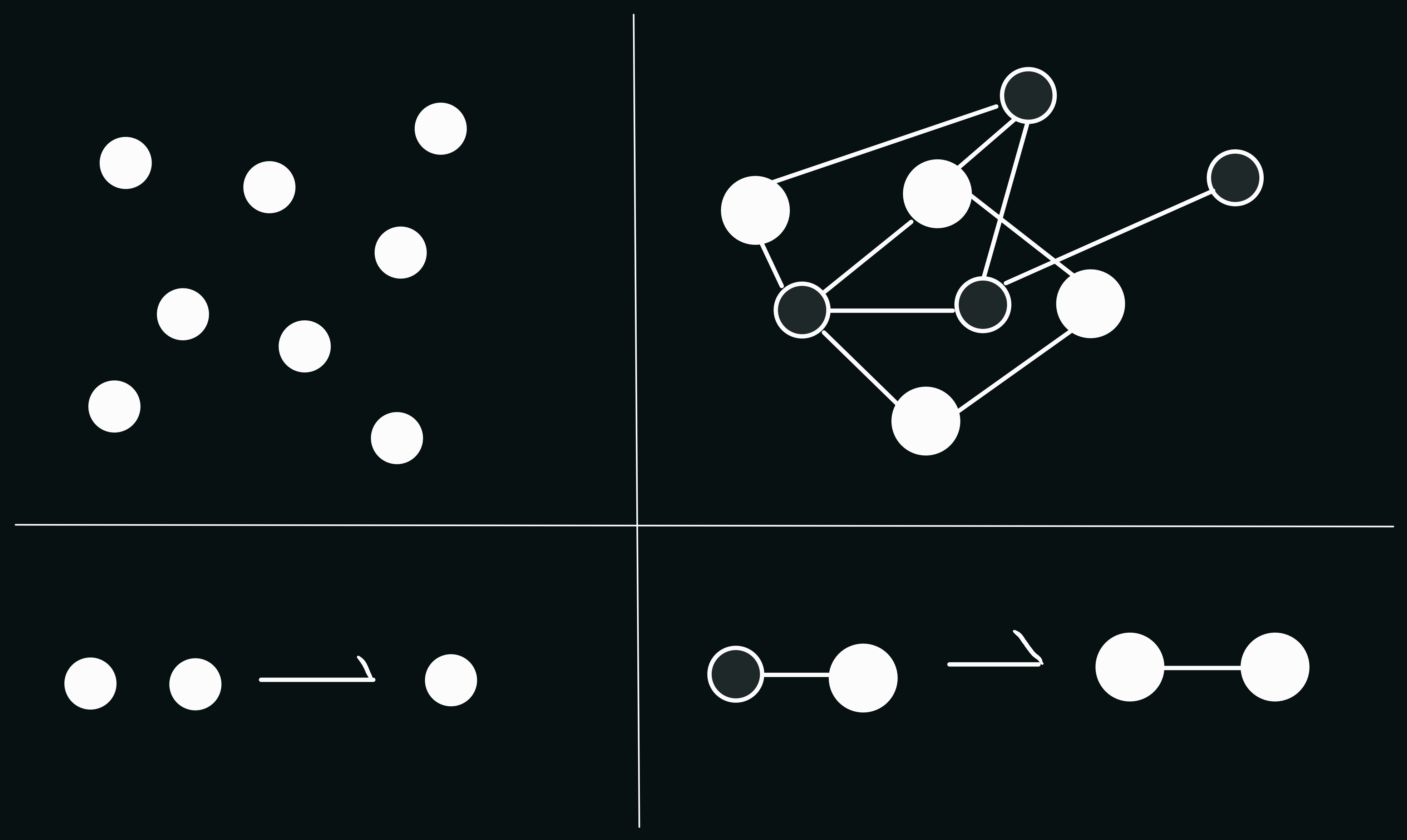
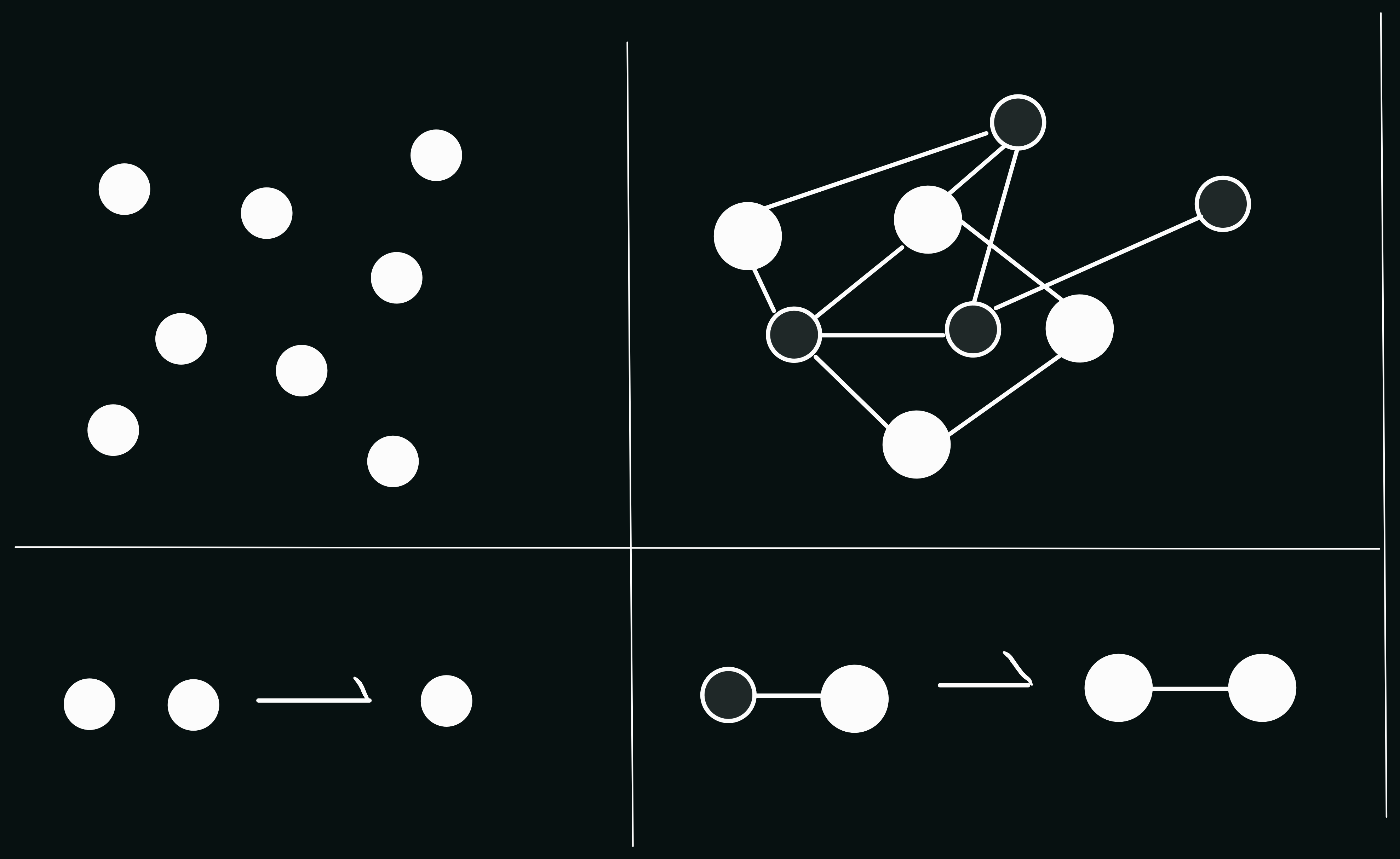
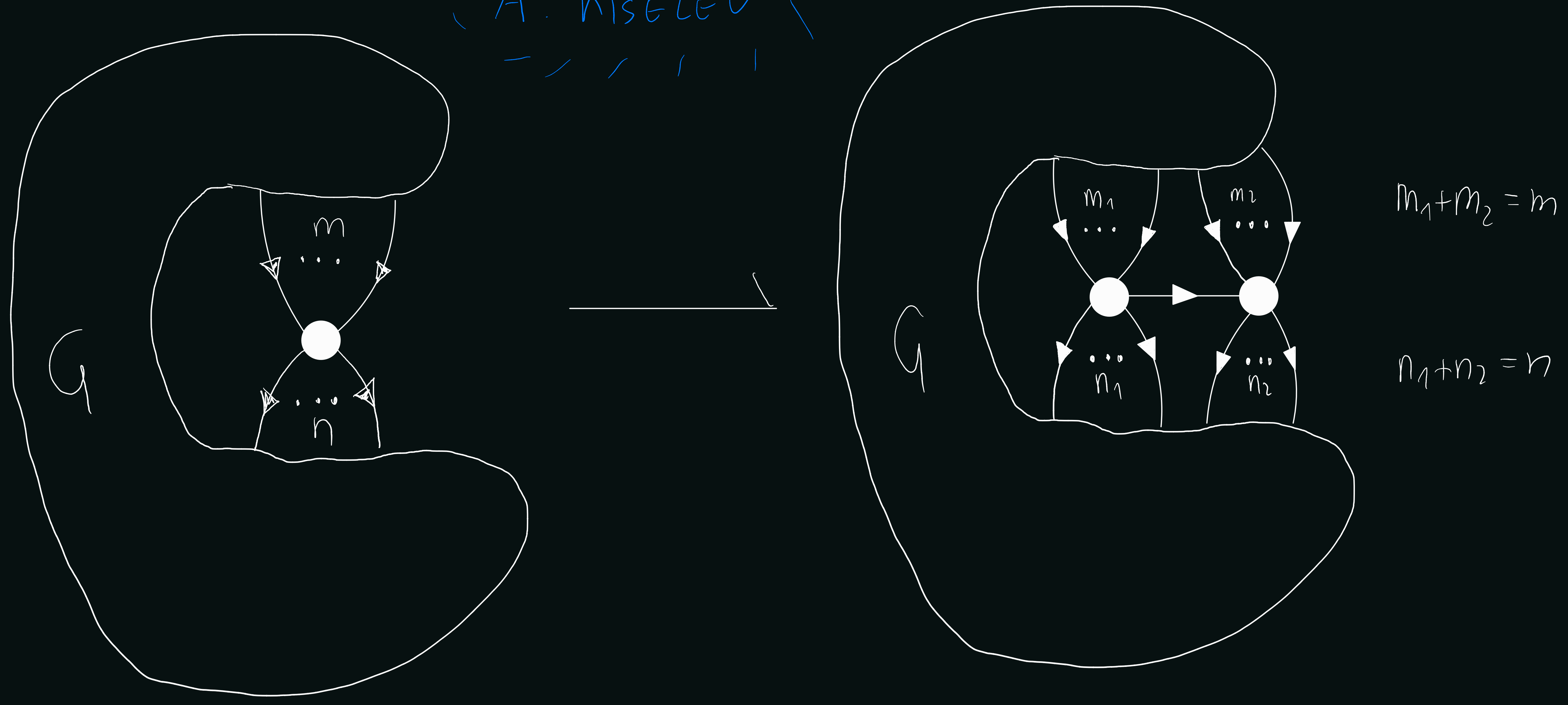


① MOTIVATION

CAP'19
A. Kiselev





① MOTIVATION

CAP'19
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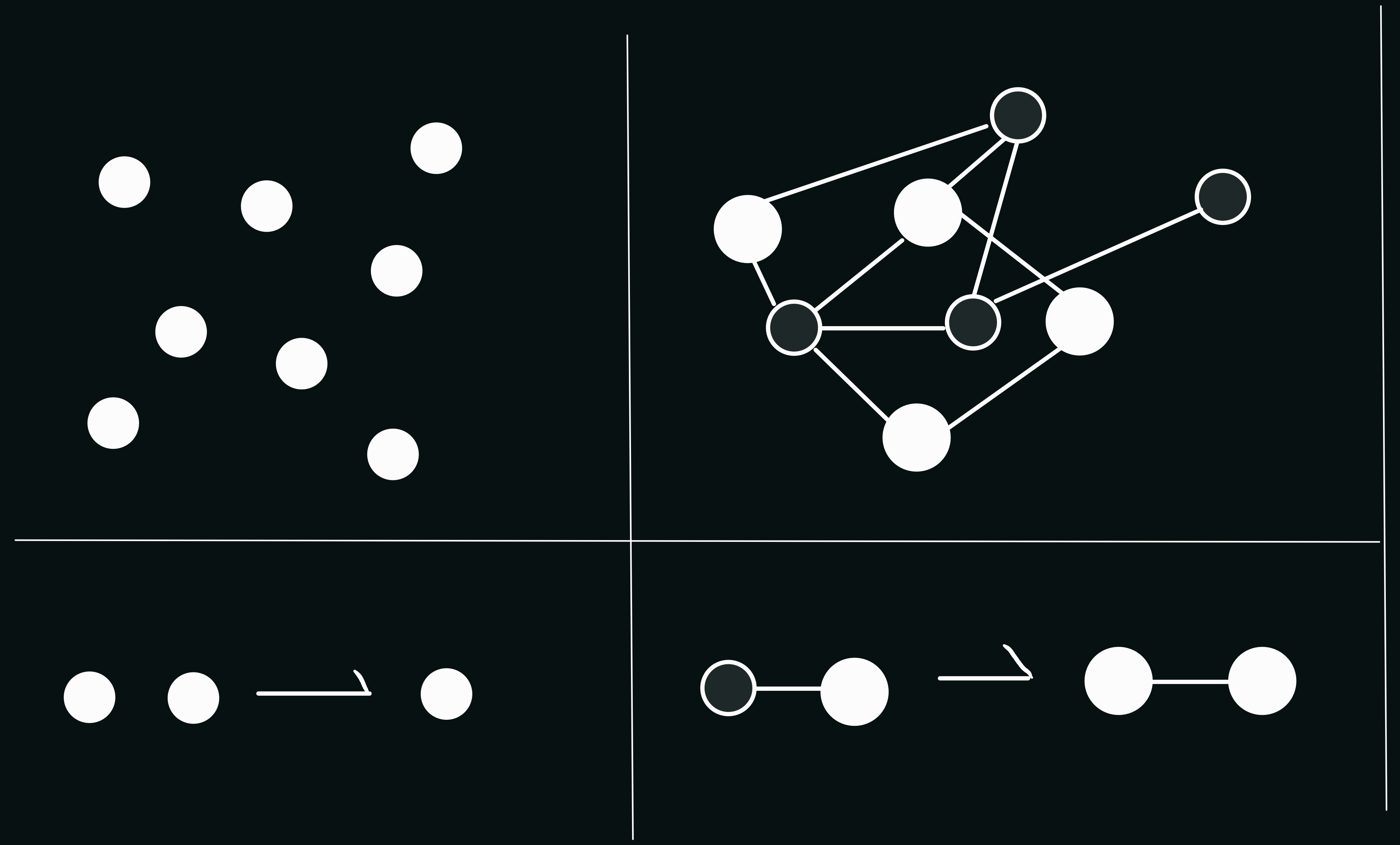

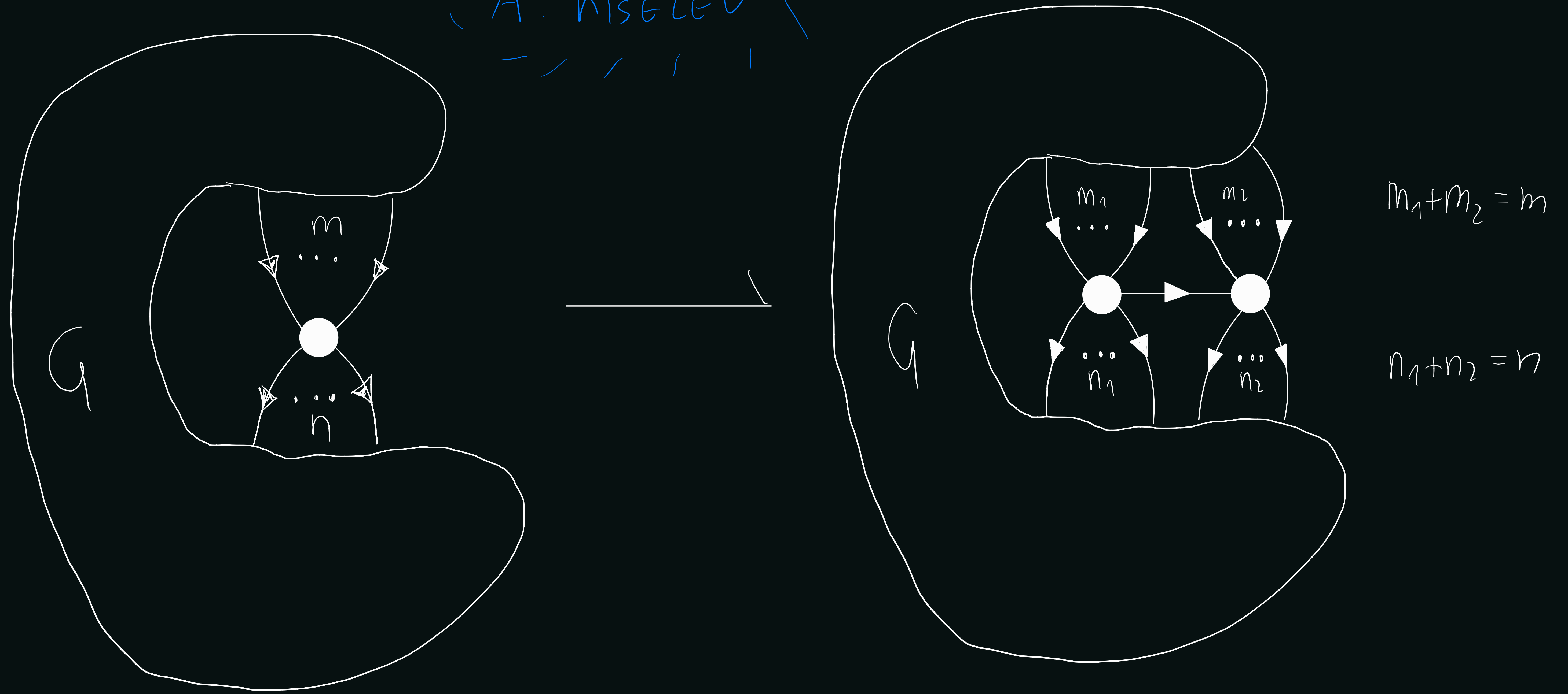

↪ ALL FORMALIZABLE IN DOUBLE-PUSHOUT (DPO) SEMANTICS:

$$
\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
r_\alpha(X) & X \arrow[l]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\
r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

PO — PUSHOUT

② Motivation: Formalization of Rewriting + Combinatorics!

- Inspiration from combinatorial species theory:

(i) $\hat{x}(x^n) = x^{n+1}$, $\quad \frac{d}{dx}(x^n) = (n)_1 x^{n-1} \equiv \begin{cases} 0, & n=0 \\ n x^{n-1}, & \text{else} \end{cases}$

↳ (ii) $x^n = \hat{x}^n(1)$

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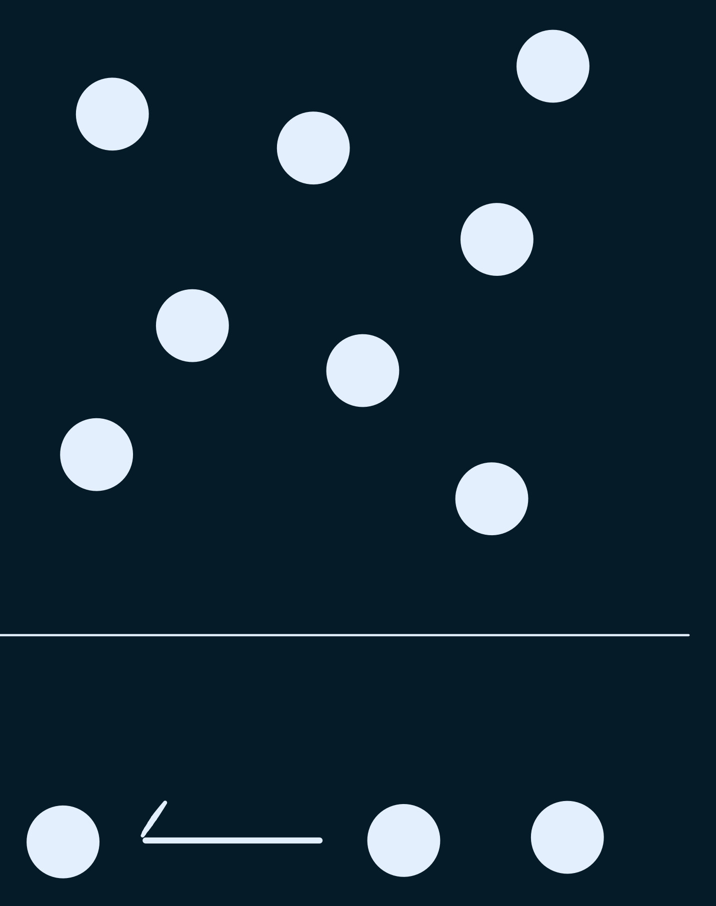
↳ (ii) $x^n = \hat{x}^n(1)$

$$\hat{x}^p \left(\frac{d}{dx}\right)^q (x^n) = (n)_q\, x^{n-q+p}$$

# of ways to remove q elements from a set of n elements



$$\hat{x}\left(\frac{d}{dx}\right)^2 x^n = (n)_2\, x^{n-1}$$

② Motivation: Formalization of Rewriting + Combinatorics!

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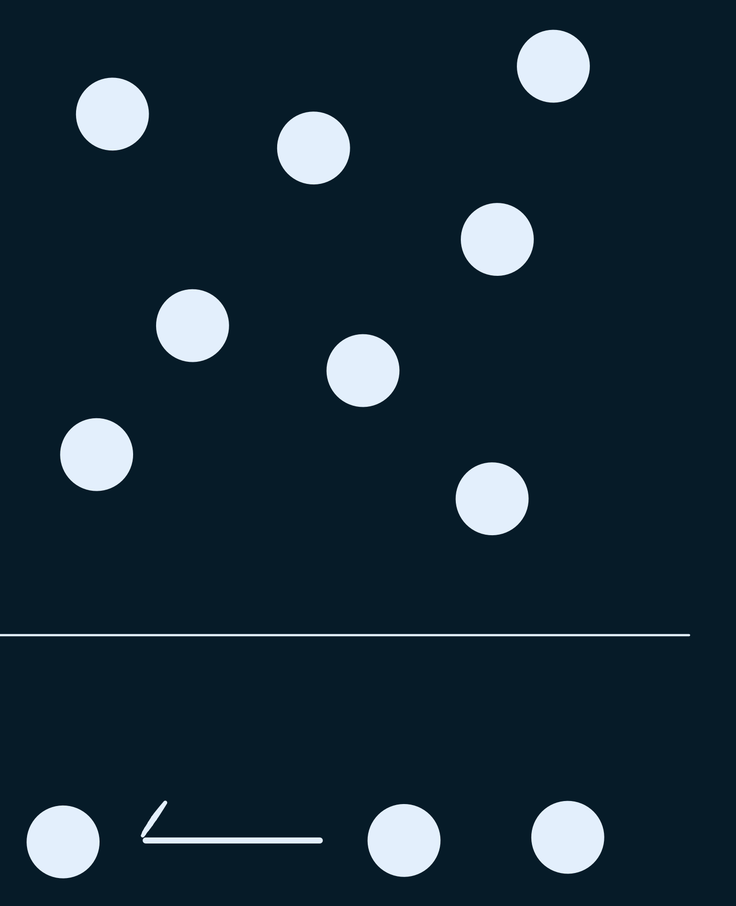
(i) $\hat{X}(x^n) = x^{n+1}$, $\quad \frac{d}{dx}(x^n) = (n)_1 \, x^{n-1} \equiv \begin{cases} 0, & n=0 \\ n x^{n-1}, & \text{else} \end{cases}$

↪ (ii) $x^n = \hat{x}^n(1)$

$$\hat{X}^p \left(\frac{d}{dx}\right)^q (x^n) = (n)_q \, x^{n-q+p}$$

\# of ways to remove q elements from a set of n elements

(iii) $$\hat{X}^p \left(\frac{d}{dx}\right)^q \hat{X}^r \left(\frac{d}{dx}\right)^s = \sum_{k \geq 0} \underbrace{\binom{q}{k} k! \binom{r}{k}}_{\in \mathbb{Z}_{\geq 0}} \hat{X}^{p+r-k} \left(\frac{d}{dx}\right)^{q+s-k}$$



$$\hat{X} \left(\frac{d}{dx}\right)^2 x^n = (n)_2 \, x^{n-1}$$

② MOTIVATION: FORMALIZATION OF REWRITING + COMBINATORICS!

- INSPIRATION FROM COMBINATORIAL SPECIES THEORY:

(i) $\hat{x}(x^n) = x^{n+1}$, $\quad \frac{d}{dx}(x^n) = (n)_1\, x^{n-1} \equiv \begin{cases} 0, & n=0 \\ n x^{n-1}, & \text{else} \end{cases}$

↳ (ii) $x^n = \hat{x}^n(1)$

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\# OF WAYS TO REMOVE q ELEMENTS FROM A SET OF n ELEMENTS

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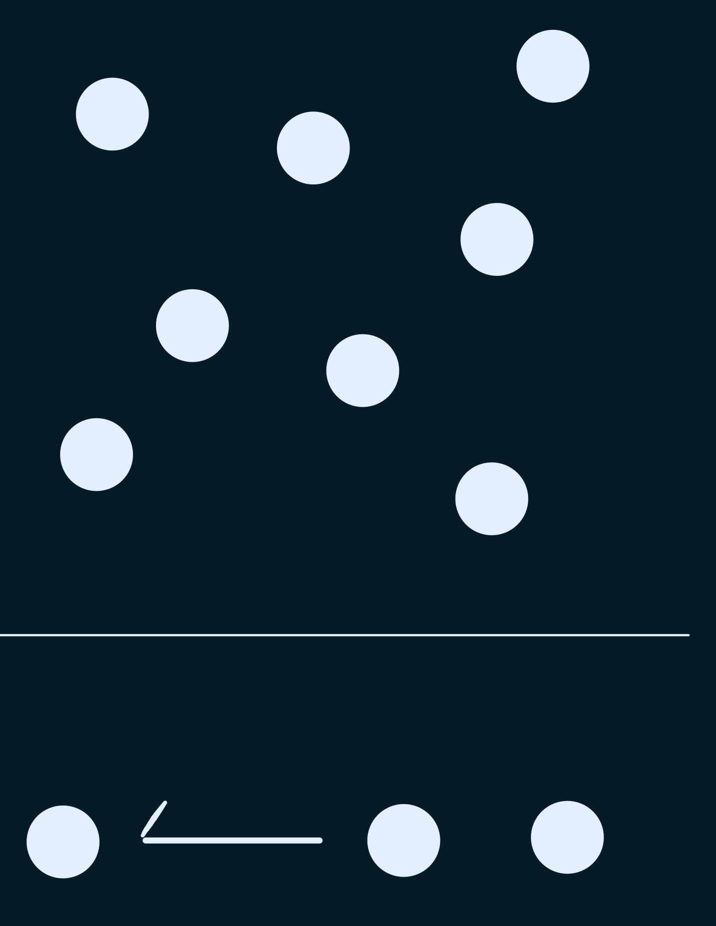

$$\hat{x}\left(\frac{d}{dx}\right)^2 x^n = (n)_2\, x^{n-1}$$

GOAL:

(i) " $\rho(\delta(r))|X\rangle := \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \overbrace{\mu_X^Y}^{\in \mathbb{Z}_{\geq 0}} |Y\rangle$ "

(ii) " $|X\rangle = \rho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$ "

(iii) " $\rho(\delta(r_2))\rho(\delta(r_1)) = \sum_{r_u} \underbrace{\mu_{r_1,r_2}^{r_u}}_{\in \mathbb{Z}_{\geq 0}} \rho(\delta(r_u))$ "

(iv) " $\rho(\delta(r_2))\rho(\delta(r_1)) = \rho(\delta(r_2) \odot \delta(r_1))$ "

$\odot$ – RULE ALGEBRA PRODUCT

③ Conceptual Obstacle: Essential Uniqueness of Universal Constructions

Recap:

$$\begin{tikzcd} O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\ r_\alpha(X) & X \arrow[l] \end{tikzcd} \quad := \quad \begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\ r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X \end{tikzcd}$$

PO — Pushout

Definition:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \text{ is a PO } :\Leftrightarrow \forall \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & D & \\ & & X \end{tikzcd} : \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!"] & \\ & & X \end{tikzcd}$$

$$\hookrightarrow \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd} \wedge \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\ C \arrow[r] & D' \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\ C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!", "\cong"'] & \\ & & D' \end{tikzcd}$$

Example: (in $\underline{\text{FinSet}}$)

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\ C \arrow[r, hook] & \text{"}B \cup_I C\text{"} \end{tikzcd}$$

④ ANSATZ: CATEGORIFICATION

GOAL:

(i) "$\rho(\delta(r))|X\rangle := \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{y} \overbrace{\mu_X^{\,y}}^{\in \mathbb{Z}_{\geq 0}} |y\rangle$" (ii) "$|X\rangle = \rho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$"

(iii) "$\rho(\delta(r_2))\rho(\delta(r_1)) = \sum_{r_u} \underbrace{\mu_{r_1,r_2}^{\,r_u}}_{\in \mathbb{Z}_{\geq 0}} \rho(\delta(r_u))$" (iv) "$\rho(\delta(r_2))\rho(\delta(r_1)) = \rho(\delta(r_2) \odot \delta(r_1))$"

$\odot$ – RULE ALGEBRA PRODUCT

I. FORMALIZE
$$\begin{tikzcd} O \arrow[d] & I \arrow[l, "r"'] \arrow[d] \arrow[dl, phantom, "\Downarrow\alpha"] \\ r_\alpha(X) & X \arrow[l] \end{tikzcd}$$
AS 2-CELLS IN A DOUBLE CATEGORY

II. FORMALIZE $\mathbb{Z}_{\geq 0}$-COEFFICIENTS AS CARDINALITIES (OF SUITABLE SETS...)

METHODS: DOUBLE CATEGORIES, PRESHEAVES, FIBRATIONS, COENDS, MULTISUMS ...

5 DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A WEAKLY INTERNAL CATEGORY IN $\underline{CAT}$

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"']
\end{tikzcd}
$$

$\mathbb{D}_0$:

$$
\begin{tikzcd}
X \arrow[d, "f"] \\
Y
\end{tikzcd}
$$

"0-cells" — objects of $\mathbb{D}_0$

"vertical morphisms" — morphisms of $\mathbb{D}_0$

$\mathbb{D}_1$:

$$
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
$$

"horizontal morphisms" — objects of $\mathbb{D}_1$

"2-cells" - morphisms of $\mathbb{D}_1$

⑤ DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A WEAKLY INTERNAL CATEGORY IN CAT

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right=40, "S"'] \arrow[l, bend left=40, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"'] \end{tikzcd}$$

$\mathbb{D}_0$:
$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$$\begin{tikzcd} O \arrow[d, "n"'] \\ Y \end{tikzcd} \xleftarrow{\ T\ } \begin{tikzcd} O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \xmapsto{\ S\ } \begin{tikzcd} I \arrow[d, "m"] \\ X \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \xmapsto{\ u\ } \begin{tikzcd} A \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow id"] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\ B & B \arrow[l, "id_B"] \end{tikzcd}$$

$\mathbb{D}_1$:
$$\begin{tikzcd} O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd}$$

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\end{tikzcd}
$$

$\mathbb{D}_0$:
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\end{tikzcd}
$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

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\begin{tikzcd}
O \arrow[d, "n"'] \\
Y
\end{tikzcd}
\xleftarrow{\;T\;}\!\!\shortmid
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
\overset{S}{\longmapsto}
\begin{tikzcd}
I \arrow[d, "m"] \\
X
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[d, "f"'] \\
B
\end{tikzcd}
\overset{u}{\longmapsto}
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow id"] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\
B & B \arrow[l, "id_B"]
\end{tikzcd}
$$

$\mathbb{D}_1$:
$$
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

$$
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
O' & I' \arrow[l, "r'"]
\end{tikzcd}
\;\diamond_v\;
\begin{tikzcd}
O' \arrow[d, "n'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & I' \arrow[l, "r'"'] \arrow[d, "m'"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n' \circ n"'] \arrow[r, phantom, "\Downarrow\alpha' \circ \alpha"] & I \arrow[l, "r"'] \arrow[d, "m' \circ m"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}
$$

"vertical composition"

— composition in $\mathbb{D}_1$

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$$\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"']
\end{tikzcd}$$

$$\mathbb{D}_0 : \quad \begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$$\begin{tikzcd}
O \arrow[d, "n"'] \\ Y
\end{tikzcd}
\overset{T}{\longmapsfrom}
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
\overset{S}{\longmapsto}
\begin{tikzcd}
I \arrow[d, "m"] \\ X
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] \\ B
\end{tikzcd}
\overset{u}{\longmapsto}
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow id"] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\
B & B \arrow[l, "id_B"]
\end{tikzcd}$$

$$\mathbb{D}_1 : \quad \begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

$$\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
O' & I' \arrow[l, "r'"]
\end{tikzcd}
\; \diamond_v \;
\begin{tikzcd}
O' \arrow[d, "n'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & I' \arrow[l, "r'"'] \arrow[d, "m'"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n' \circ n"'] \arrow[r, phantom, "\Downarrow\alpha' \circ \alpha"] & I \arrow[l, "r"'] \arrow[d, "m' \circ m"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}$$

"vertical composition"

— composition in $\mathbb{D}_1$

$$\begin{tikzcd}
C \arrow[d, "c"'] \arrow[r, phantom, "\Downarrow\alpha_2"] & B \arrow[l, "r_2"'] \arrow[d, "b"] \\
C' & B' \arrow[l, "s_2"]
\end{tikzcd}
\; \diamond_h \;
\begin{tikzcd}
B \arrow[d, "b"'] \arrow[r, phantom, "\Downarrow\alpha_1"] & A \arrow[l, "r_1"'] \arrow[d, "a"] \\
B' & A' \arrow[l, "s_1"]
\end{tikzcd}
:=
\begin{tikzcd}
C \arrow[d, "c"'] \arrow[r, phantom, "\Downarrow\alpha_2 \diamond_h \alpha_1"] & A \arrow[l, "r_2 \diamond_h r_1"'] \arrow[d, "a"] \\
C' & A' \arrow[l, "s_2 \diamond_h s_1"]
\end{tikzcd}$$

"horizontal composition" ← IN GENERAL ONLY WEAKLY ASSOCIATIVE!

6 DEFINITION: A **PRESENTATION** OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n\geq 0}$ OF FUNCTORS $h_n : \mathbb{D}_n \rightarrow \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := \mathcal{U}, \quad h_1 := id, \quad h_2(-_2,-_1) := -_2 \diamond_h -_1,$$

$$\forall n \geq 2: \; h_{n+1}(-_{n+1}, \ldots, -_1) \cong h_2(-_{n+1}, h_n(-_n, \ldots, -_1))$$

- NOTATIONAL CONVENTION:

$$\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l, "r_n"'] & \cdots & \cdot & \cdot \arrow[l, "r_1"'] \arrow[d] \\
\cdot & & \arrow[u, phantom, "\Downarrow\sigma"] & & \cdot \arrow[llll, "s"]
\end{tikzcd}
\equiv
\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l, "r_n"'] & \cdots & \cdot & \cdot \arrow[l, "r_1"'] \arrow[d] \arrow[llll, bend left, "h_n(r_n,\ldots,r_1)"] \\
\cdot & & \arrow[u, phantom, "\Downarrow\sigma"] & & \cdot \arrow[llll, bend left, "s"]
\end{tikzcd}$$

6 DEFINITION: A PRESENTATION OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n\geq 0}$

OF FUNCTORS $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := U, \quad h_1 := id, \quad h_2(-_2,-_1) := -_2 \diamond_h -_1,$$

$$\forall n \geq 2: \; h_{n+1}(-_{n+1},\dots,-_1) \cong h_2(-_{n+1}, h_n(-_n,\dots,-_1))$$

▸ NOTATIONAL CONVENTION:

$$\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l, "r_n"'] & \cdots & \cdot & \cdot \arrow[l, "r_1"'] \arrow[d] \\
\cdot & & \Downarrow\sigma & & \cdot \arrow[llll, "s"]
\end{tikzcd}
\equiv
\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l, "r_n"'] & \cdots & \cdot & \cdot \arrow[l, "r_1"'] \arrow[d] \arrow[llll, bend left, "h_n(r_n,\dots,r_1)"] \\
\cdot & & \Downarrow\sigma & & \cdot \arrow[llll, bend left, "s"]
\end{tikzcd}$$

▸ EXAMPLE:

$$\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
:=
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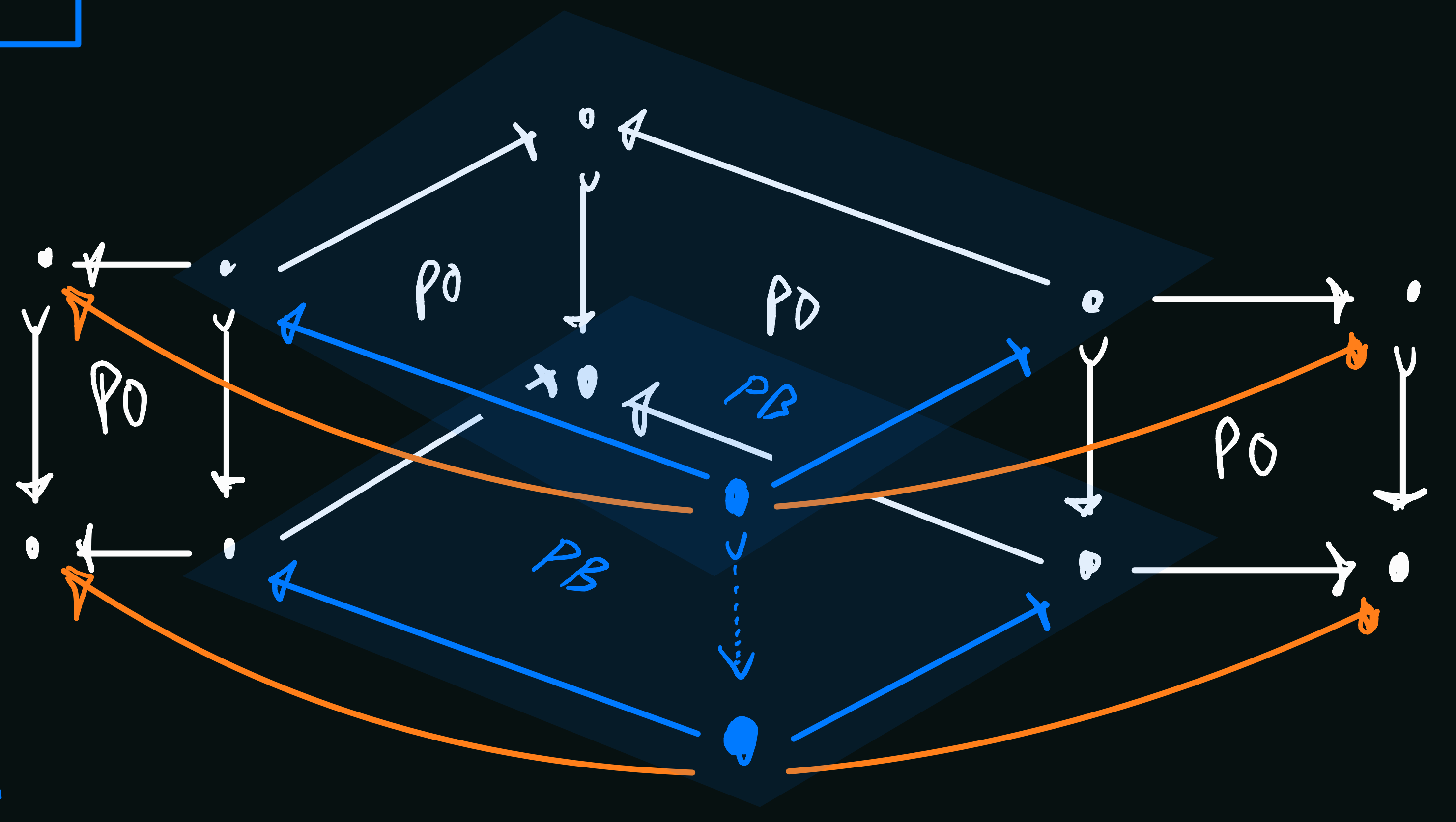
O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\
Y & K_s \arrow[l, "o_s"] \arrow[r, "i_s"'] & X
\end{tikzcd}$$

↗ HORIZONTAL COMPOSITION:

≙ CHOICE OF PULLBACKS (PBs)!



⑦ KEY CONCEPT: (COVARIANT) PRESHEAVES $F : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

↳ IDEA: $\forall r \in \mathbb{D}_1 : \quad \hat{\Delta}_r := \mathbb{D}_1(r, -)$

↳ $\left|\hat{\Delta}_r(Y \xleftarrow{s} X)\right| = \left| \left\{ \begin{array}{ccc} O & \xleftarrow{r} & I \\ n\downarrow & \Downarrow\alpha & \downarrow m \\ Y & \xleftarrow[s]{} & X \end{array} \in \mathbb{D}_1 \right\} \right| \propto$ "# ways to rewrite X into Y along $Y \xleftarrow{s} X$ with rule $O \xleftarrow{r} I$"

▸ BUT: we want "$\rho(\delta(r))|X\rangle = \underbrace{\rho(\delta(r))\rho(\delta(\overset{?}{\overbrace{X \leftarrow \emptyset}}))|\overset{?}{\emptyset}\rangle}_{?} = \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \underbrace{\overset{\in \mathbb{Z}_{\geq 0}}{\overbrace{M_{r,X}{}^{Y}}}}_{?} |Y\rangle$"

⑦ KEY CONCEPT: (COVARIANT) PRESHEAVES $F : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

↳ <u>IDEA</u>: $\forall r \in \mathbb{D}_1 : \quad \hat{\Delta}_r := \mathbb{D}_1(r, -)$

↳ $\left|\hat{\Delta}_r(Y \xleftarrow{s} X)\right| = \left| \left\{ \begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \in \mathbb{D}_1 \right\} \right| \propto$ "# ways to rewrite X into Y along $Y \xleftarrow{s} X$ with rule $O \xleftarrow{r} I$"

▸ BUT: we want "$\rho(\delta(r))|X\rangle = \underbrace{\rho(\delta(r))\,\rho(\delta(\overbrace{X \leftarrow \emptyset}^{?}))\,|\overset{?}{\emptyset}\rangle}_{?} = \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \underbrace{\overbrace{M_{r,X}{}^{Y}}^{\in \mathbb{Z}_{\geq 0}}}_{?}\, |Y\rangle$"

▸ <u>ASSUMPTION</u>: $\mathbb{D}_0$ HAS A STRICT INITIAL OBJECT $\emptyset$ (i.e., $\forall X \in \mathbb{D}_0 : \exists!\ \emptyset \rightarrowtail X \ \wedge\ \forall X \xrightarrow{f} \emptyset : X = \emptyset$),

AND SUCH THAT (i) $\forall X \in \mathbb{D}_0 : \exists!\ (X \leftarrow \emptyset) \in ob(\mathbb{D}_1) \ \wedge\ \exists!\ (\emptyset \leftarrow X) \in ob(\mathbb{D}_1)$

(ii) $\forall \begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \in \mathbb{D}_0 : \left| \left\{ \begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \emptyset \arrow[l] \arrow[d, equal] \\ Y & \emptyset \arrow[l] \end{tikzcd} \in \mathbb{D}_1 \right\} \right| \leq 1 \ \wedge\ \left| \left\{ \begin{tikzcd} \emptyset \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\beta"] & X \arrow[l] \arrow[d, "f"] \\ \emptyset & Y \arrow[l] \end{tikzcd} \in \mathbb{D}_1 \right\} \right| \leq 1$

8 DEFINITION: A COEND FOR A FUNCTOR $F : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathbf{Set}}$

IS DEFINED AS $\displaystyle\int^{C \in \mathcal{C}} F(C,C) = \left(\coprod_{C \in \mathcal{C}} F(C,C) \right) \Big/ \sim$

$$\text{with: } (\underset{\in F(C,C)}{C, x}) \sim (\underset{\in F(C',C')}{C', x'}) :\Leftrightarrow \exists\, C \xrightarrow{\gamma} C',\ y \in F(C',C) :\ x = F(\gamma, id)y \ \wedge\ x' = F(id, \gamma)y$$

(8) DEFINITION: A COEND FOR A FUNCTOR $F: \mathcal{C}^{op}\times\mathcal{C}\to\underline{Set}$

IS DEFINED AS
$$\int^{C\in\mathcal{C}} F(C,C) = \left(\coprod_{C\in\mathcal{C}} F(C,C)\right)\Big/\sim$$

with: $(C,\overset{\substack{F(C,C)\\ \Downarrow}}{x})\sim(C',\overset{\substack{F(C',C')\\ \Downarrow}}{x'}) :\Leftrightarrow \exists\, C\xrightarrow{\gamma}C',\ y\in F(C',C):\ x=F(\gamma,id)y\ \wedge\ x'=F(id,\gamma)y$

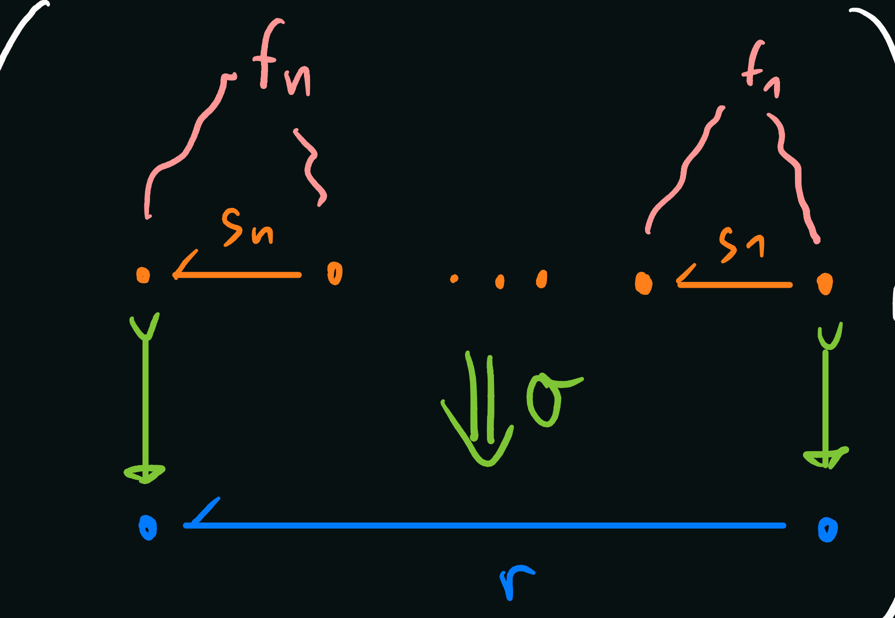
KEY CONCEPT: CONVOLUTION PRODUCTS OF PRESHEAVES $F_n,\dots,F_1:\mathbb{D}_1\to\underline{Set}$

$$(F_n*\cdots*F_1) := r\mapsto\int^{S=(S_n,\dots,S_1)\in\mathbb{D}_n}\mathbb{D}_1(h_n(S),r)\times\mathbb{F}_n(S)$$

$(\cong \mathrm{Lan}_{h_n}(\mathbb{F}_n))$

- $\mathbb{D}_1(h_n(-),r):\mathbb{D}_n^{op}\to\underline{Set}$
- $\mathbb{F}_n := F_n\times\ldots\times F_1:\mathbb{D}_n\to\underline{Set}$

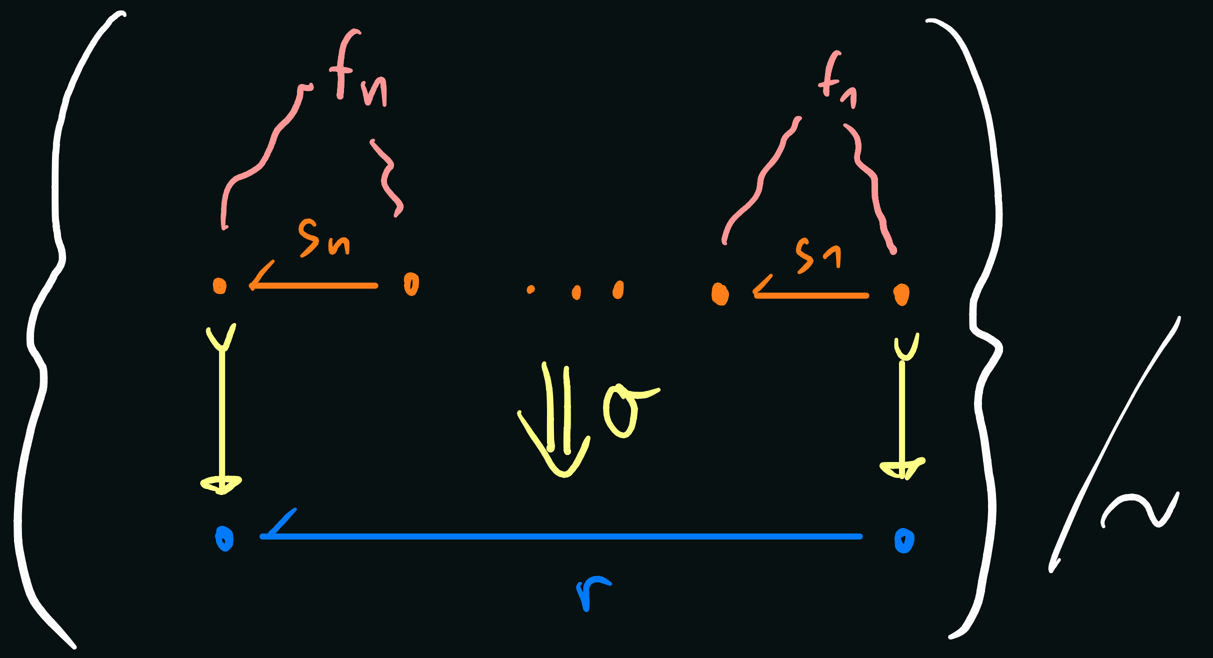
$$=\left\{(S,(\sigma,f))\ \middle|\ \begin{array}{l}S\in\mathbb{D}_n\\ \sigma\in\mathbb{D}_1(h_n(S),r)\\ f\in\mathbb{F}_n(S)\end{array}\right\}\Big/\sim\ \equiv\ \left\{\ \right.$$



$\left.\right\}\Big/\sim$

9

$$(F_n * \cdots * F_1)(r) = \left\{ (S, (\sigma, f)) \,\middle|\, \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \quad \equiv$$

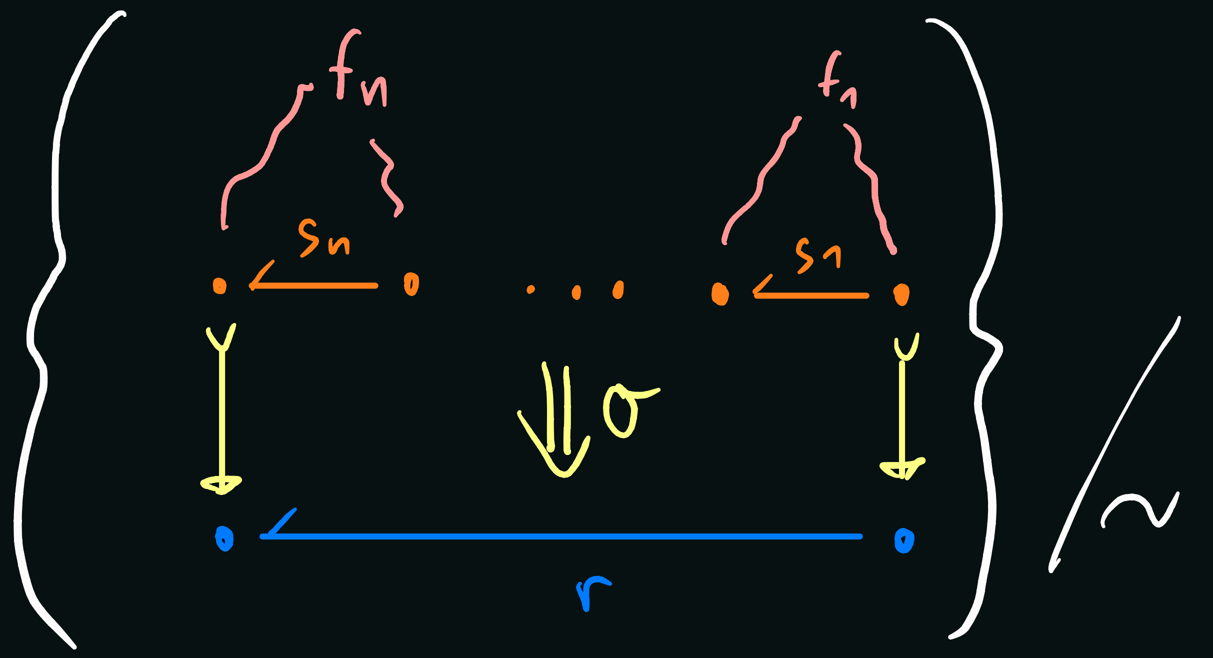


- $(S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g) \in \mathbb{D}_1(h_n(S'), r) \times \mathbb{F}_n(S):$

$$(\sigma,f) = (\mathbb{D}_1(h_n(A), r)\tau,\ g) \ \wedge\ (\sigma',f') = (\tau,\ \mathbb{F}_n(A)\, g)$$

9

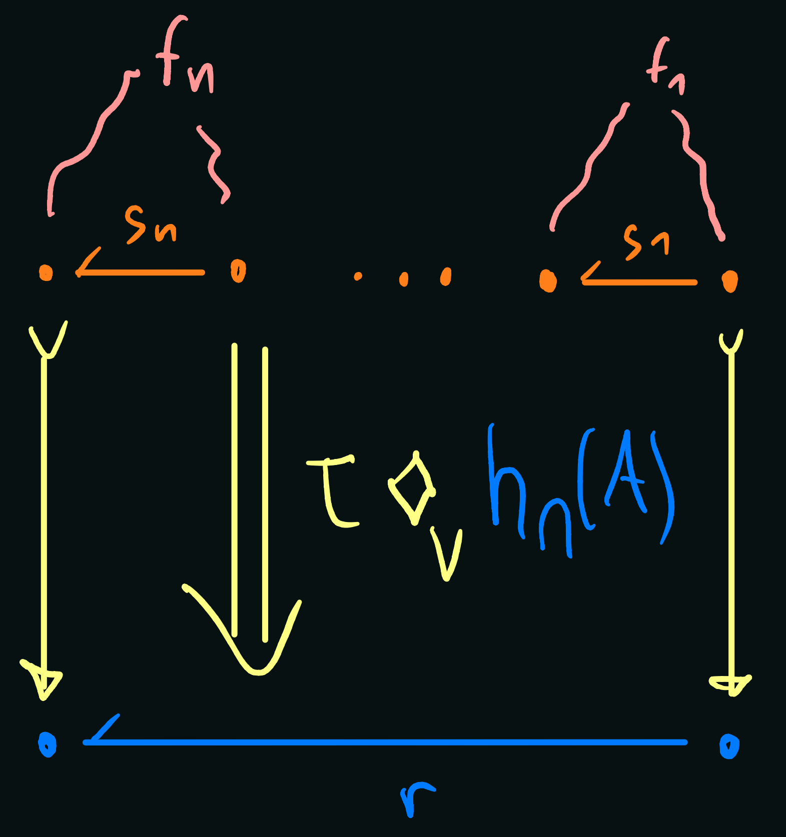
$$(F_n * \cdots * F_1)(r) = \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S\in \mathbb{D}_n \\ \sigma\in \mathbb{D}_1(h_n(S),r) \\ f\in \mathbb{F}_n(S)\end{array}\right\}\Big/\sim \;\equiv$$



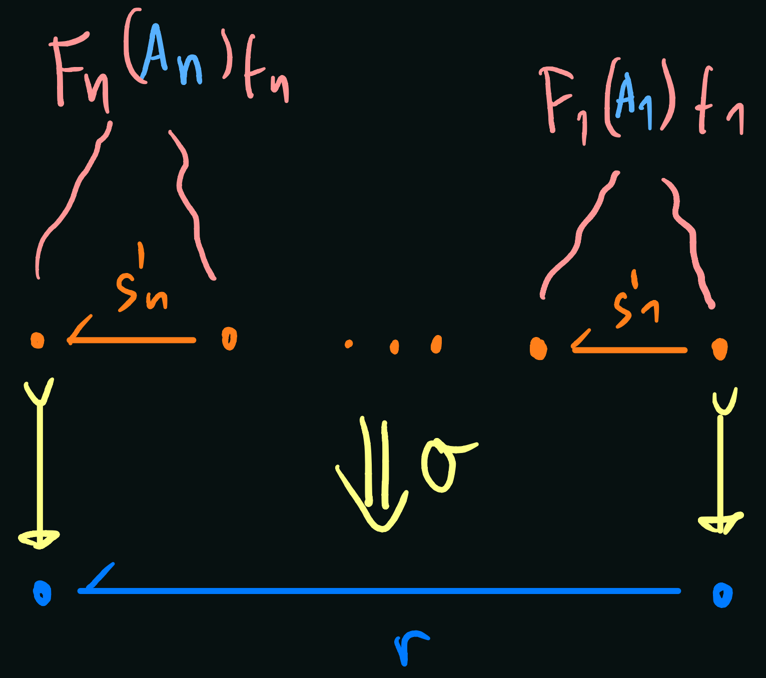
$$\cdot\ (S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g)\in \mathbb{D}_1(h_n(S'),r)\times \mathbb{F}_n(S):$$

$$(\sigma,f) = (\mathbb{D}_1(h_n(A),r)\tau,\ g) \;\wedge\; (\sigma',f') = (\tau,\ \mathbb{F}_n(A)g)$$

↳ $\forall\, S\xrightarrow{A}S'\in\mathbb{D}_n,\ h_n(S')\xrightarrow{\tau} r\in\mathbb{D}_1$:

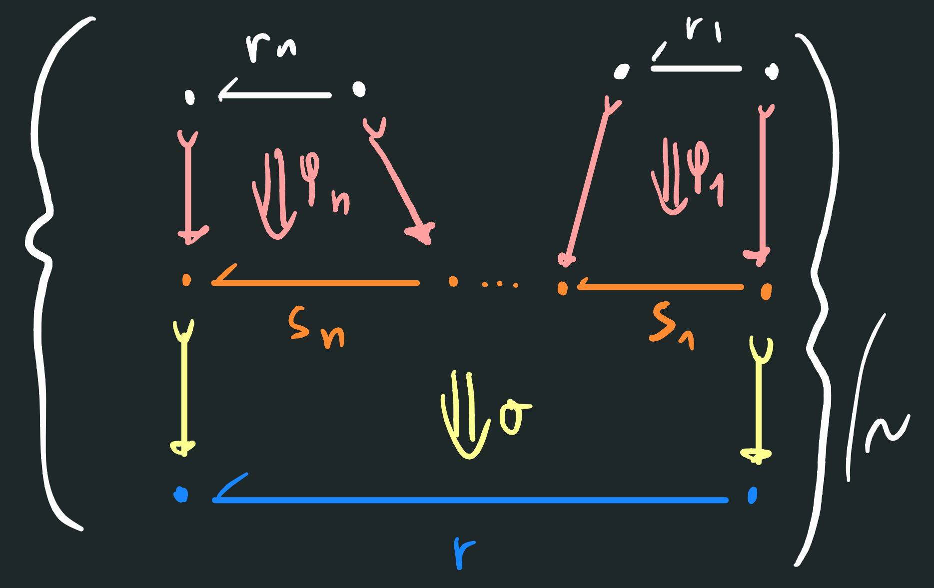


$\sim$



EXAMPLE: $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j,-):\mathbb{D}_1 \to \underline{Set}$ $\quad(j=1,\dots,n)$

↳ $(\hat{\Delta}_{r_n} * \cdots * \hat{\Delta}_{r_1})(r) =$



10 KEY CONCEPT: FIBRATIONAL STRUCTURES

- DEFINITION: A FUNCTOR $G:\mathcal{E}\to\mathcal{B}$ IS A GROTHENDIECK OPFIBRATION IFF

$$
\forall\ \begin{tikzcd} & e \arrow[d, mapsto, "G"'] & \\ & b \arrow[r, "f"'] & b' \end{tikzcd}
\ :\ \exists\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] & e' \arrow[d, mapsto, "G"] \\ b \arrow[r, "f"'] & b' \end{tikzcd}
\ :\ \forall\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, mapsto, "G"'] & e'' \arrow[d, mapsto, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd}
\ :\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, mapsto, "G"'] \arrow[r, dashed, "\beta", "\exists!"'] & e'' \arrow[d, mapsto, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=G(\beta)"'] & b'' \end{tikzcd}
$$

⑩ Key concept: Fibrational structures

• Definition: A functor $G:\mathcal{E}\to\mathcal{B}$ is a Grothendieck opfibration iff

$$\forall\;\begin{tikzcd} e \arrow[d, "G"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd} \;:\; \exists\;\begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] & e' \arrow[d, "G"] \\ b \arrow[r, "f"'] & b' \end{tikzcd} \;:\; \forall\;\begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[rr, bend left, "\varepsilon"] \arrow[d, "G"'] & e' \arrow[d, "G"'] & e'' \arrow[d, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd} \;:\; \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[rr, bend left, "\varepsilon"] \arrow[d, "G"'] & e' \arrow[r, dashed, "\beta"', "\exists!"] \arrow[d, "G"'] & e'' \arrow[d, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=G(\beta)"'] & b'' \end{tikzcd}$$

• Definition: A functor $S:\mathcal{E}\to\mathcal{B}$ is a Street opfibration iff

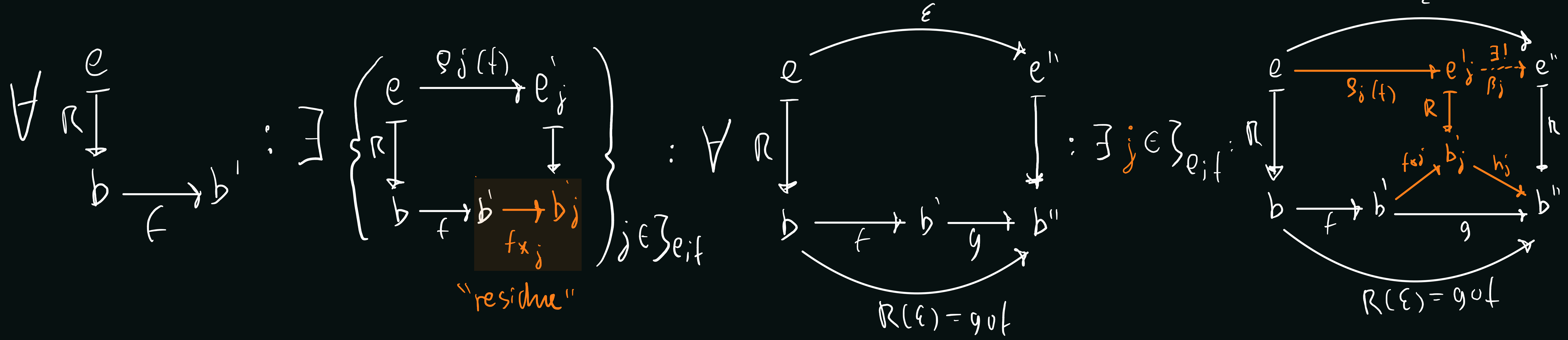
$$\forall\;\begin{tikzcd} e \arrow[d, "S"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd} \;:\; \exists\;\begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] & & \tilde{e} \arrow[d, "S"] \\ b \arrow[r, "f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \end{tikzcd} \;:\; \forall\;\begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[rrr, bend left, "\varepsilon"] \arrow[d, "S"'] & & \tilde{e} \arrow[d, "S"] & e'' \arrow[d, "S"] \\ b \arrow[r, "f"'] \arrow[rrr, bend right, "S(\varepsilon)=g\circ\varphi_f\circ f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \arrow[r, "g"'] & b'' \end{tikzcd} \;:\; \begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[rrr, bend left, "\varepsilon"] \arrow[d, "S"'] & & \tilde{e} \arrow[r, dashed, "\beta"] \arrow[d, "S"] & e'' \arrow[d, "S"] \\ b \arrow[r, "f"'] \arrow[rrr, bend right, "S(\varepsilon)=g\circ\varphi_f\circ f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \arrow[r, "g=S(\beta)"'] & b'' \end{tikzcd}$$

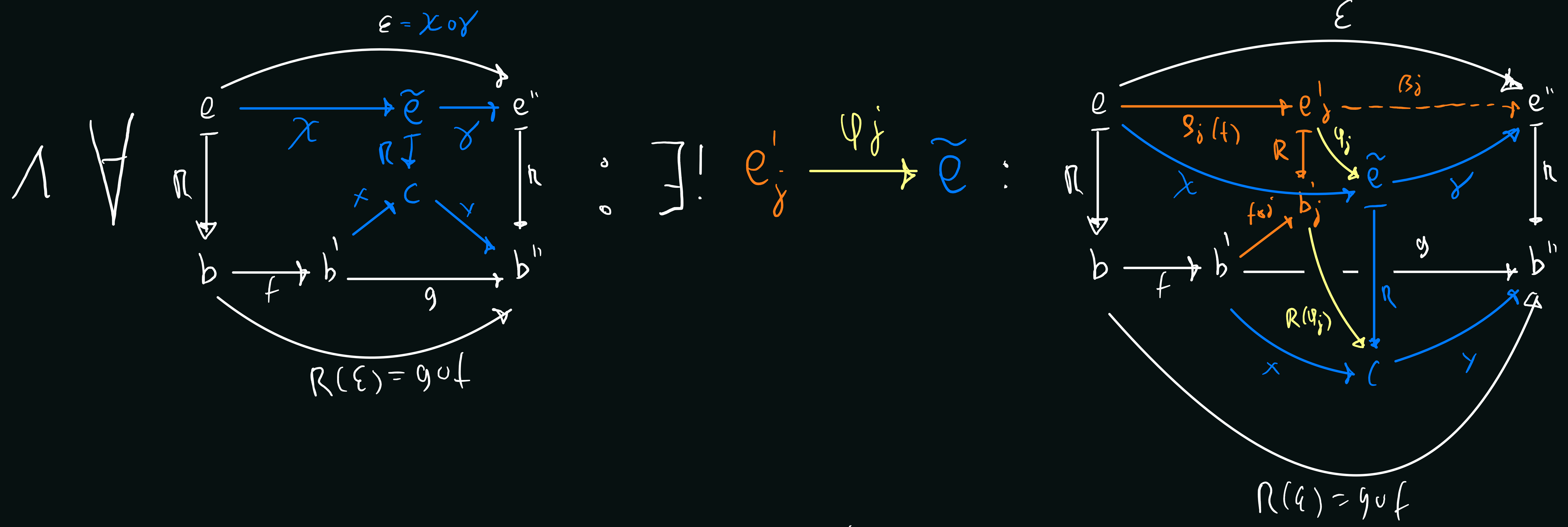
(11) DEFINITION: A FUNCTOR $M: \mathcal{E} \to \mathcal{B}$ IS A MULTI-OPFIBRATION IFF

$$
\forall \begin{array}{c} e \\ {\scriptstyle M}\downarrow \\ b \xrightarrow{f} b' \end{array} : \exists \left\{ \begin{tikzcd} e \arrow[r, "\mu_j(f)"] \arrow[d, mapsto, "M"'] & e_j \arrow[d, mapsto, "M"] \\ b \arrow[r, "f"'] & b' \end{tikzcd} \right\}_{j \in J_{e,f}} : \forall \begin{tikzcd} e \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "M"'] & & e'' \arrow[d, mapsto, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon) = g \circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd} : \exists j \in J_{e,f} : \begin{tikzcd} e \arrow[r, "\mu_j(f)"'] \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "M"'] & e'_j \arrow[r, dashed, "\beta_j"', "\exists !"] \arrow[d, mapsto, "M"] & e'' \arrow[d, mapsto, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon) = g \circ f"'] & b' \arrow[r, "g = M(\beta_j)"'] & b'' \end{tikzcd}
$$

$$
\wedge \ \forall \begin{tikzcd} e \arrow[r, "x"] \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "M"'] & X \arrow[r, "\gamma"] \arrow[d, mapsto, "M"] & e'' \arrow[d, mapsto, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon) = g \circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd} : \begin{tikzcd} e \arrow[r, "\mu_j(f)"'] \arrow[rrr, bend left, "\varepsilon"] \arrow[rrd, bend right, "x"'] \arrow[ddd, mapsto, "M"'] & e'_j \arrow[rr, dashed, "\beta_j"] \arrow[rd, dashed, "\exists !"] \arrow[dd, mapsto, "M"] & & e'' \arrow[ddd, mapsto, "M"] \\ & & X \arrow[ru, "\gamma"'] \arrow[ld, mapsto, "M"] & \\ & b' & & \\ b \arrow[ru, "f"'] \arrow[rrr, bend right, "M(\varepsilon) = g \circ f"'] & & & b'' \arrow[from=3-2, "g"'] \end{tikzcd}
$$

(12) DEFINITION: A FUNCTOR $R : \mathcal{E} \to \mathcal{B}$ IS A RESIDUAL MULTI-OPFIBRATION IFF





13 **DEFINITION:** LET $X:\mathcal{E}\to\mathcal{B}$ BE AN X-OPFIBRATION $(X\in\{G,S,M,R\})$.

THEN A CLEAVAGE FOR X IS DEFINED AS A CHOICE OF REPRESENTATIVE FOR EACH X-OPCARTESIAN LIFTING:

$$G^*\left(\begin{tikzcd} e \arrow[d, |->, "G"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \begin{tikzcd} e \arrow[r, "G^*(f)"] \arrow[d, |->, "G"'] & b \arrow[d, |->, "G"] \\ b \arrow[r, "f"'] & b \end{tikzcd}$$

$$S^*\left(\begin{tikzcd} e \arrow[d, |->, "S"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \begin{tikzcd} e \arrow[rr, "\sigma^*(f)"] \arrow[d, |->, "S"'] & & \tilde{e} \arrow[d, |->, "S"] \\ b \arrow[r, "f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \end{tikzcd}$$

$$M^*\left(\begin{tikzcd} e \arrow[d, |->, "M"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \left\{\begin{tikzcd} e \arrow[r, "\mu_j^*(f)"] \arrow[d, |->, "M"'] & e'_j \arrow[d, |->, "M"] \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right\}_{j\in\mathfrak{Z}^*_{e;f}}$$

$$R^*\left(\begin{tikzcd} e \arrow[d, |->, "R"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \left\{\begin{tikzcd} e \arrow[rr, "\rho_j^*(f)"] \arrow[d, |->, "R"'] & & e'_j \arrow[d, |->, "R"] \\ b \arrow[r, "f"'] & b' \arrow[r, "f x_j"'] & b'_j \end{tikzcd}\right\}_{j\in\mathfrak{Z}^*_{e;f}}$$

ONE REPRESENTATIVE PER EQUIVALENCE CLASS IN $\mathfrak{Z}_{e;f}$!

(14) EMPIRICAL RESULT: $\mathbb{D}$ FOR COMPOSITIONAL* REWRITING SEMANTICS

*2204.07175

- $h_2 = \diamond_n : \mathbb{D}_2 \longrightarrow \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION, i.e.,

$$\forall \begin{tikzcd} R=(r_2,r_1) \arrow[d, "h_2"'] & \\ r \arrow[r, "\alpha"'] & s \end{tikzcd} : \exists \begin{tikzcd} R \arrow[rr, "A"] \arrow[d, "h_2"'] & & T \arrow[d, "h_2"] \\ r \arrow[r, "\alpha"'] & s \arrow[r, "\cong", "\varphi_\alpha"'] & t \end{tikzcd} : S(\varphi_\alpha) = id_{S(s)} \wedge T(\varphi_\alpha) = id_{T(s)} \wedge$$

(STREET OPFIBRATION CONDITIONS)

$$\forall \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "r_2"'] \arrow[d, phantom, "\Downarrow\alpha"] & \bullet \arrow[l, "r_1"'] \arrow[d] \\ \bullet & & \bullet \arrow[ll, "s"] \end{tikzcd} : \exists \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "r_2"'] \arrow[d] & \bullet \arrow[l, "r_1"'] \arrow[d] \\ \bullet \arrow[r, phantom, "\Downarrow A_2"] & \bullet \arrow[l, bend right, "t_2"'] \arrow[d, phantom, "\cong\Downarrow\varphi_\alpha^{-1}"] \arrow[r, phantom, "\Downarrow A_1"] & \bullet \arrow[l, bend right, "t_1"'] \arrow[ll, bend left=60, "s"] \\ & & \end{tikzcd} : \varphi_\alpha^{-1} \diamond_v (A_2 \diamond A_1) = \alpha$$

"GLOBULAR" ISOMORPHISM

BY INDUCTION ON n, ONE FINDS THAT

$\forall n \geq 2$: $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$ ARE "GLOBULAR" STREET OPFIBRATIONS

15

$\mathbb{D}_0$ HAS MULTI-SUMS:

$$\forall\ (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 :\ \exists \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j \in \mathcal{J}(A,B)} :\ \forall \begin{tikzcd} A \arrow[dr, rightarrowtail, "x_A"'] & & B \arrow[dl, rightarrowtail, "x_B"] \\ & X & \end{tikzcd} :\ \exists j \in \mathcal{J}(A,B) : \begin{tikzcd} A \arrow[dr, rightarrowtail, "a_j"] \arrow[ddr, rightarrowtail, bend right, "x_A"'] & & B \arrow[dl, rightarrowtail, "b_j"'] \arrow[ddl, rightarrowtail, bend left, "x_B"] \\ & M_j \arrow[d, rightarrowtail, dashed, "\exists!\ m_j"] & \\ & X & \end{tikzcd}$$

$$\wedge\ \forall \begin{tikzcd} A \arrow[dr, rightarrowtail, "y_A"] \arrow[ddr, rightarrowtail, bend right, "x_A"'] & & B \arrow[dl, rightarrowtail, "y_B"'] \arrow[ddl, rightarrowtail, bend left, "x_B"] \\ & Y \arrow[d, rightarrowtail, "y_X"] & \\ & X & \end{tikzcd} :\ \exists!\ M_j \overset{m}{\rightarrowtail} Y :\ \begin{tikzcd} A \arrow[r, rightarrowtail, "a_j"] \arrow[dr, rightarrowtail, "y_A"] \arrow[ddr, rightarrowtail, bend right, "x_A"'] & M_j \arrow[d, rightarrowtail, dashed, "m"] & B \arrow[l, rightarrowtail, "b_j"'] \arrow[dl, rightarrowtail, "y_B"'] \arrow[ddl, rightarrowtail, bend left, "x_B"] \\ & Y \arrow[d, rightarrowtail, "y_X"] & \\ & X & \end{tikzcd}$$

DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall\ (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 :\ ms(A,B) = \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{\mathcal{J}^{\circ}(A,B)}$$

15 $\mathbb{D}_0$ HAS MULTI-SUMS:

$\forall\ (A,B)\in \mathbb{D}_0\times\mathbb{D}_0 : \exists \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j\in J_{(A,B)}} : \forall$

$$\begin{tikzcd} A \arrow[dr, bend right, "x_A"'] & & B \arrow[dl, bend left, "x_B"] \\ & X & \end{tikzcd}$$

$: \exists j\in J_{(A,B)} :$

$$\begin{tikzcd} A \arrow[r, "a_j"] \arrow[dr, bend right, "x_A"'] & M_j \arrow[d, dotted, "\exists!\, m_j"] & B \arrow[l, "b_j"'] \arrow[dl, bend left, "x_B"] \\ & X & \end{tikzcd}$$

$\wedge\ \forall$

$$\begin{tikzcd} A \arrow[dr, "y_A"] \arrow[ddr, bend right, "x_A"'] & & B \arrow[dl, "y_B"'] \arrow[ddl, bend left, "x_B"] \\ & Y \arrow[d, "y_X"] & \\ & X & \end{tikzcd}$$

$: \exists!\ M_j \xrightarrow{m} Y :$

$$\begin{tikzcd} A \arrow[r, "a_j"] \arrow[dr, "y_A"'] \arrow[ddr, bend right, "x_A"'] & M_j \arrow[d, "m"] & B \arrow[l, "b_j"'] \arrow[dl, "y_B"] \arrow[ddl, bend left, "x_B"] \\ & Y \arrow[d, "y_X"] & \\ & X & \end{tikzcd}$$

DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall (A,B)\in\mathbb{D}_0\times\mathbb{D}_0 :\ ms(A,B) = \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{J^{\circ}_{(A,B)}}$$

$S : \mathbb{D}_1 \to \mathbb{D}_0$ IS A MULTI-OPFIBRATION

$\forall$

$$\begin{tikzcd} \cdot \arrow[dd, "n''"'] & \cdot \arrow[l, "r"'] \arrow[d, "m"] \arrow[ddl, phantom, "\Downarrow\alpha''"] \\ & \cdot \arrow[d, "m'"] \\ \cdot & \cdot \arrow[l, "r''"] \end{tikzcd}$$

$: \exists$

$$\begin{tikzcd} \cdot \arrow[d, "n"'] & \cdot \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\ \cdot \arrow[d, "n'"'] & \cdot \arrow[l, "r'"'] \arrow[d, "m'"] \arrow[dl, phantom, "\Downarrow\alpha'"] \\ \cdot & \cdot \arrow[l, "r''"] \end{tikzcd}$$

$: n'\circ n = n''$

$\wedge\ \alpha'\circ_v \alpha = \alpha''$

$T : \mathbb{D}_1 \to \mathbb{D}_0$ IS A RESIDUAL MULTI-OPFIBRATION

$\forall$

$$\begin{tikzcd} \cdot \arrow[d, "n"'] & \cdot \arrow[l, "r"'] \arrow[dd, "m''"] \arrow[ddl, phantom, "\Downarrow\alpha''"] \\ \cdot \arrow[d, "n'"'] & \\ \cdot & \cdot \arrow[l, "r''"] \end{tikzcd}$$

$: \exists$

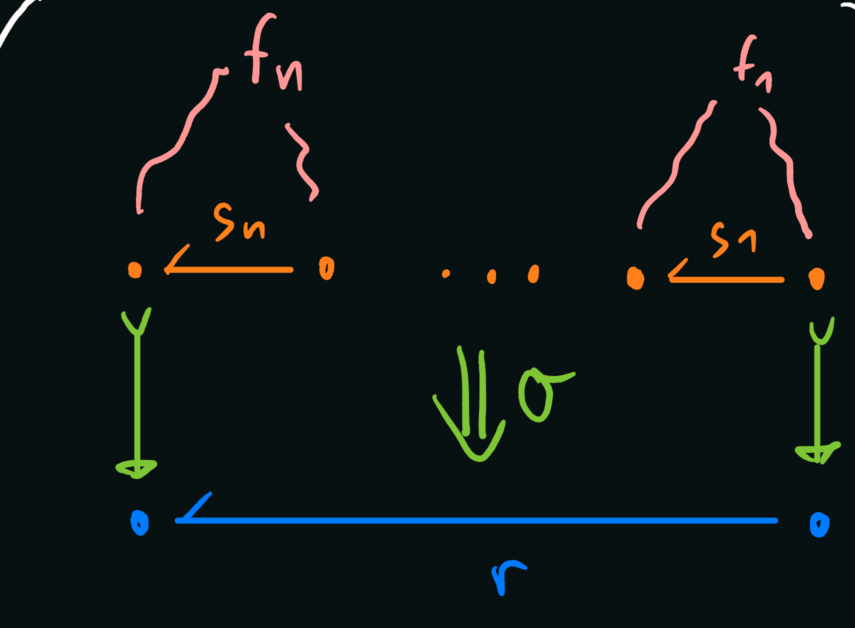
$$\begin{tikzcd} \cdot \arrow[d, "n"'] & & \cdot \arrow[ll, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\beta"] \\ \cdot \arrow[d, "n'"'] \arrow[r, "n_x"] & \cdot \arrow[dl, "\bar{n}"] & \cdot \arrow[l, "r_x"'] \arrow[d, "m'"] \arrow[dl, phantom, "\Downarrow\beta'"] \\ \cdot & & \cdot \arrow[ll, "r''"] \end{tikzcd}$$

$: \bar{n}\circ n_x = n'$

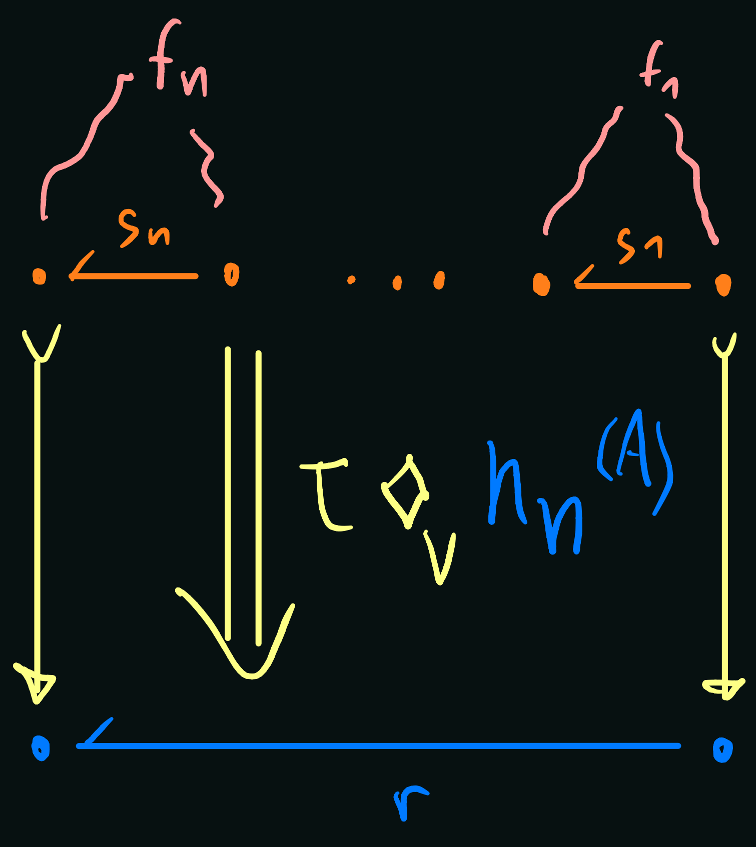
$\wedge\ m'\circ m = m''$

$\wedge\ \beta'\circ_v \beta = \beta''$

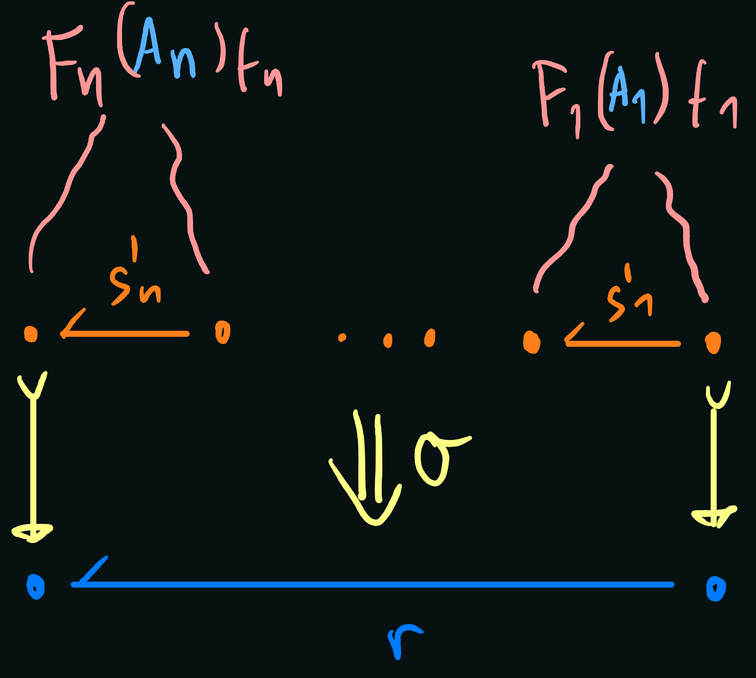
16 CONVOLUTION PRODUCTS REVISITED

RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(S_n,\dots,S_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) = \left\{ \right.$  $\left. \right\} / \sim$

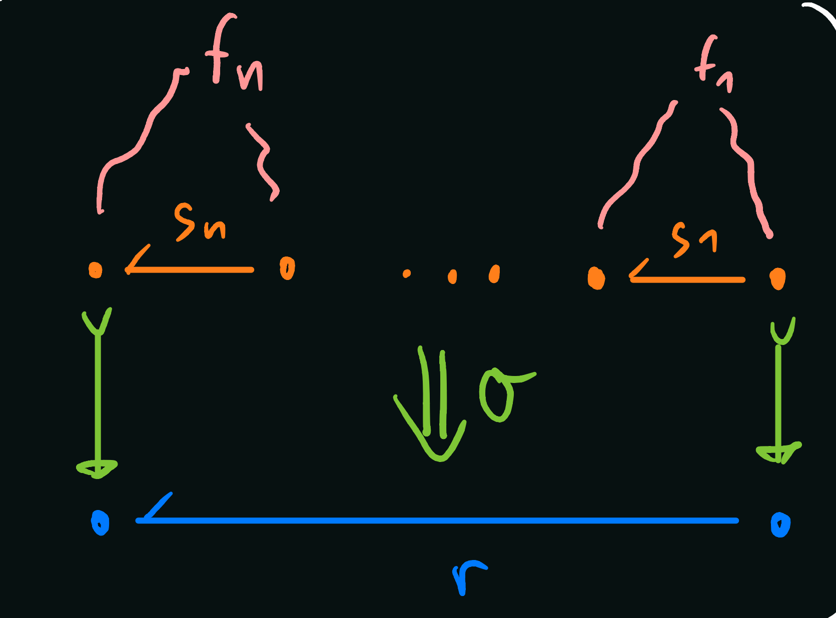
$\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,$
$h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:



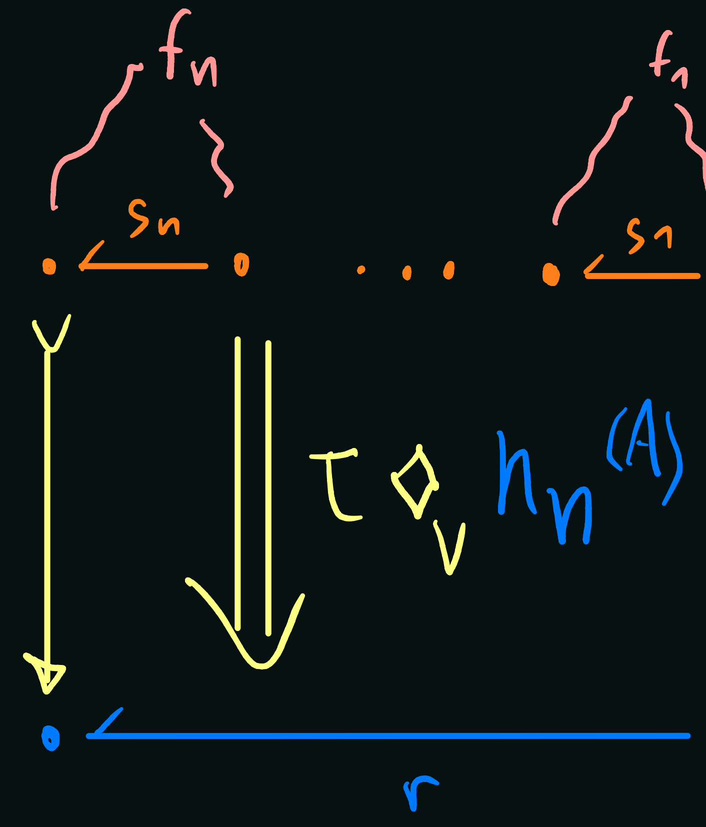
$\sim$



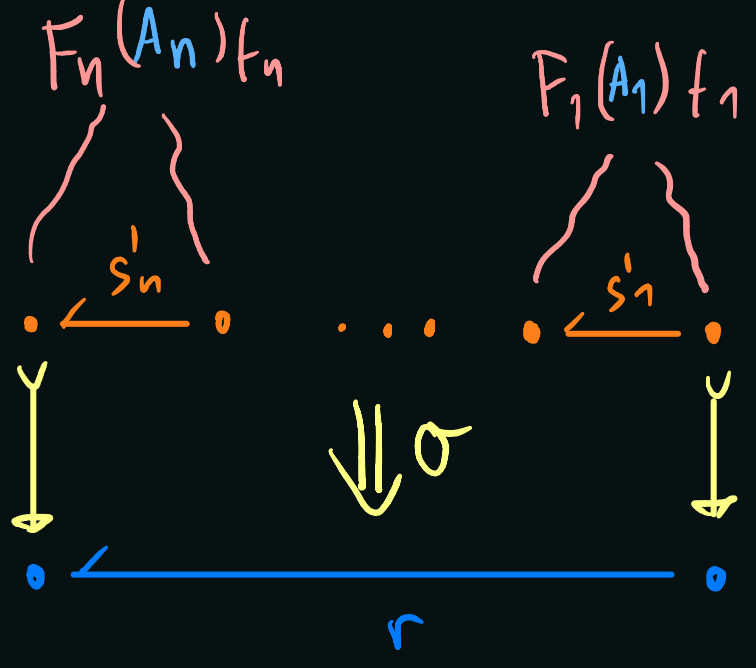
(16) CONVOLUTION PRODUCTS REVISITED

RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(S_n,\dots,S_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) = \left\{ \right.$  $\left. \right\} / \sim$

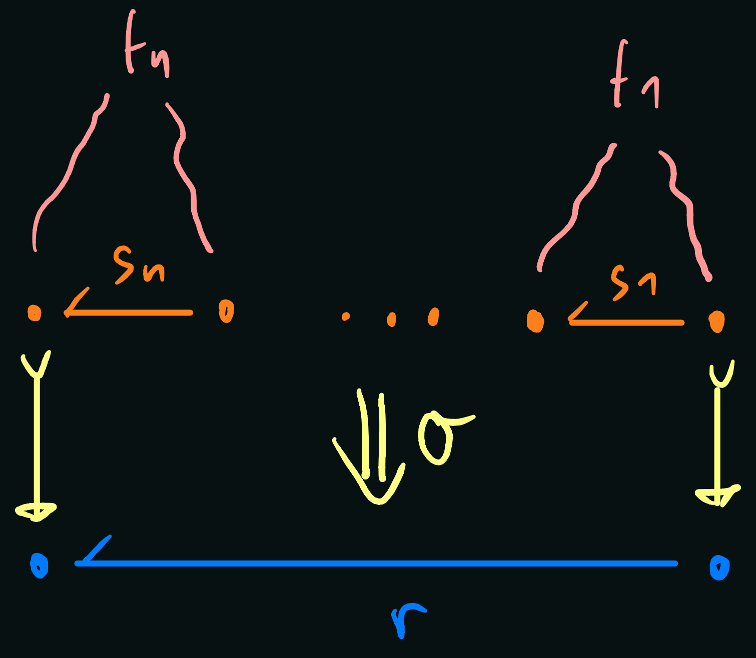
$\forall\ S \xrightarrow{A} S' \in \mathbb{D}_n,$
$h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:



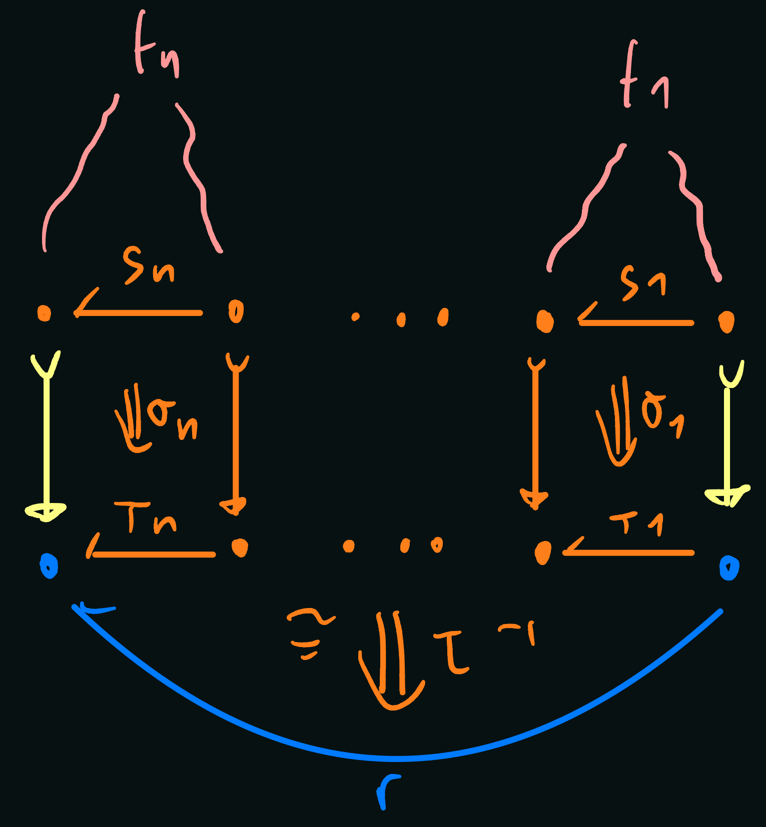
$\sim$



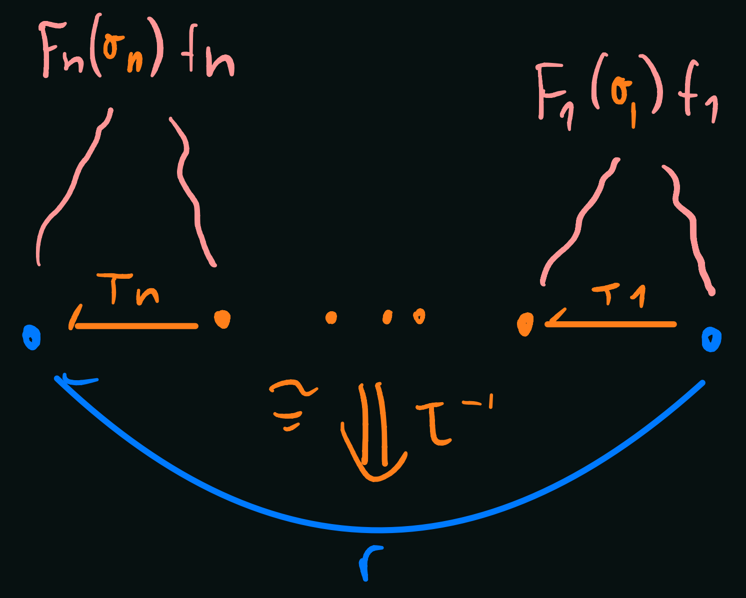
<u>NOW</u>: $h_n : \mathbb{D}_n \to \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION $\Rightarrow \forall$



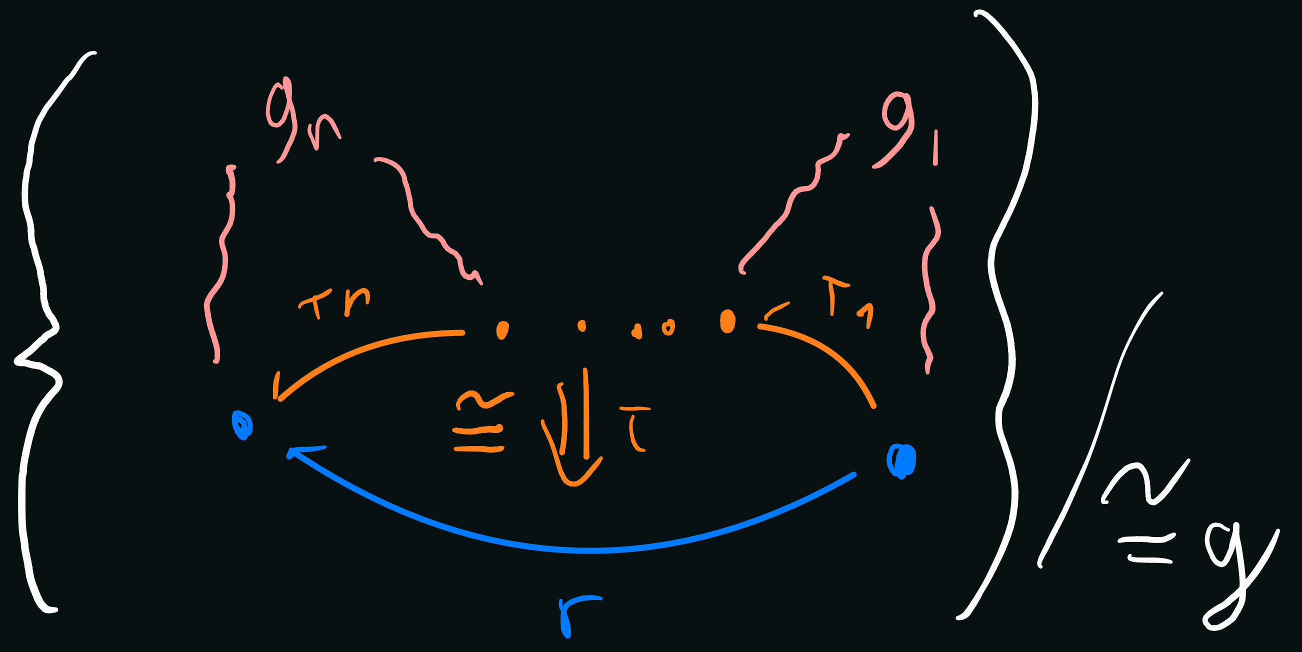
$: \exists$

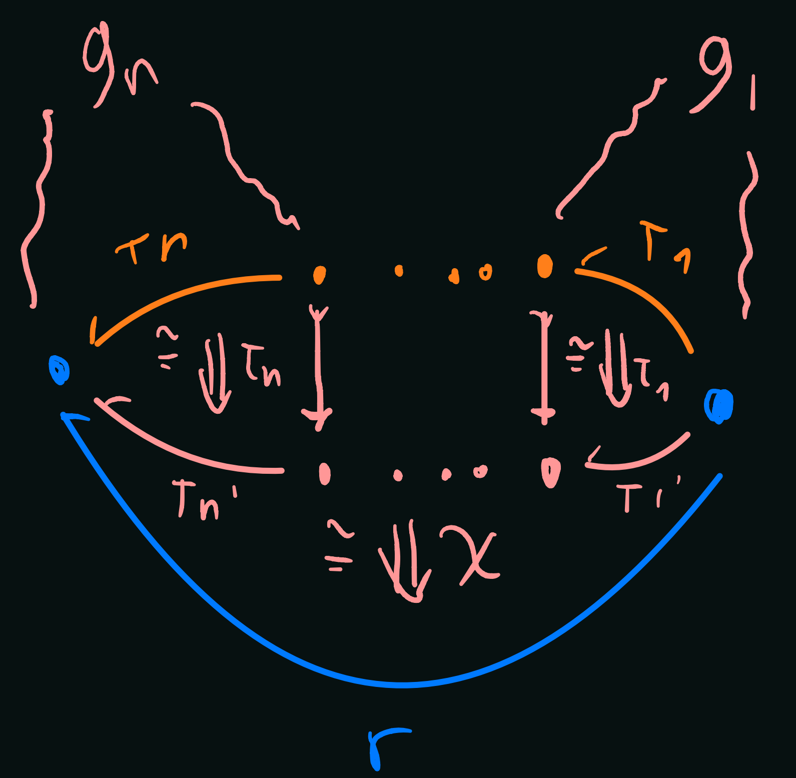


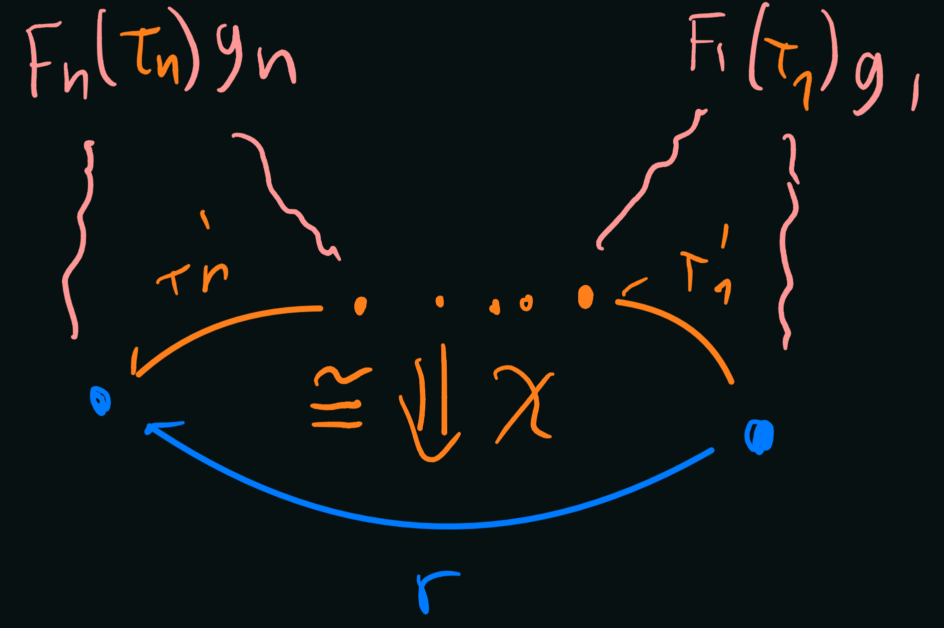
$\sim$



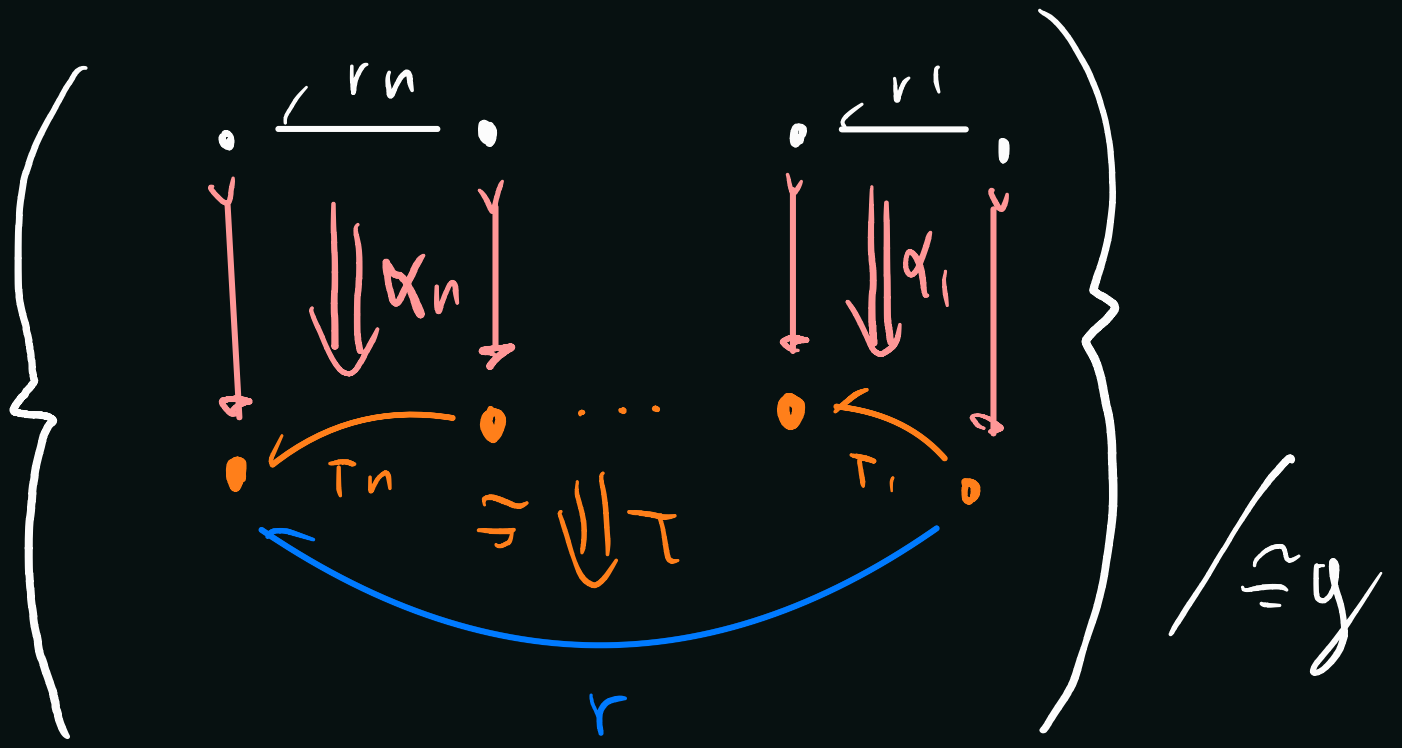
17

$\hookrightarrow \quad (F_n * \ldots * F_1)(r) \cong$ 

WHERE 

$\cong_g$ 

EXAMPLE: FOR $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -) \quad (j=1,\ldots,n)$

$(\hat{\Delta}_{r_n} * \ldots * \hat{\Delta}_{r_1})(r) \cong$ 

18 KEY RESULT: WEAK ASSOCIATIVITY OF $*$

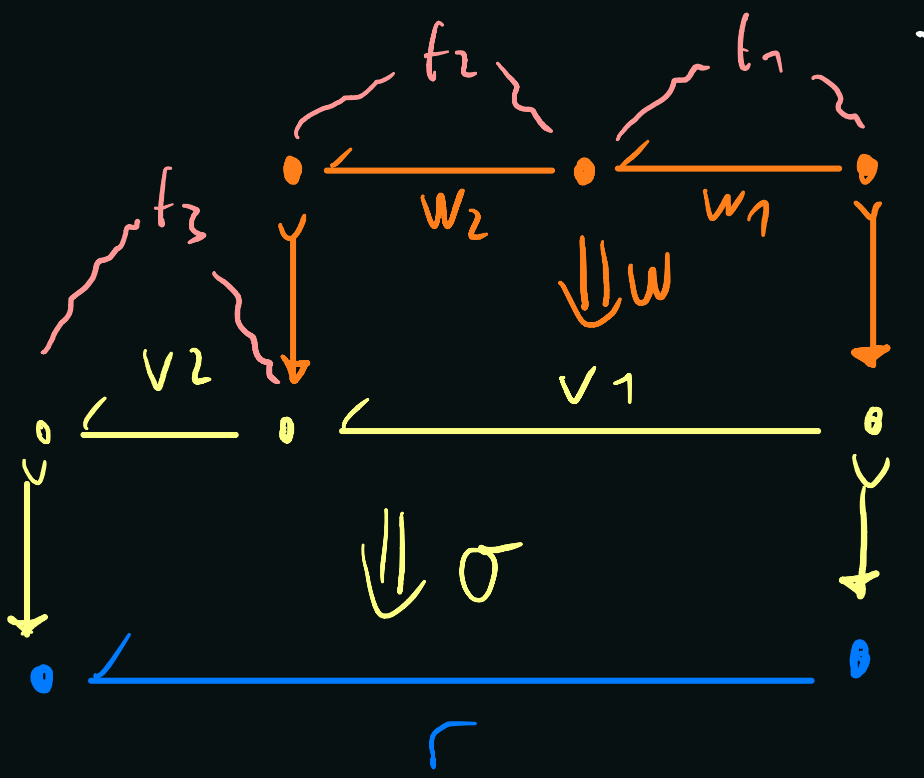
$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0\,,\ r \in \mathbb{D}_1 : \quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

(18) KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0\,,\ r \in \mathbb{D}_1 : \quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

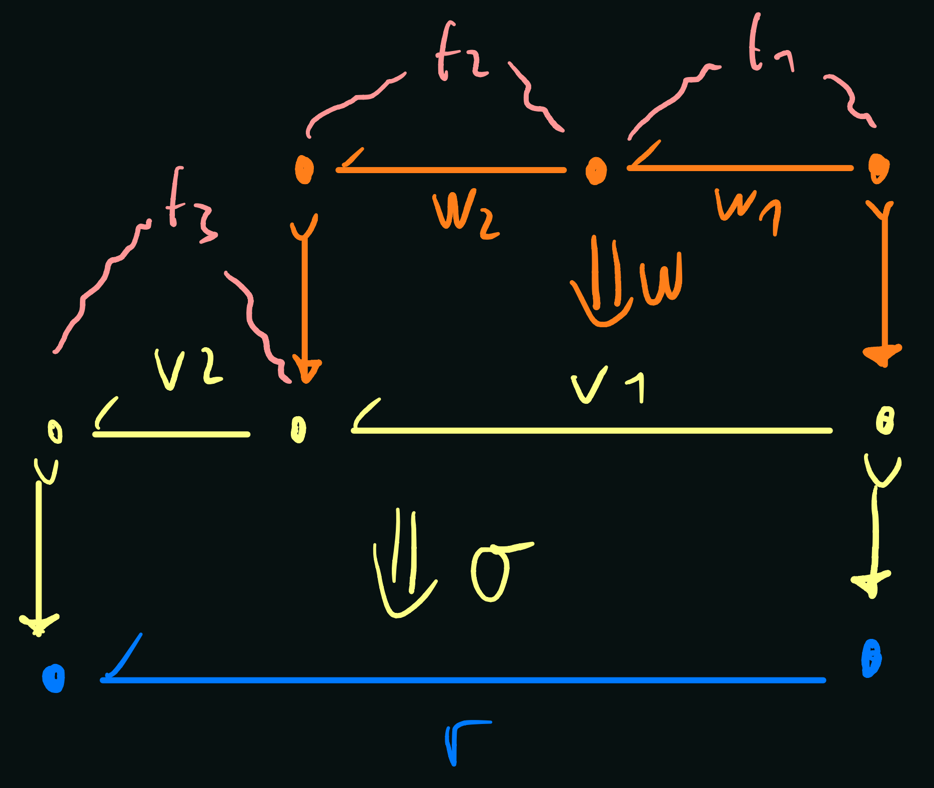
PROOF (SKETCH):

$F_3 * (F_2 * F_1)(r) = \Big\{$

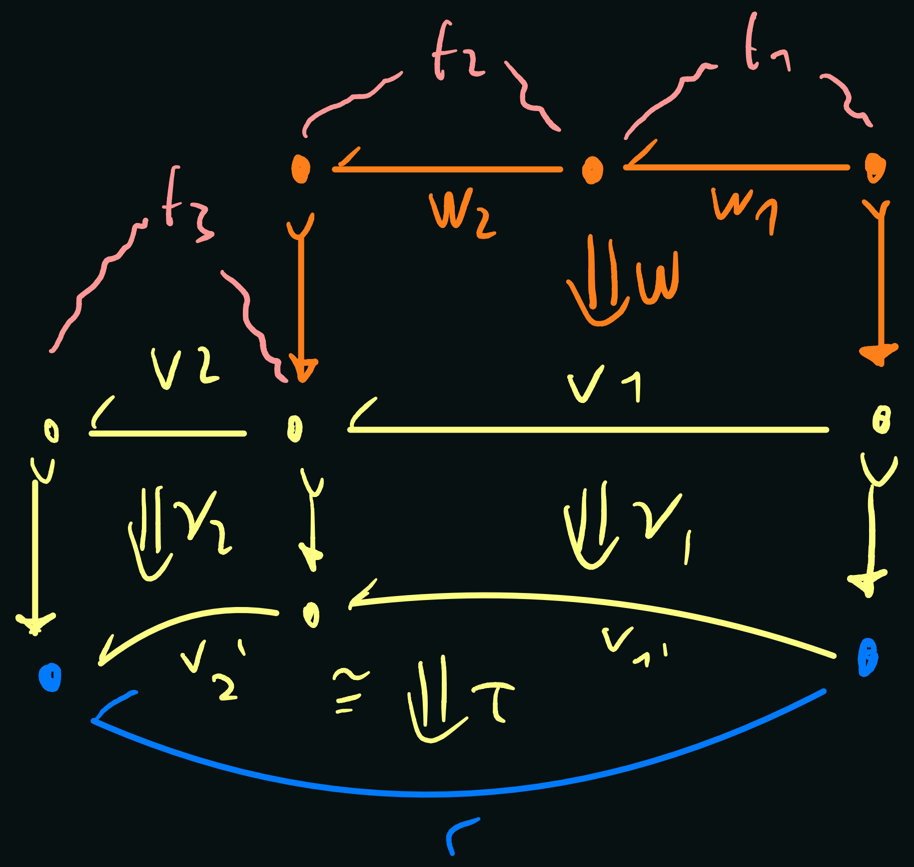


$\Big\} /{\sim_V} /{\sim_W}$

$\forall$



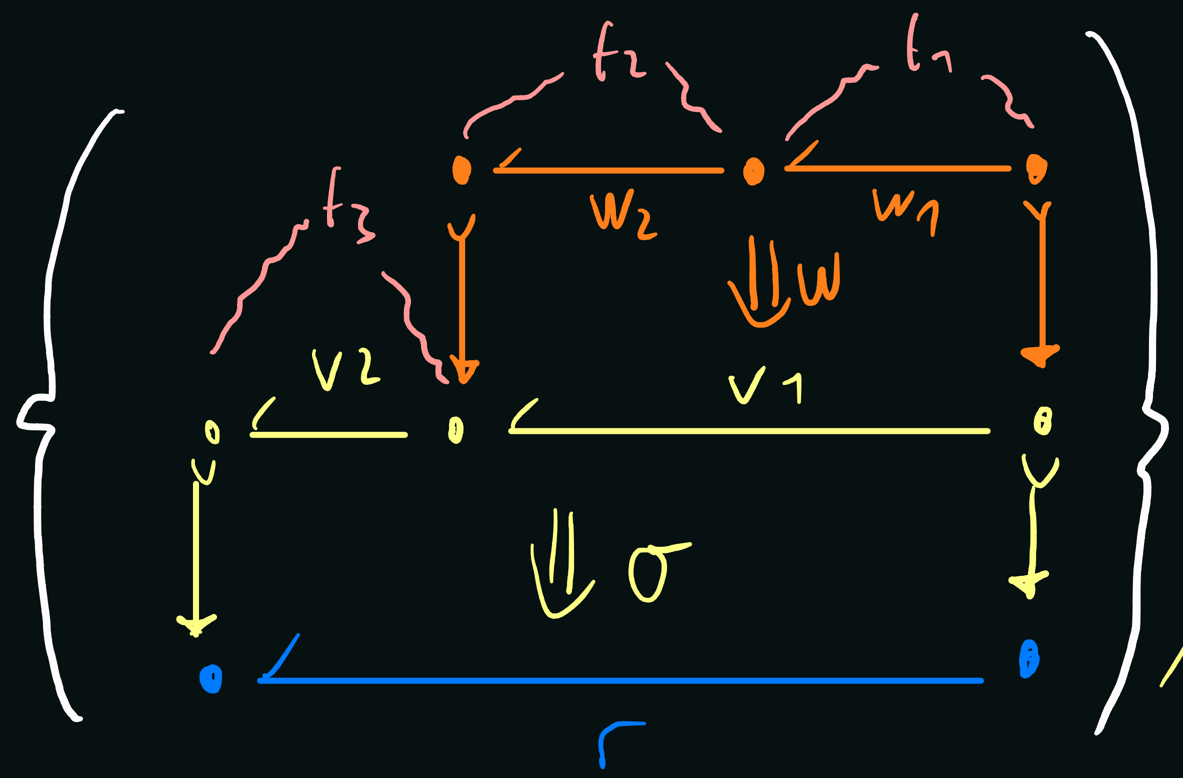
$: \exists$

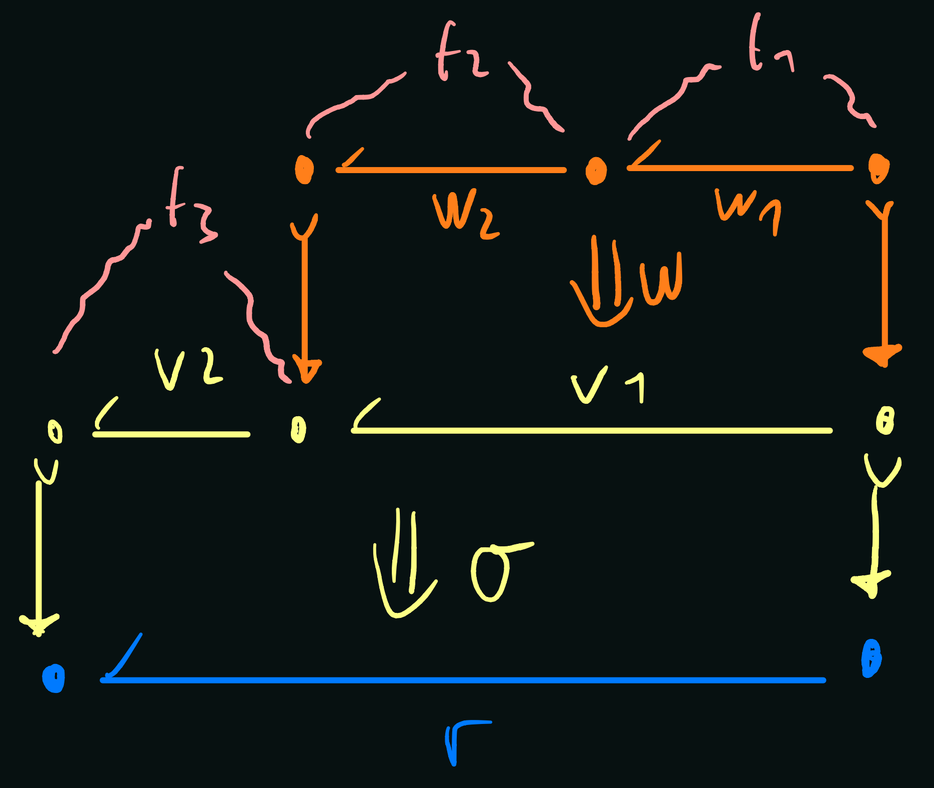


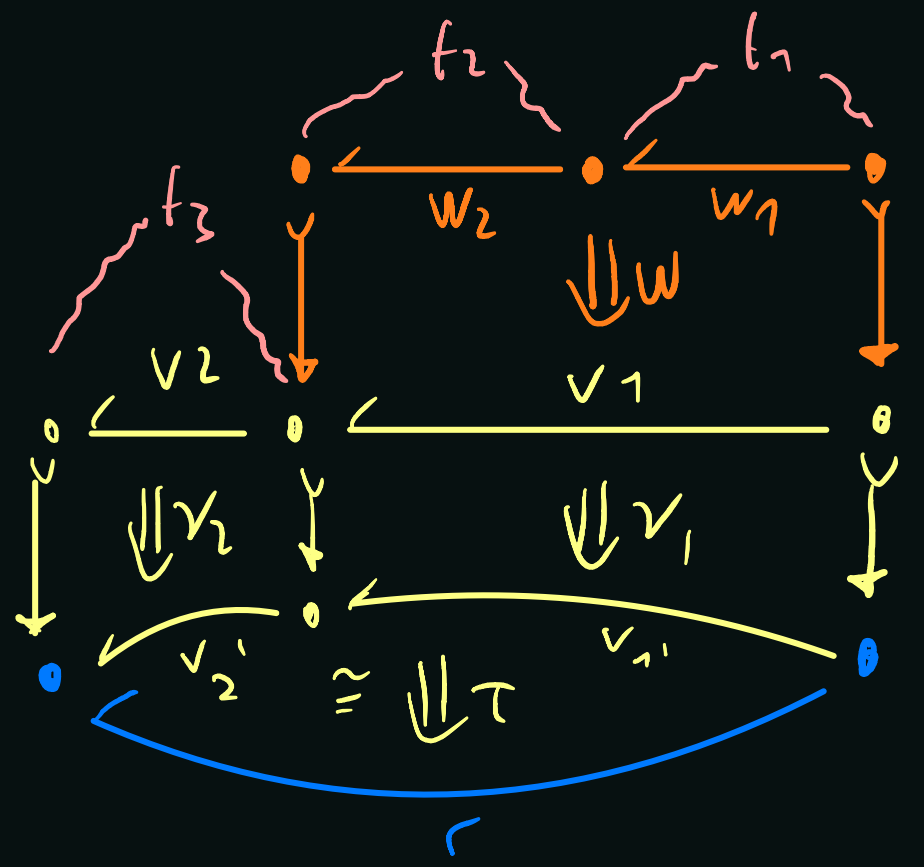
18 KEY RESULT: WEAK ASSOCIATIVITY OF $*$

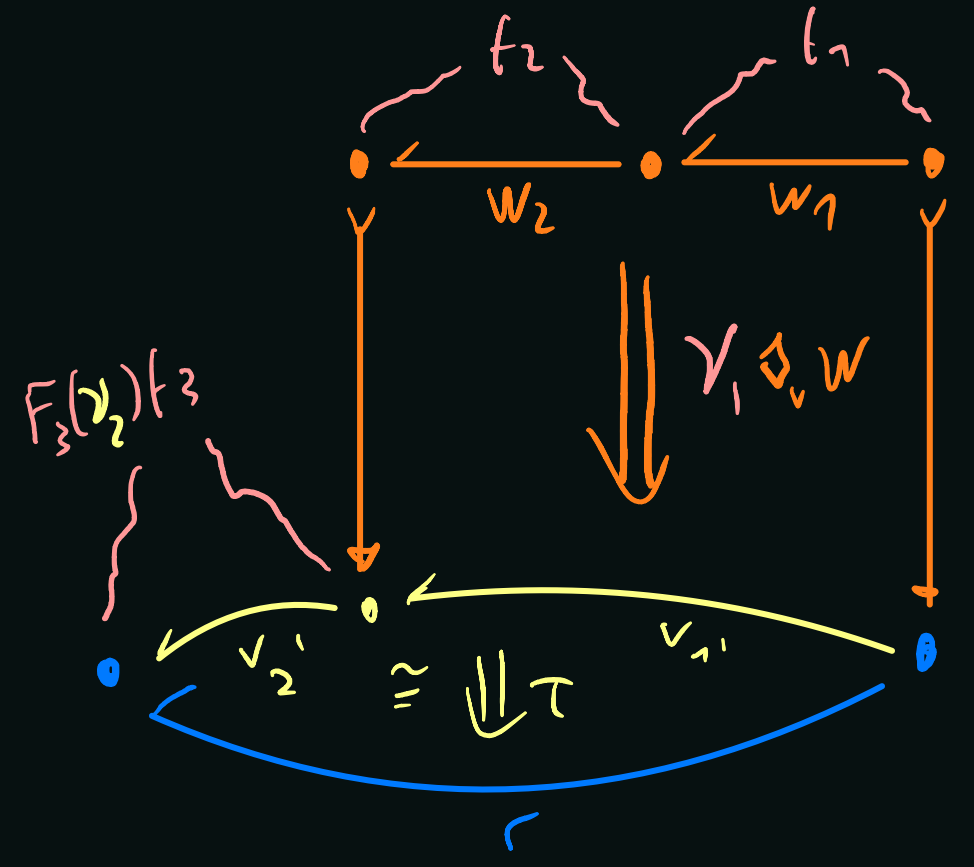
$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \to \mathbb{D}_0,\ r \in \mathbb{D}_1 : \quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$

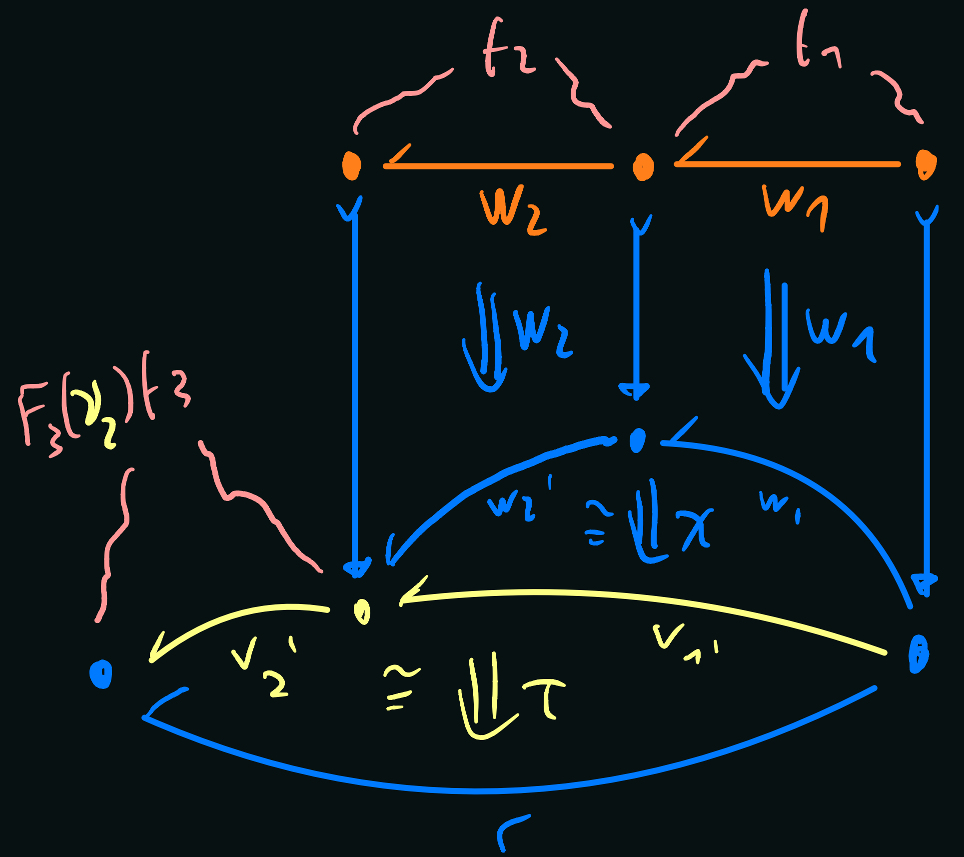
PROOF (SKETCH):

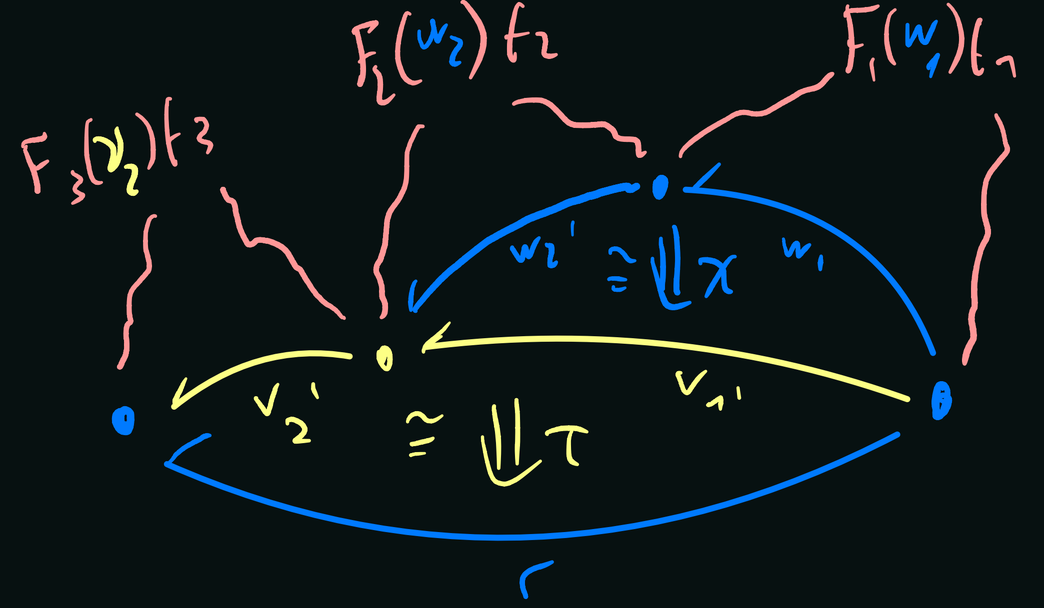
$F_3 * (F_2 * F_1)(r) = \{$  $\} /\!\sim_V /\!\sim_W$

$\forall$ 

:] 

$\sim_V$ 

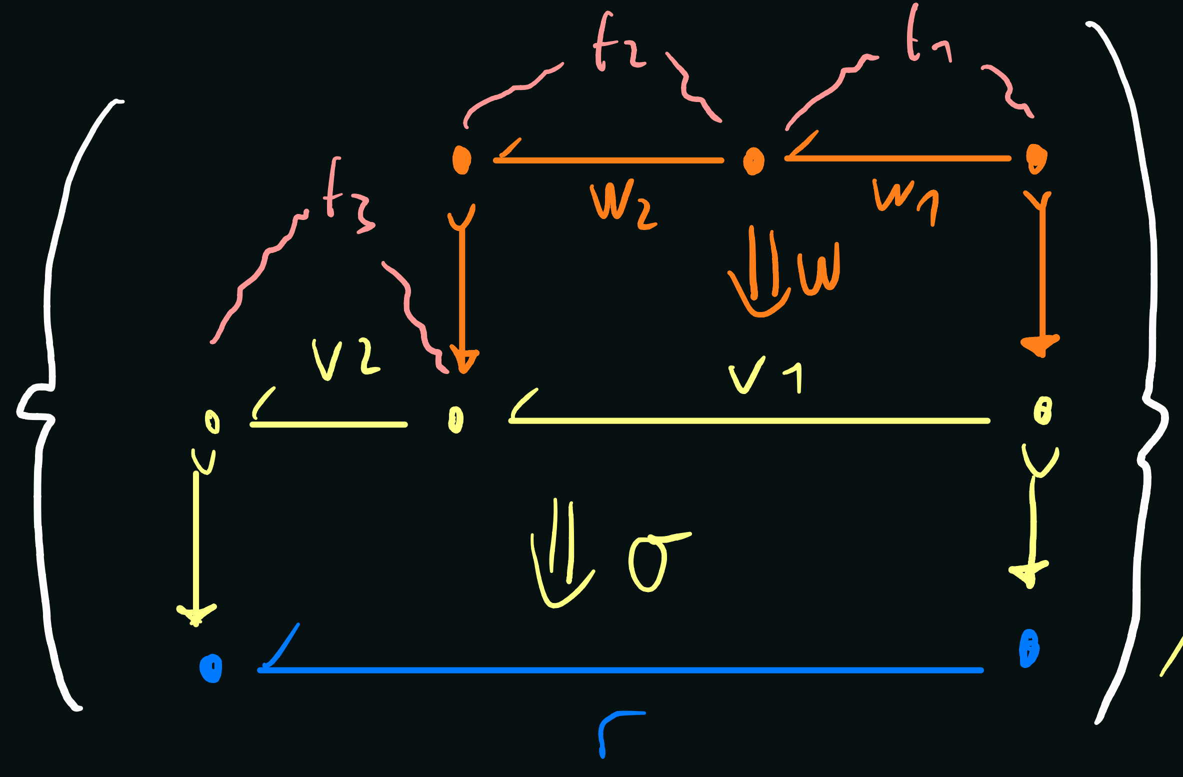
:] 

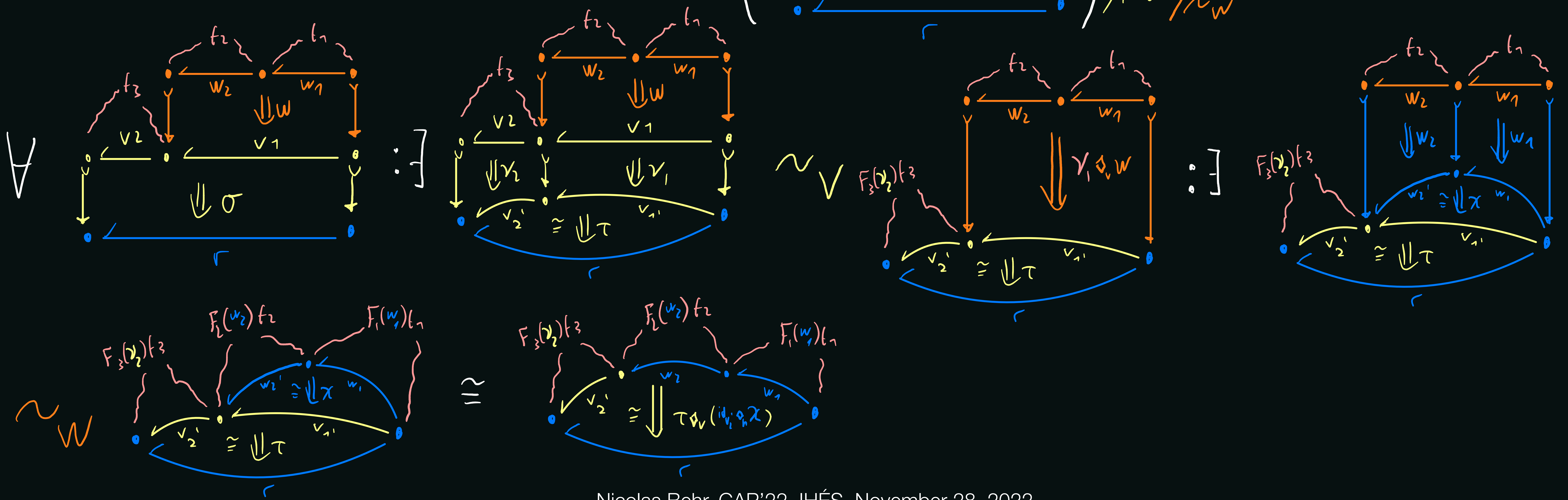
$\sim_W$ 

(18) KEY RESULT: WEAK ASSOCIATIVITY OF $*$

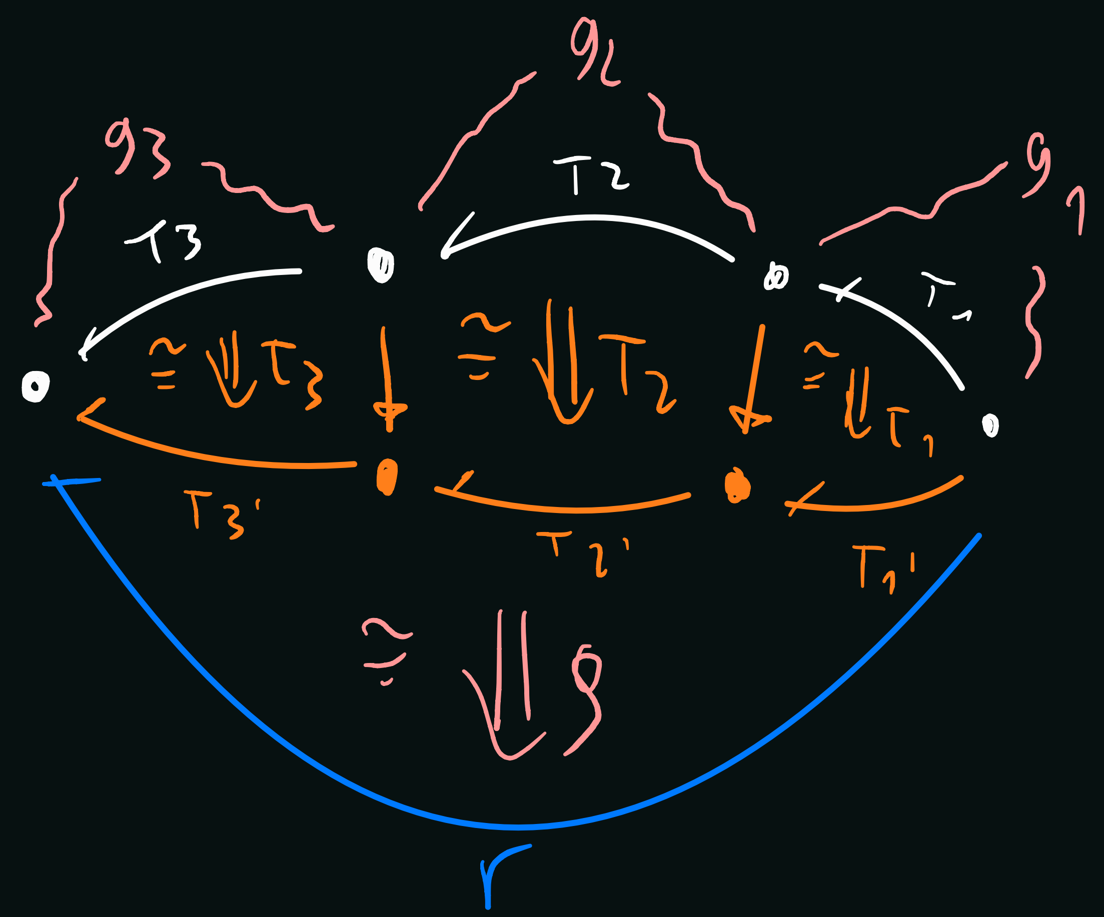
$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \mathbb{D}_0\,,\ r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

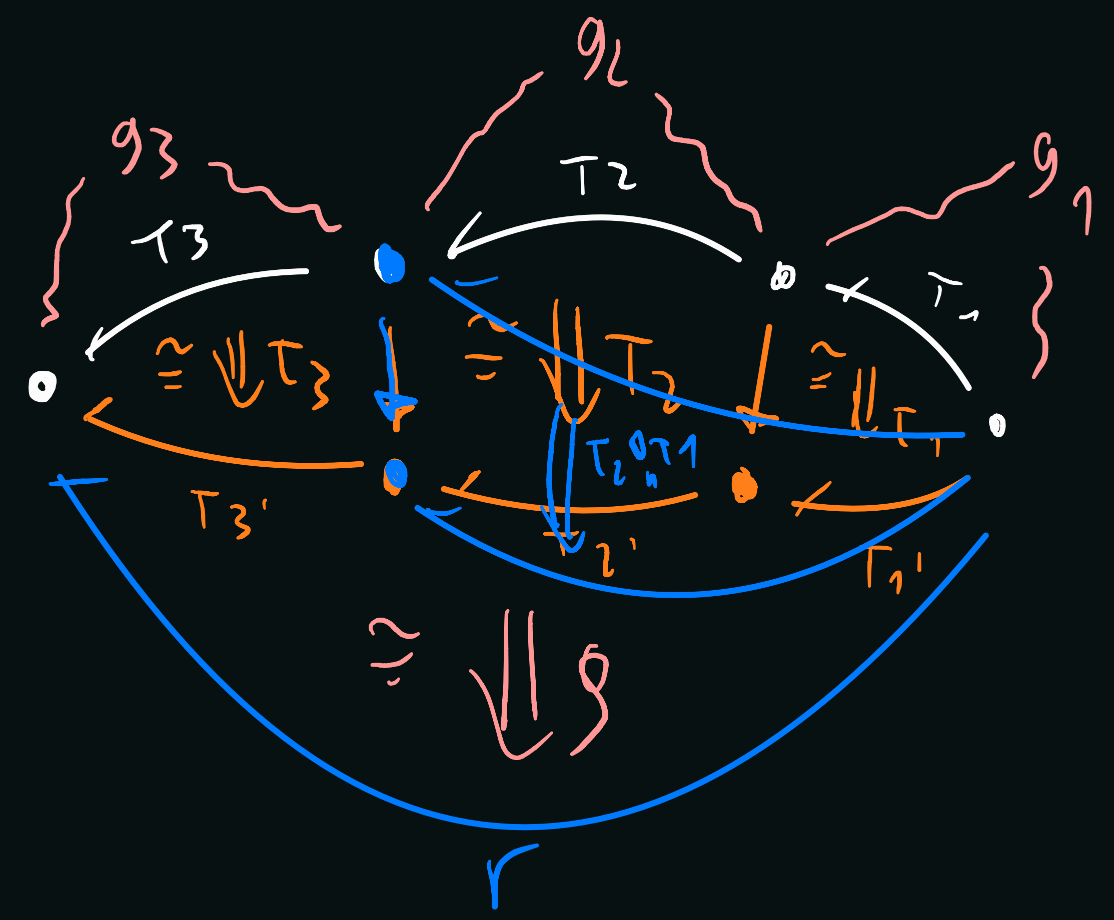
PROOF (SKETCH):

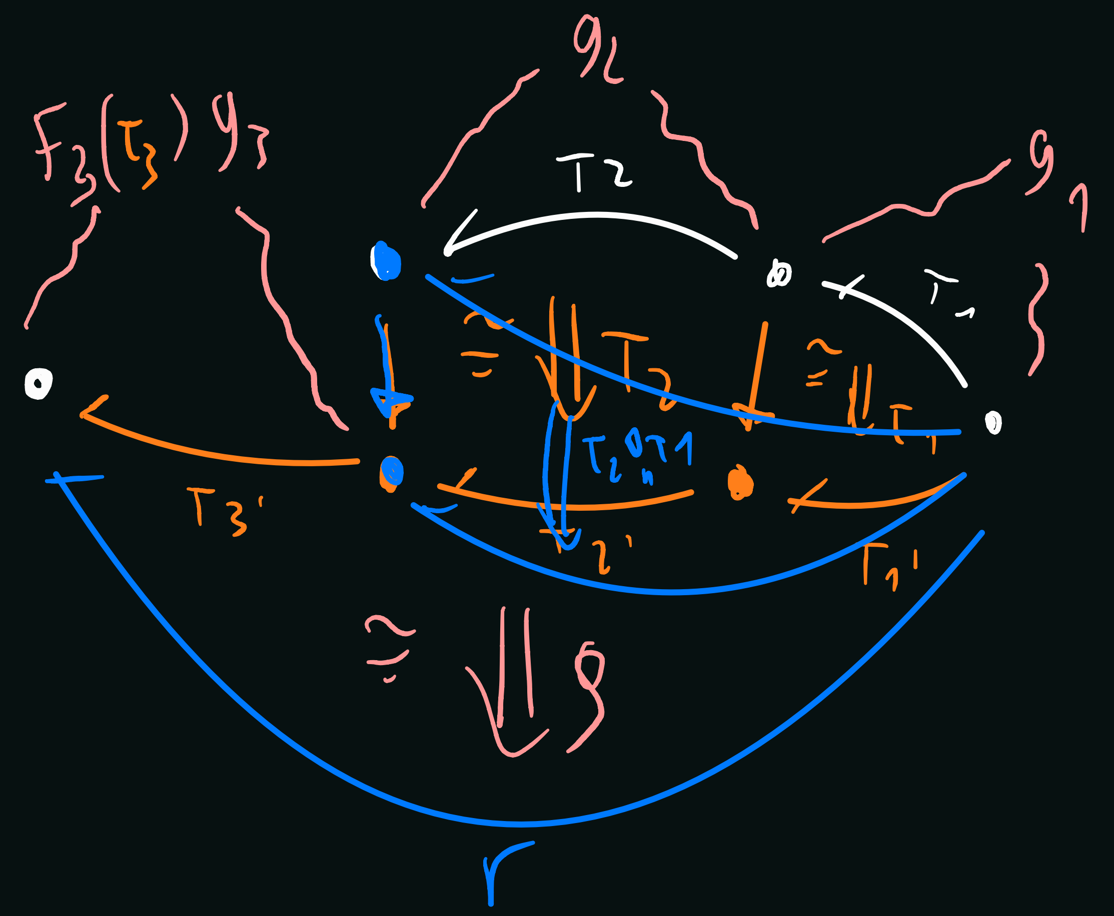
$F_3 * (F_2 * F_1)(r) =$  $/\sim_V /\sim_W$



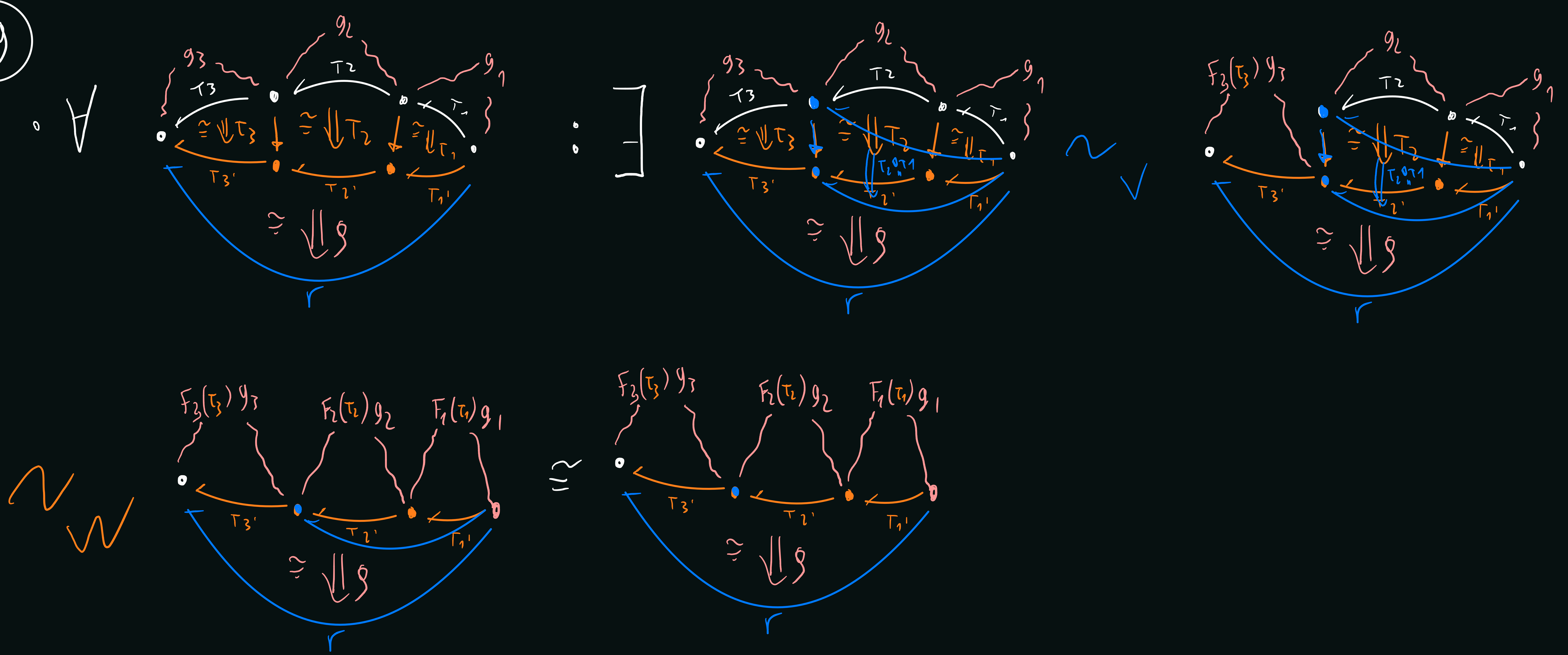
19

- $\forall$ 

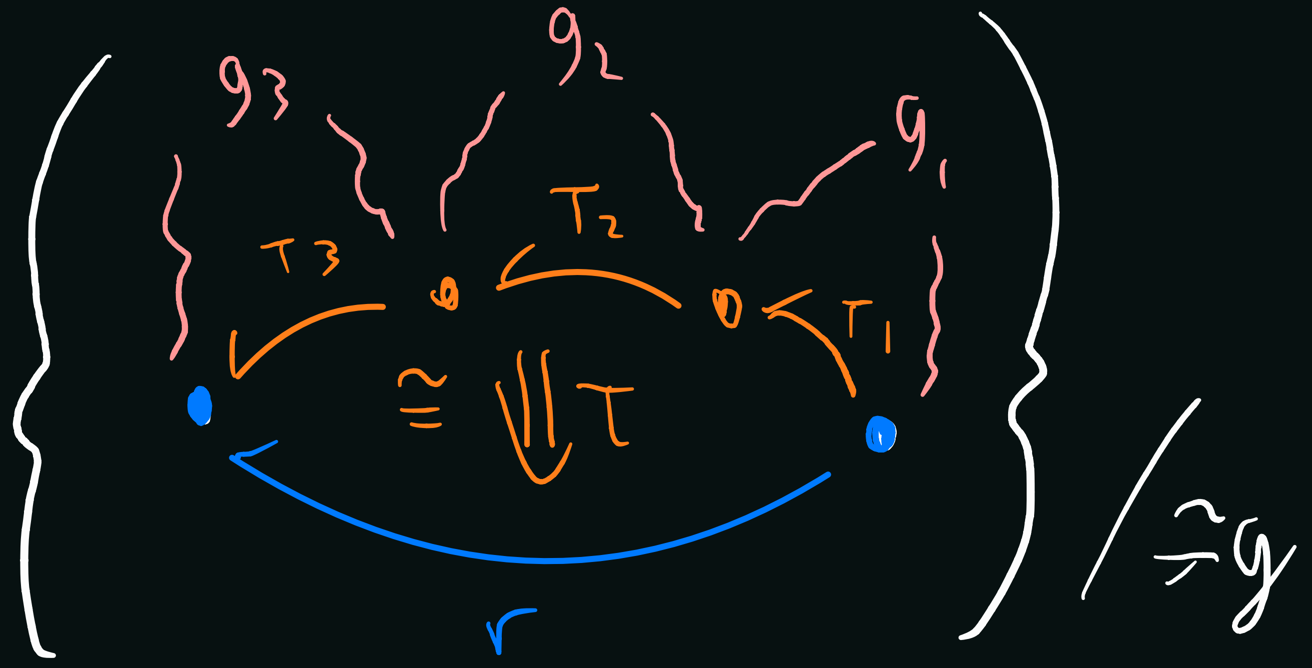
- $\exists$ 

$\sim$ $\checkmark$ 

19



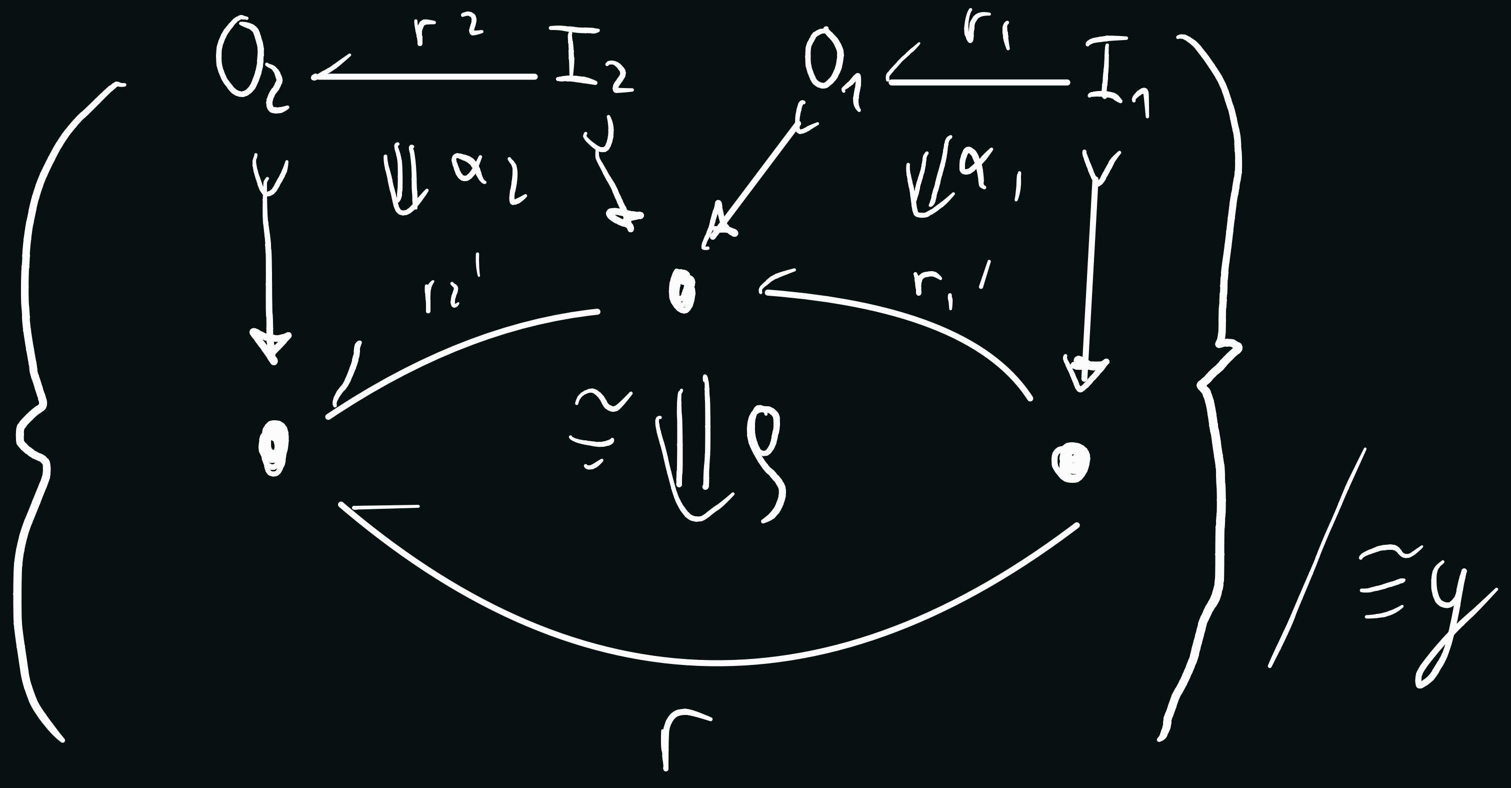
$\hookrightarrow F_3 * (F_2 * F_1)(r) \cong$



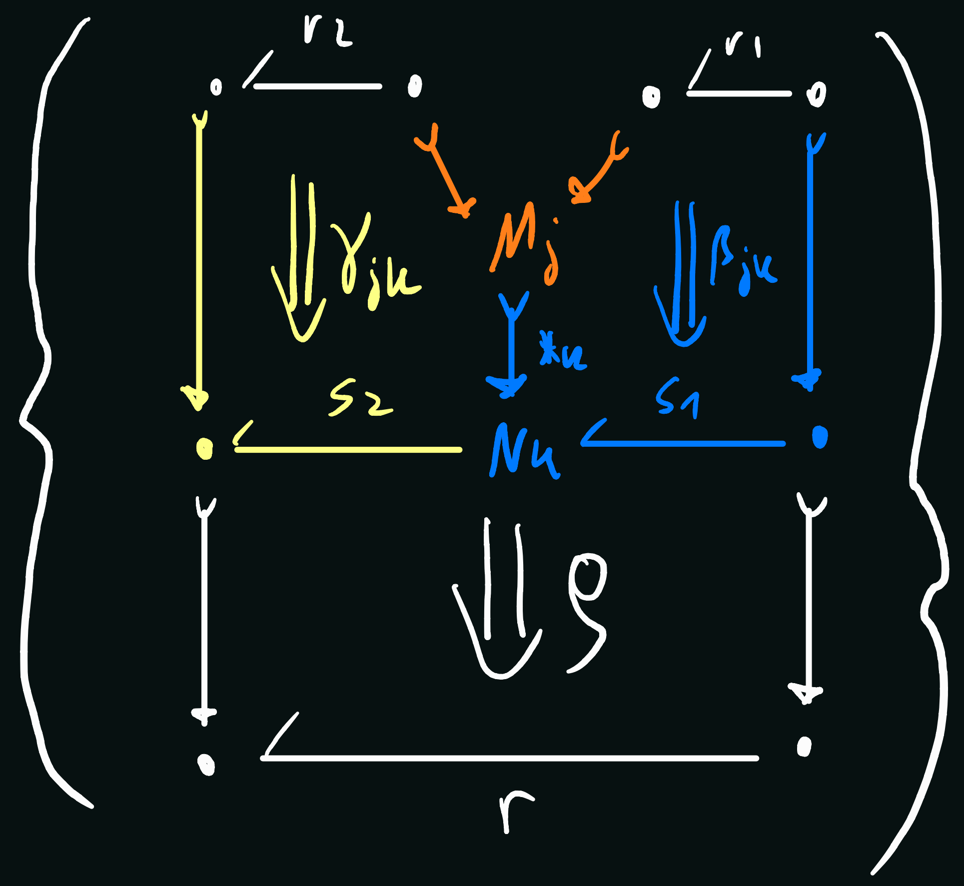
$= (F_3 * F_2 * F_3)(r)$ $\square$

20 FINAL INGREDIENT: CATEGORIFICATION OF RULE ALGEBRA

CLAIM: $(\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong$



$$\cong \coprod_{\substack{(O_2 \rightarrowtail M_j \leftarrowtail I_1) \\ \in ms(O_2, I_1)}} \; \coprod_{\beta_{jk} \in T^*(O_1 \rightarrowtail M_j)} \; \coprod_{\gamma_{jk} \in S^*(I_2 \rightarrowtail M_j \xrightarrow{*_k} N_k)}$$



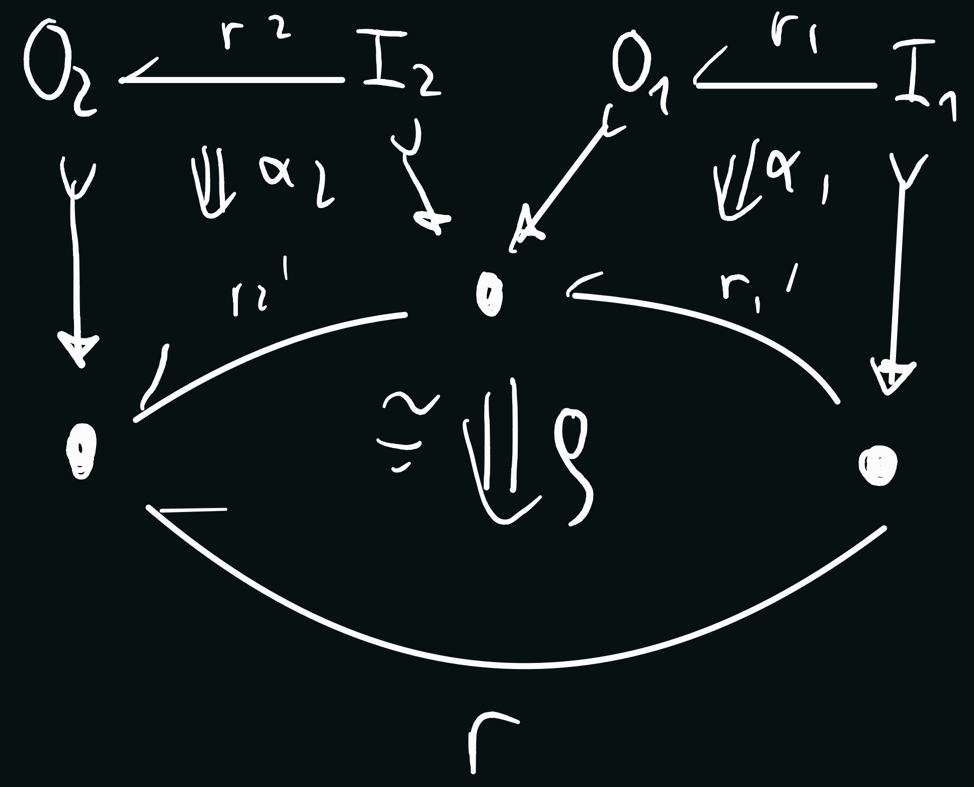
$$=: \hat{\Delta}_{r_2 \circ r_1}(r)$$

21 PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

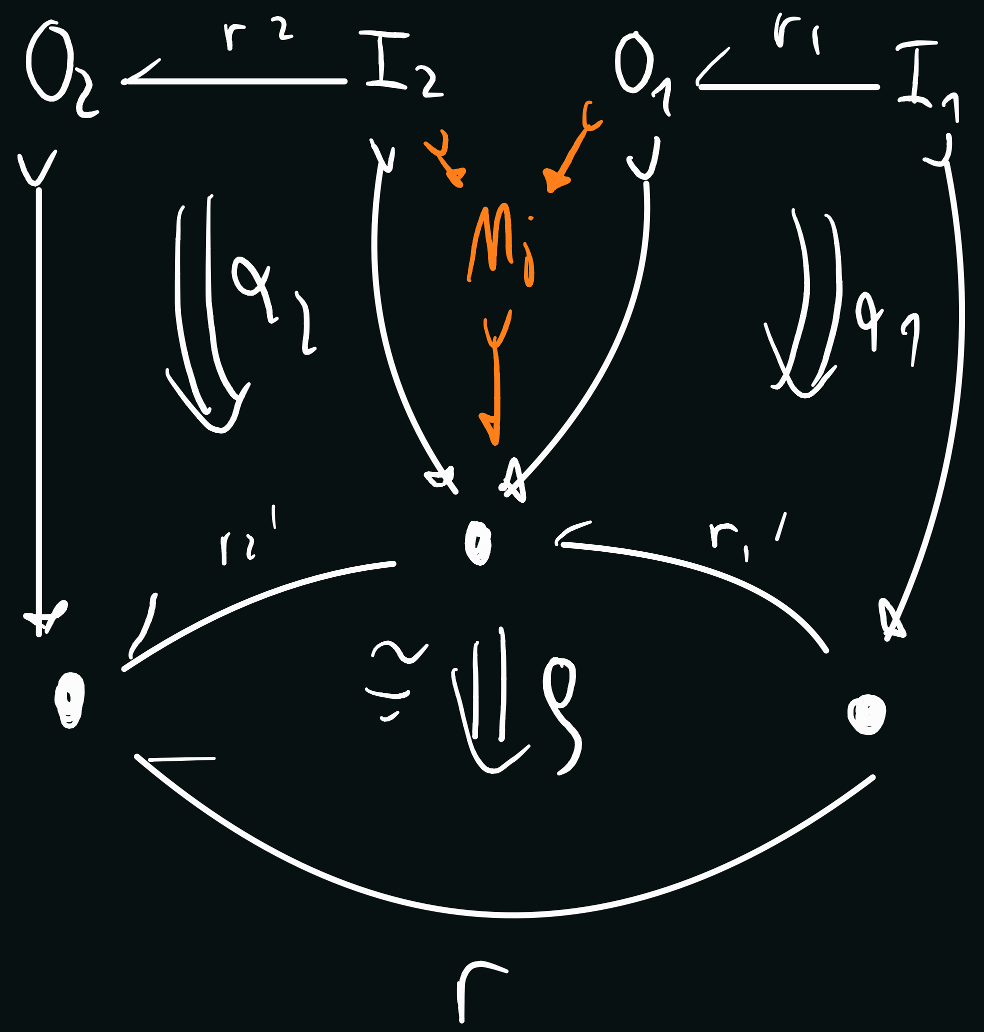
$$\forall \quad \begin{tikzcd}
O_2 \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha_2"] & I_2 \arrow[l, "r_2"'] \arrow[dr] & & O_1 \arrow[dl] \arrow[dr, phantom, "\Downarrow\alpha_1"] & I_1 \arrow[l, "r_1"'] \arrow[d] \\
\bullet & & \bullet \arrow[ll, "r_2'"'] \arrow[d, phantom, "\cong\Downarrow\rho"] & & \bullet \arrow[ll, "r_1'"'] \arrow[llll, bend left, "r"] \\
& & {} & &
\end{tikzcd} \quad : \quad \exists! \quad \begin{tikzcd}
O_2 \arrow[dd] \arrow[ddr, phantom, "\Downarrow\alpha_2"] & I_2 \arrow[l, "r_2"'] \arrow[dr] \arrow[ddr, bend right] & & O_1 \arrow[dl] \arrow[ddl, bend left] \arrow[ddr, phantom, "\Downarrow\alpha_1"] & I_1 \arrow[l, "r_1"'] \arrow[dd] \\
& & m_i \arrow[d] & & \\
\bullet & & \bullet \arrow[ll, "r_2'"'] \arrow[d, phantom, "\cong\Downarrow\rho"] & & \bullet \arrow[ll, "r_1'"'] \arrow[llll, bend left, "r"] \\
& & {} & &
\end{tikzcd}$$

(21) PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

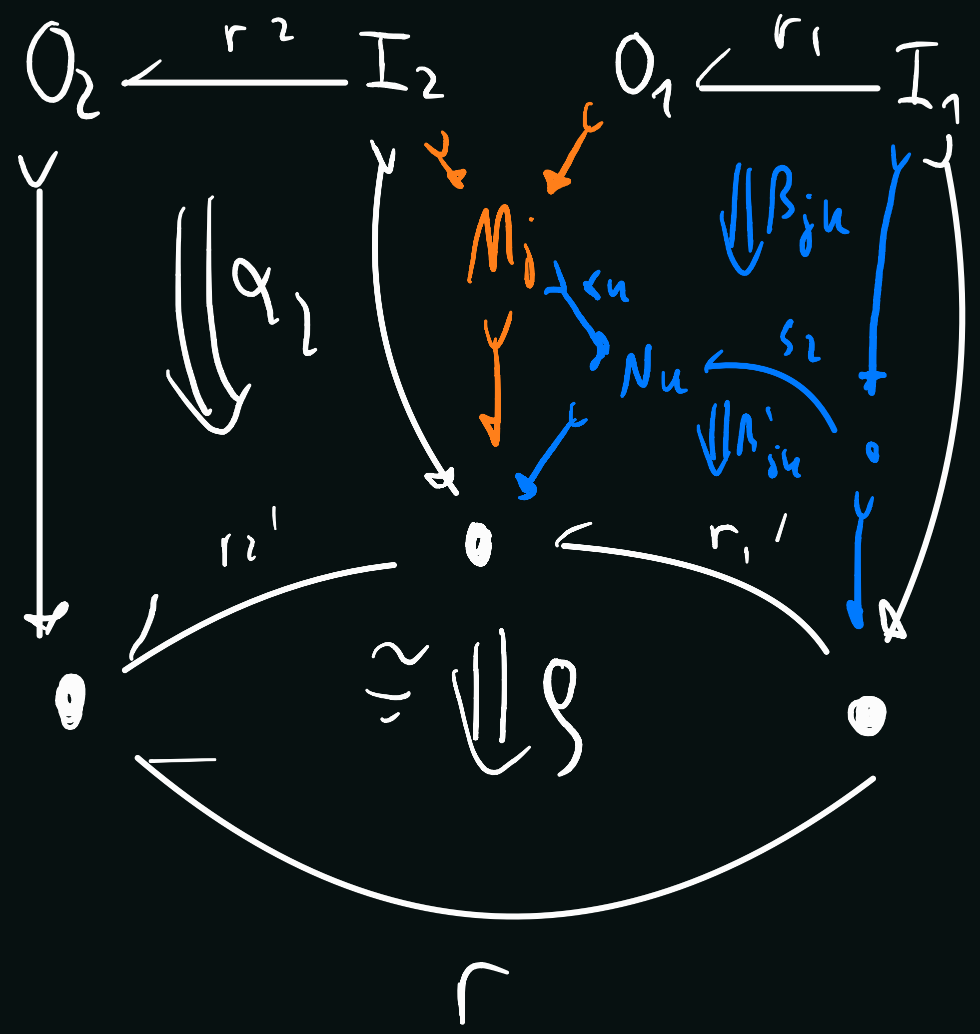
$\forall$



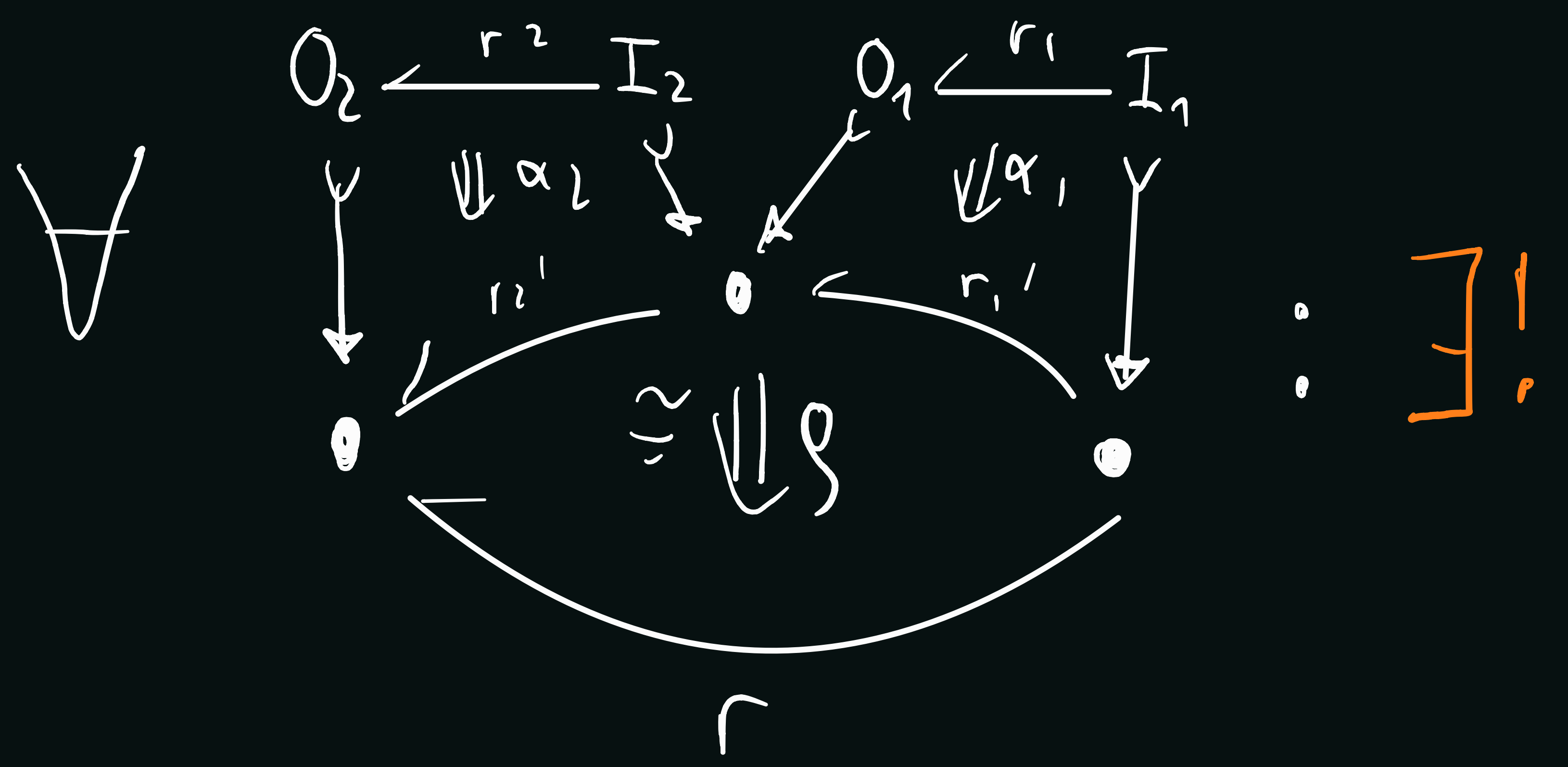
: $\exists!$

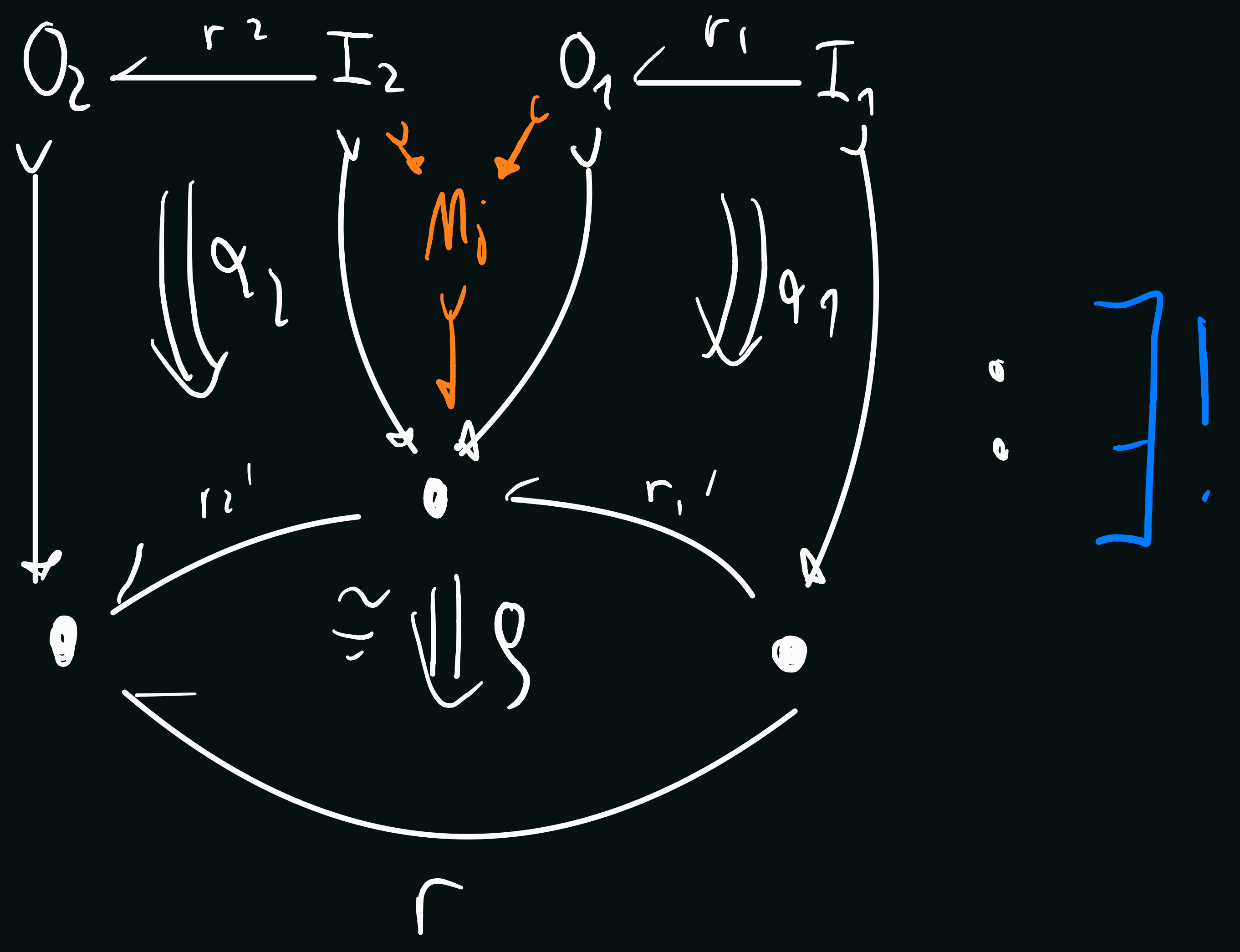


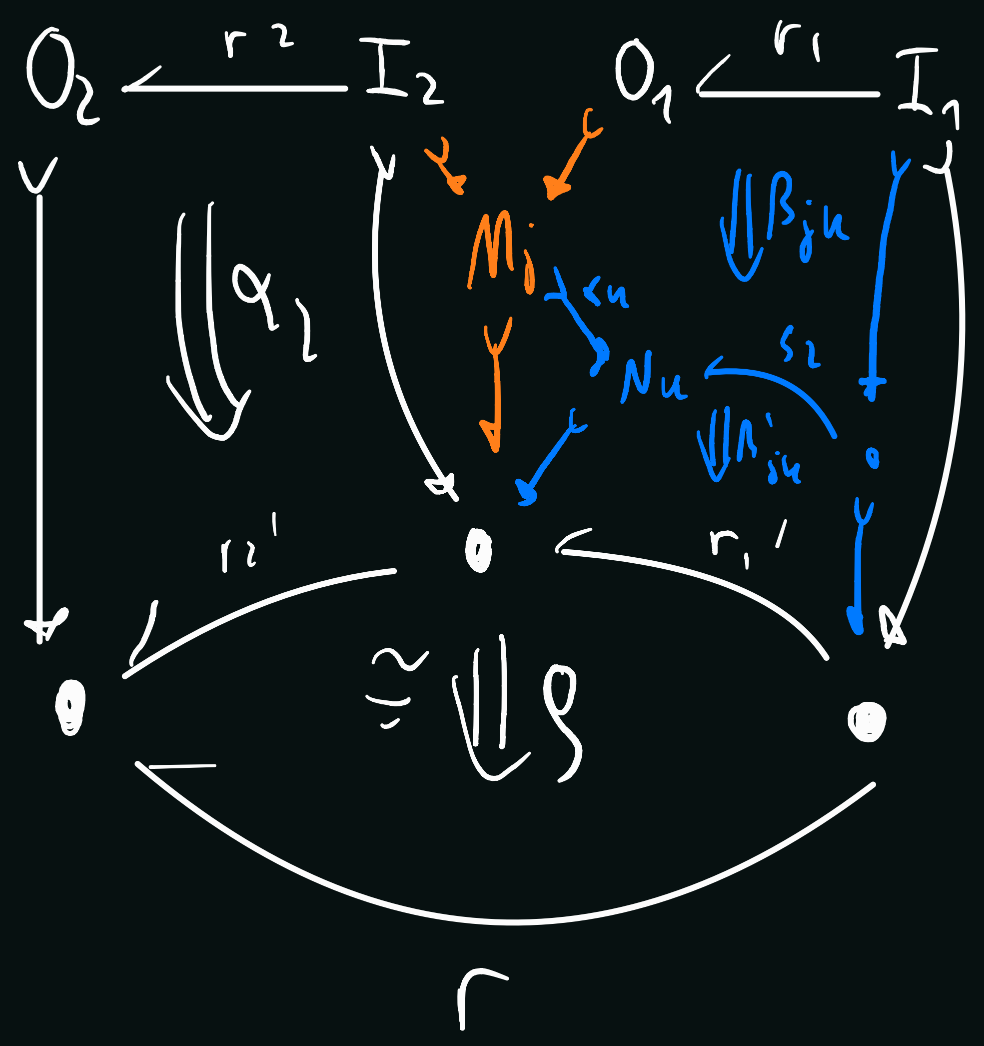
: $\exists!$

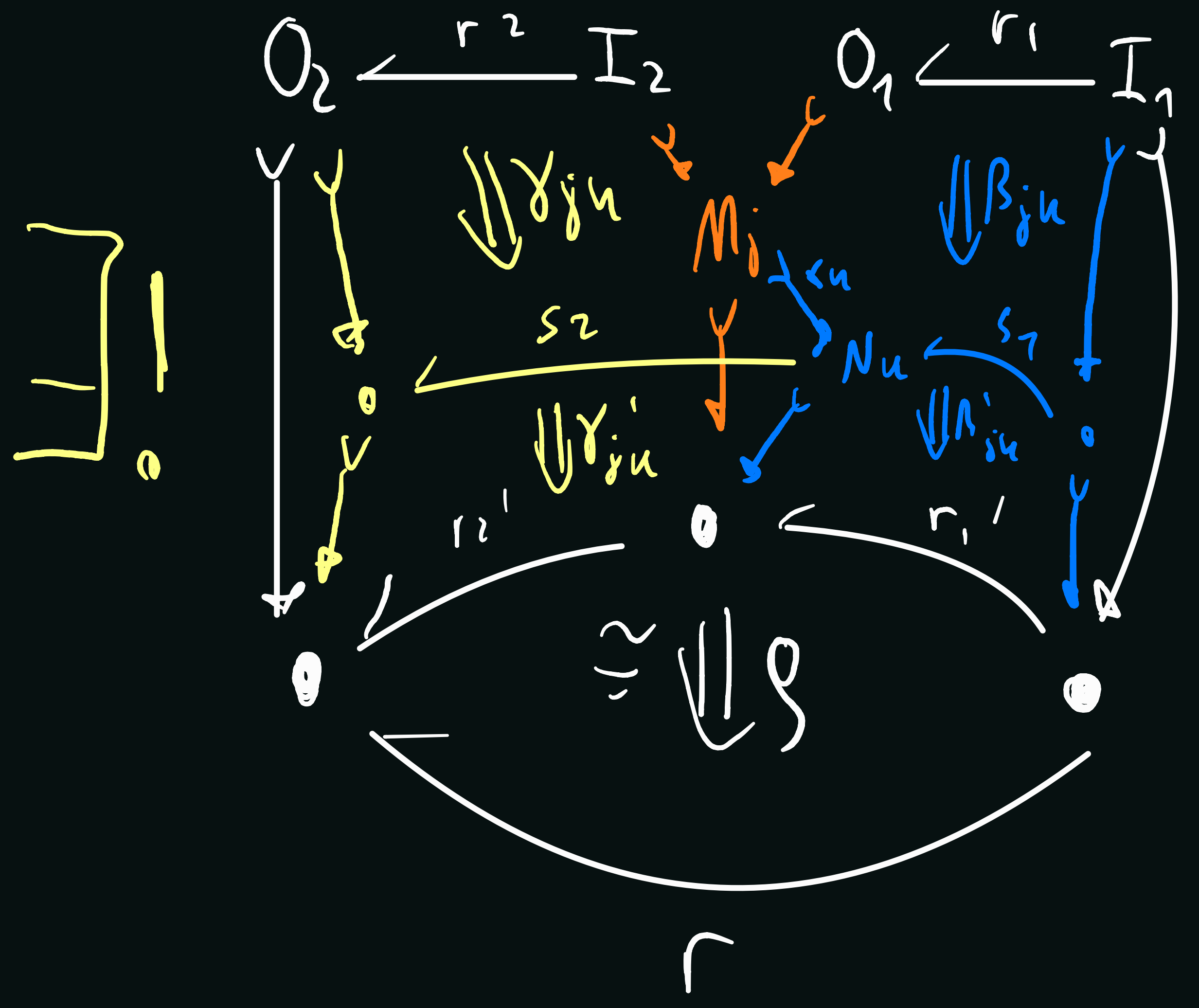


21 PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:



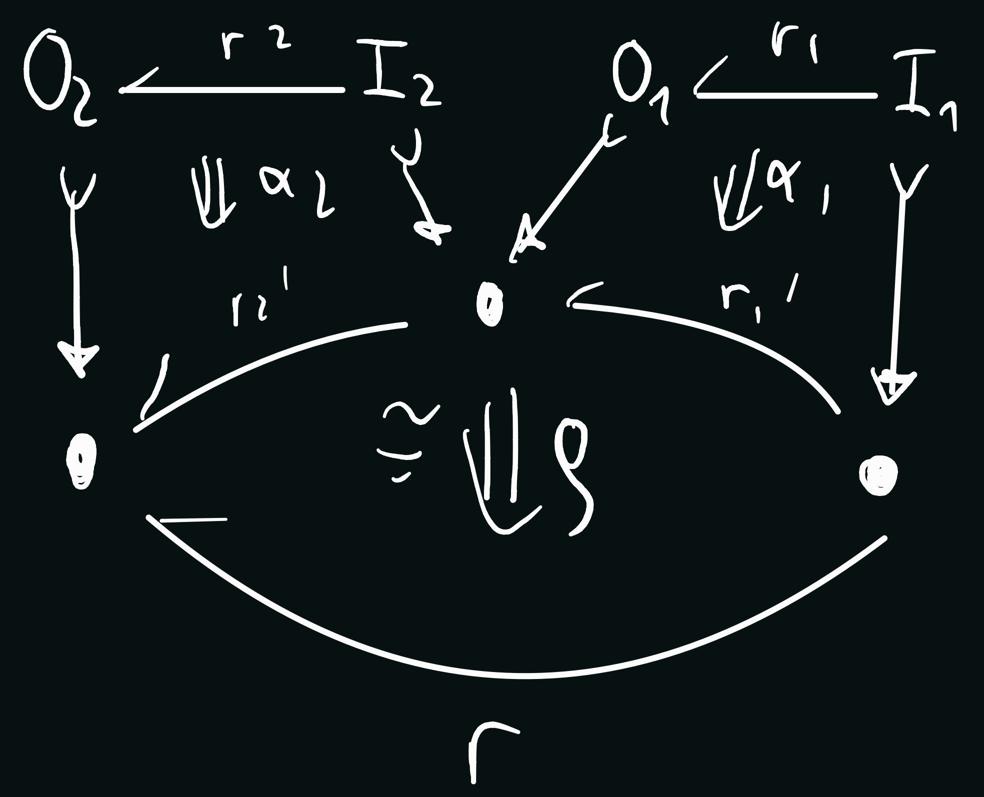




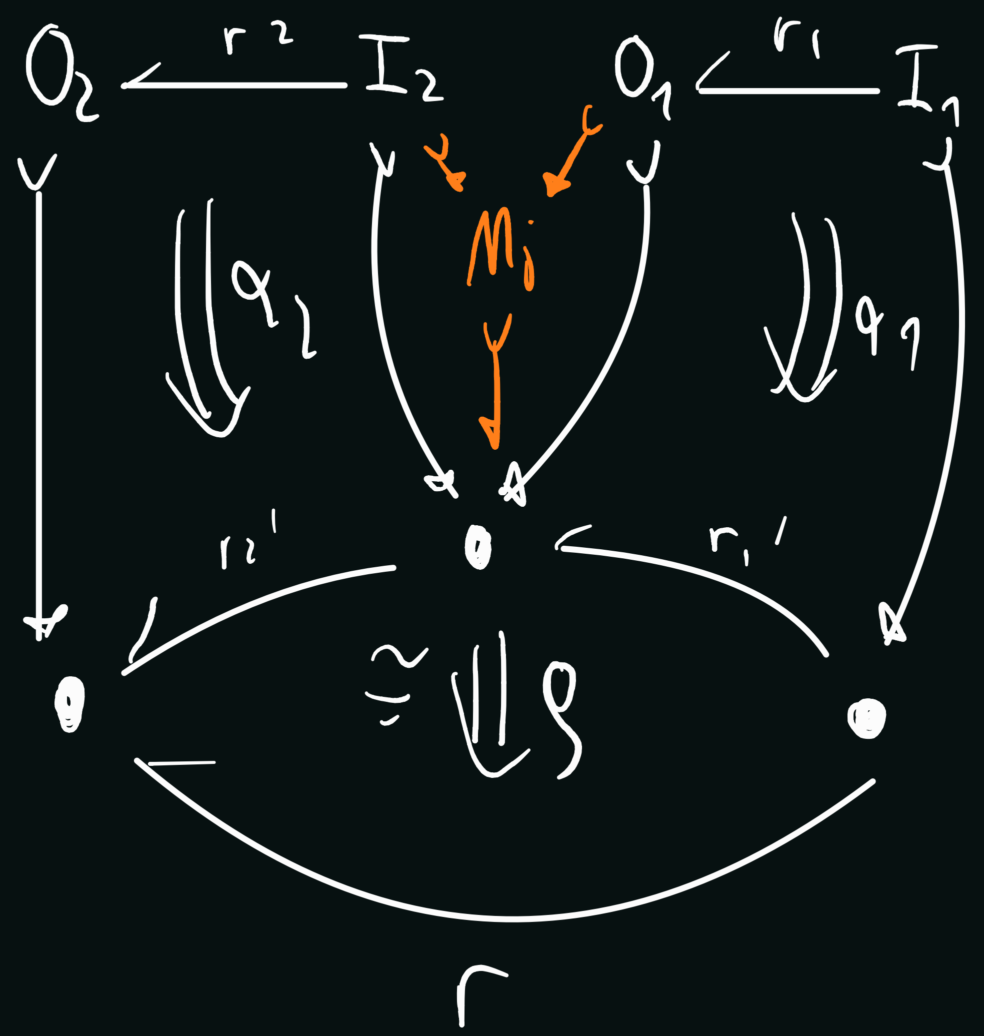


(21) PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

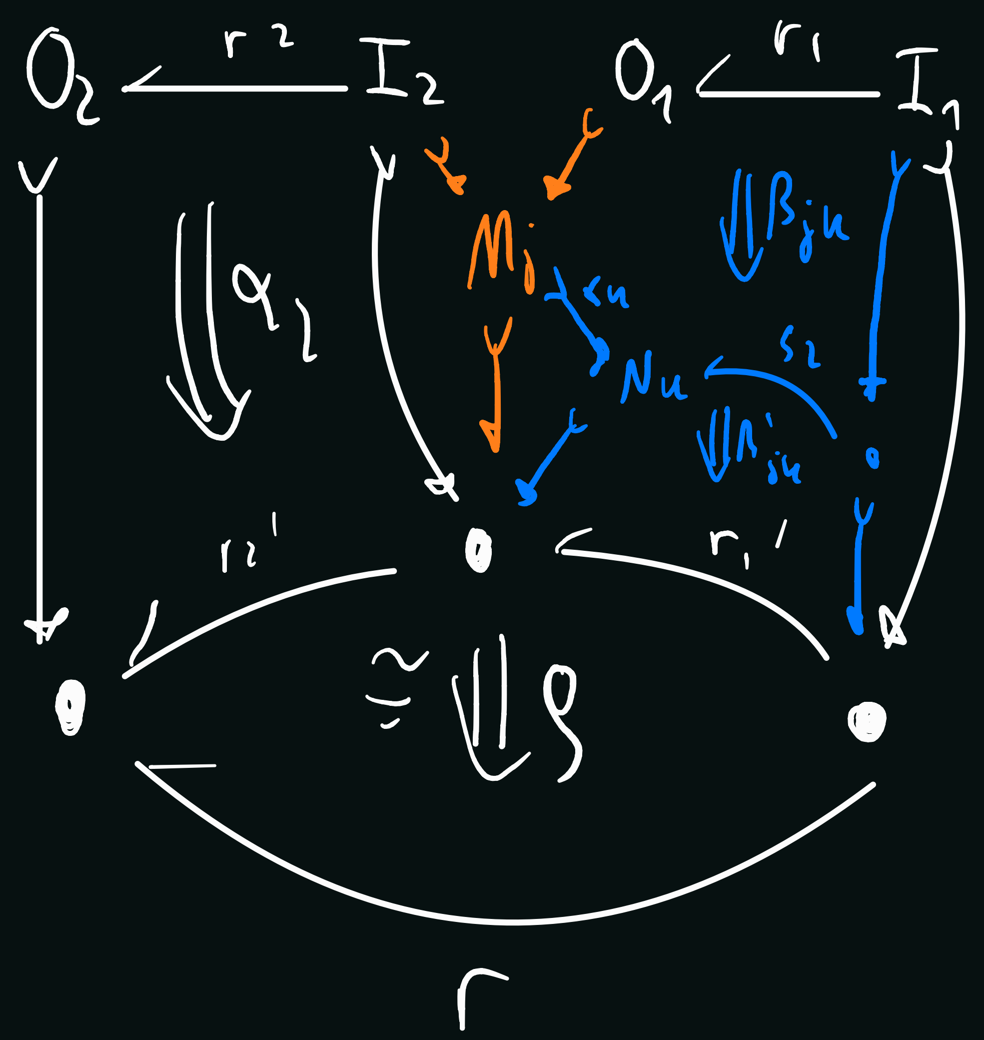
$\forall$



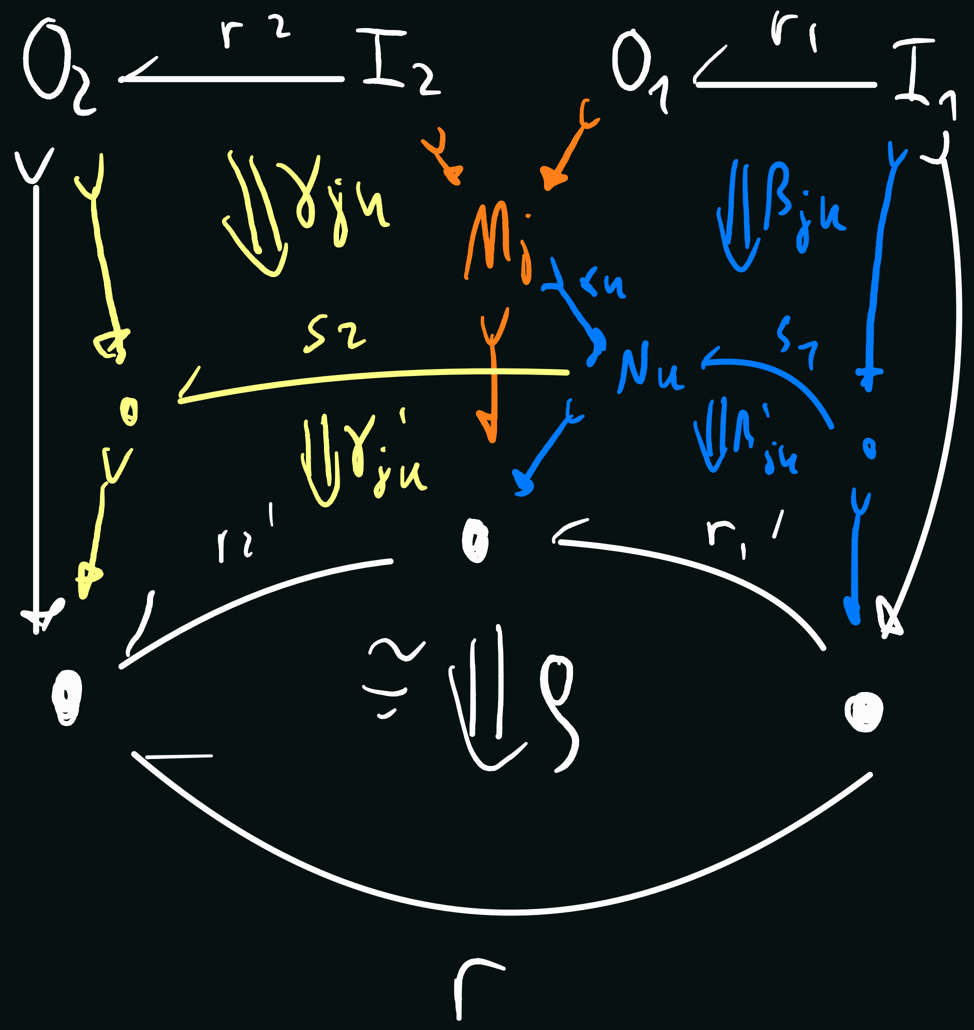
$: \exists!$



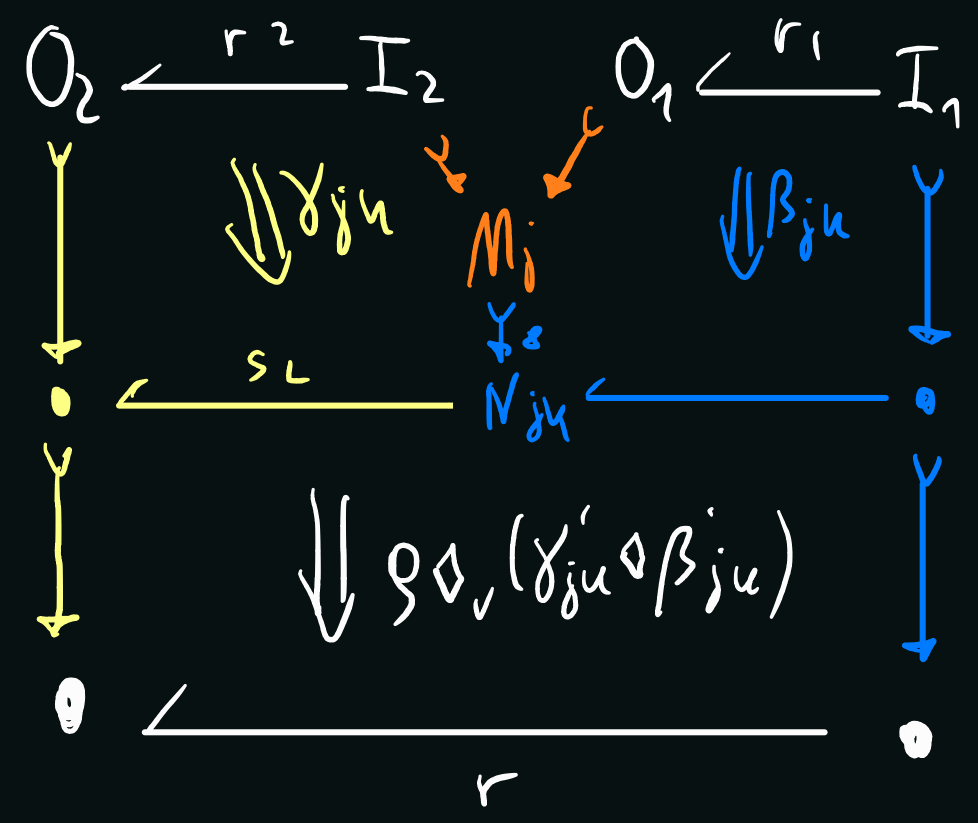
$: \exists!$



$\exists!$



$\cong$

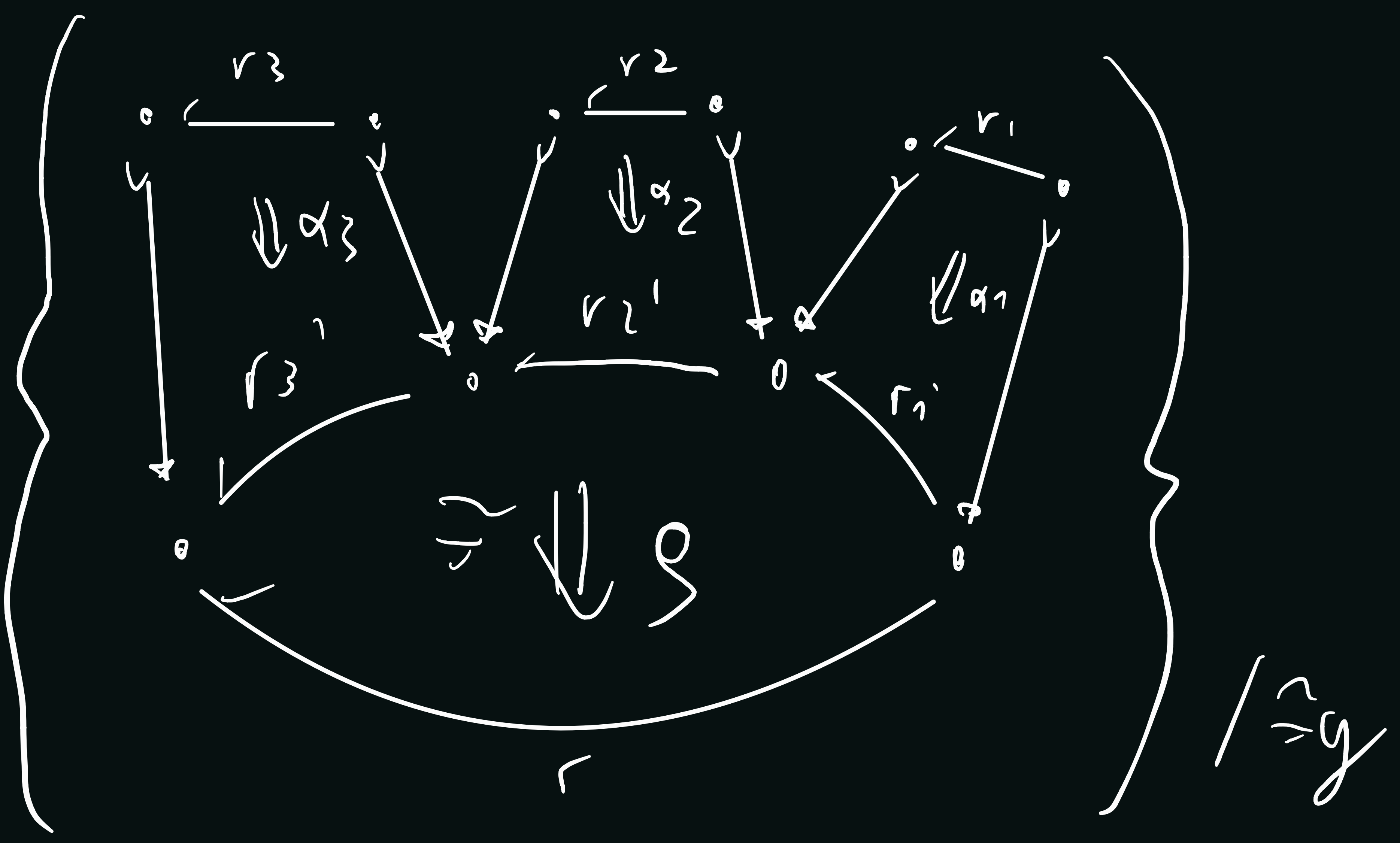


(21) PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

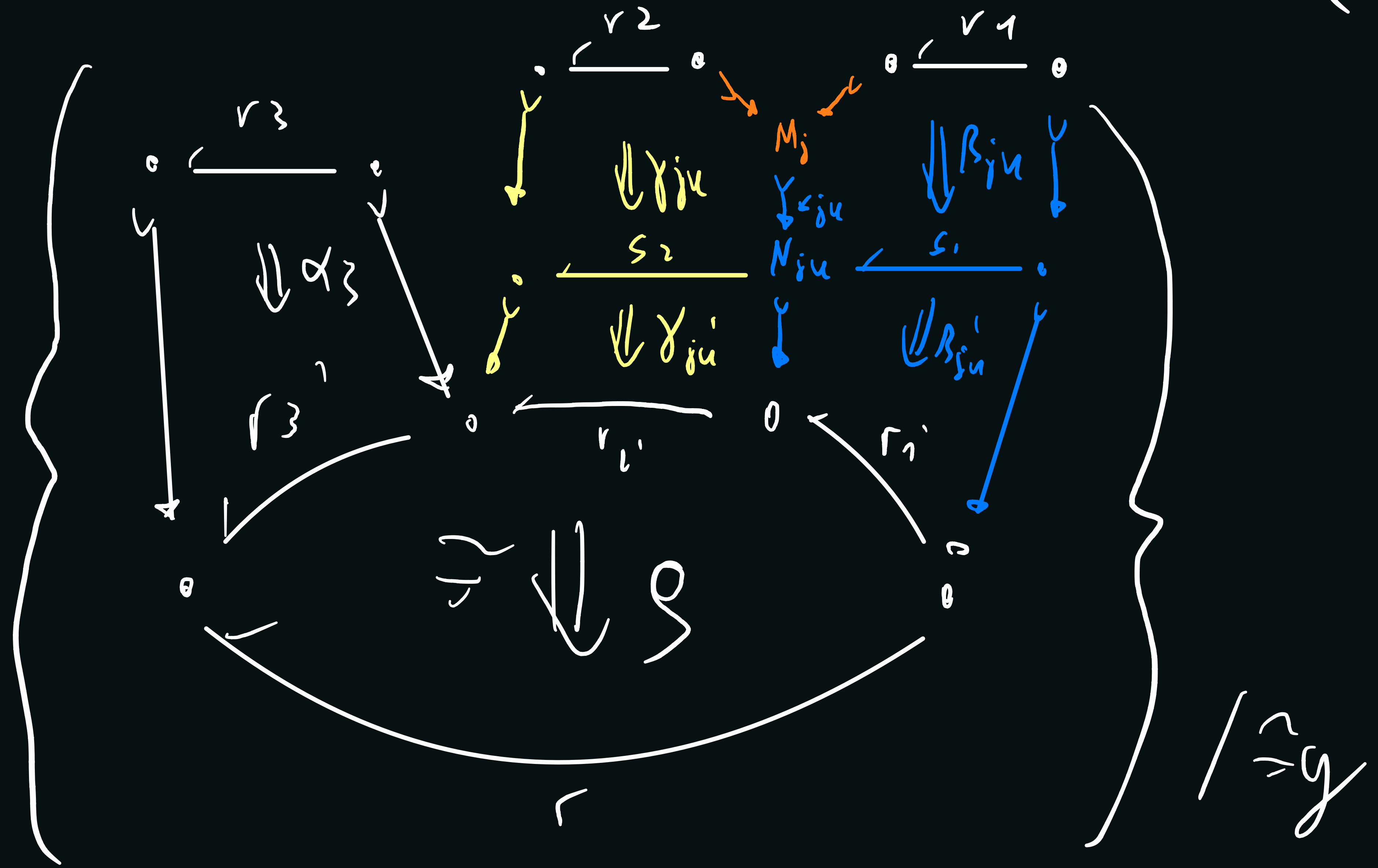
22 CLAIM: $\hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r) \cong \hat{\Delta}_{(r_3 \odot r_2) \odot r_1}$

PROOF (SKETCH):

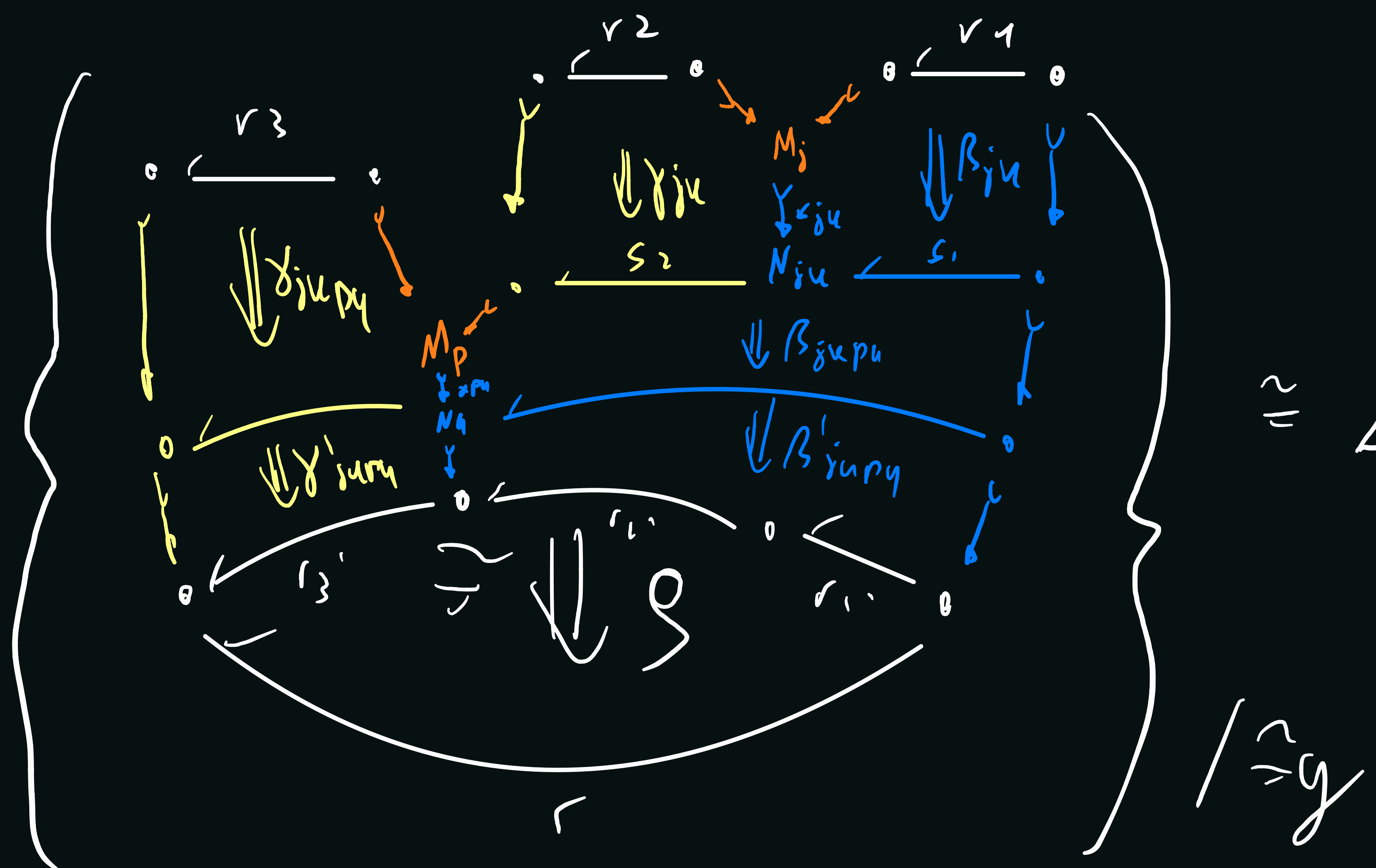
$$\left(\hat{\Delta}_{r_3} * \hat{\Delta}_{r_2} * \hat{\Delta}_{r_1}\right)(r) \cong$$



$\cong$



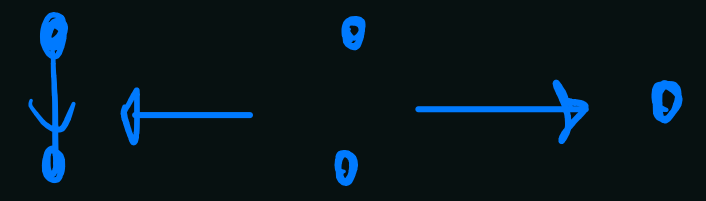
$\cong$

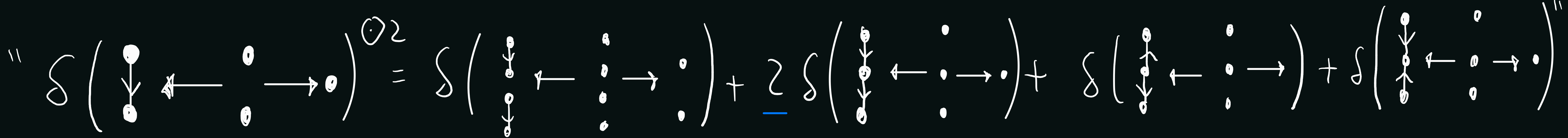


$\cong \hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r)$

$\square$

23 EXAMPLE: A.V. KISELEV'S CAP'19 RULE:



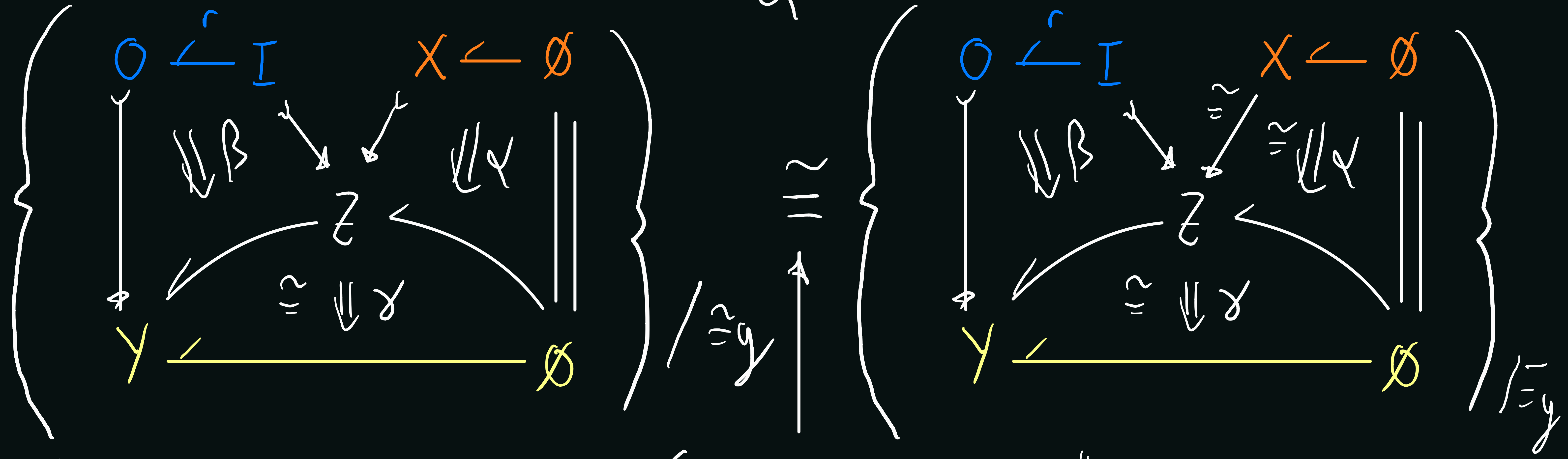


5 CONTRIBUTIONS TO $\hat{A}_{r_2 \circ r_1}$:

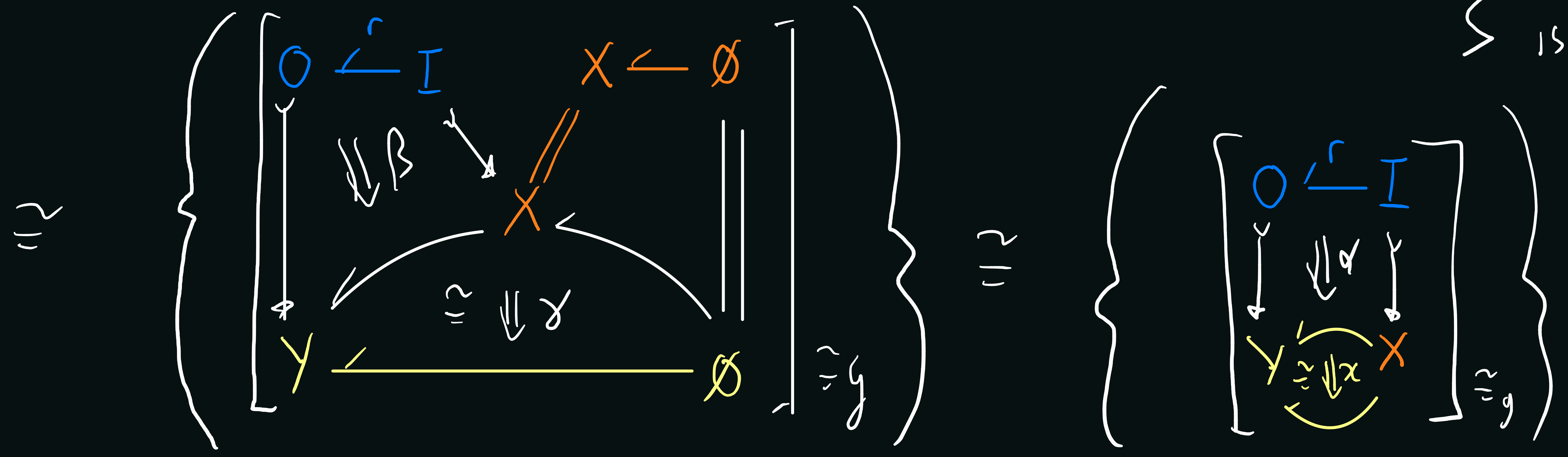
㉔ Counting rewriting sequences

- "$\rho(S(r))|X\rangle = \rho(S(r))\rho(S(X\leftarrow\emptyset))|\emptyset\rangle = \sum_{\alpha} \rho(S(r_\alpha(X)\leftarrow\emptyset))|\emptyset\rangle$"

$(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset})(Y\leftarrow\emptyset) \cong$



S is a MOPF ⇒ "lifts" isos!



(25) OUTLOOK

- "# OF WAYS TO REWRITE X VIA APPLYING RULE r":

$$\int^{Y\in\mathbb{D}_0} (\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset})(Y\leftarrow\emptyset) \cong \coprod_{\alpha\in\mathbb{D}_1} \left\{ \begin{tikzcd} O \arrow[d, tail] & I \arrow[l, "r"'] \arrow[d, tail] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \right\} \Big/ \sim_Y$$

WHERE: $\forall$

$$\begin{tikzcd} Y \arrow[d] \arrow[dr, phantom, "\cong\ \Downarrow\gamma"] & \\ Y' & X \arrow[ul, bend right, "s"'] \arrow[l, bend left, "s'"] \end{tikzcd} : \quad \begin{tikzcd} O \arrow[d, tail] & I \arrow[l, "r"'] \arrow[d, tail] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \sim_Y \begin{tikzcd} O \arrow[d, tail] & I \arrow[l, "r"'] \arrow[d, tail] \arrow[dl, phantom, "\Downarrow\gamma\diamond_v\alpha"] \\ Y' & X \arrow[l, "s'"] \end{tikzcd}$$

- STARTING MARCH 2023: ANR PROJECT CoREACT

Coq-based Rewriting: towards Executable Applied Category Theory

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