Strange gradings and elimination of generators.

From combinatorics of universal problems to usual applications.

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Part one :

Preamble, generalities and
a combinatorial example.

Which sort of elimination will we consider here ?

$$STRUCT\langle x_1, x_2, \ldots, x_n\rangle \cong NICE\langle x_1, x_2, \ldots, x_n\rangle \diamond STRUCT_1\langle x_1, \ldots, x_{n-1}\rangle \quad (1)$$

where $NICE$ et $STRUCT_1$ stand for algebraic structures generated (sometimes freely) by generators x_i. The diamond symbol being, according to the situation, a tensor product, a semi-direct product or a plain (unique) factorisation. For example, with the symmetric group $\mathfrak{S}_n$ and the pure braid group P_n [1] :

$$\mathfrak{S}_\mathfrak{n} \cong {}^{\mathbb{Z}}/_{n\mathbb{Z}} \diamond \mathfrak{S}_{\mathfrak{n}-1} \quad \text{and} \quad P_n \cong F_{n-1} \diamond P_{n-1}.$$

Here, in the first case, $\diamond$ is only a product and the iterated decomposition helps to construct a basis of $\mathbb{Q}[\mathfrak{S}_\mathfrak{n}]$ adapted to the calculation needs of *Dynkin*'s projector [4]. In the second case we have a semi-direct product (where F_{n-1} is the Free Group with $n-1$ generators.

Rewriting the factors

We recall the pattern with colors

$$STRUCT\langle x_1, x_2, \dots, x_n\rangle \cong NICE\langle x_1, x_2, \dots, x_n\rangle \diamond STRUCT_1\langle x_1, \dots, x_{n-1}\rangle$$

(when $STRUCT_1 = STRUCT$ the process can be iterated).
Let us firstly see the case of two permutable subgroups[a] (where the $\diamond$ is multiplicative), we have $G = G_1 G_2 = G_2 G_1$ (and it is required that $G = G_1.G_2$ be of unique factorisation). Then, at the level of the terms, the rewriting reads

$$g_2 g_1 \longrightarrow l(g_1, g_2) r(g_1, g_2) \qquad (2)$$

and, in the case when $r(g_1, g_2) = g_2$, we have a semidirect product i.e. for every $(g_1, g_2) \in G_1 \times G_2$, $g_2 g_1 {g_2}^{-1} \in G_1$, so that we only need to know the factor $l(g_1, g_2)$.

[a] A common occurrence in solvability.

Categories of this talk.

1. These categories are as follows
   1. **Set** the category of sets.
   2. **Mon**, the category of monoids.
   3. **k** − **Lie**, the category of **k**-Lie algebras.
   4. **Grp**, the category of groups.
   5. **k** − **AAU**, the category of **k**-associative algebras with unit.
2. Functors are as follows

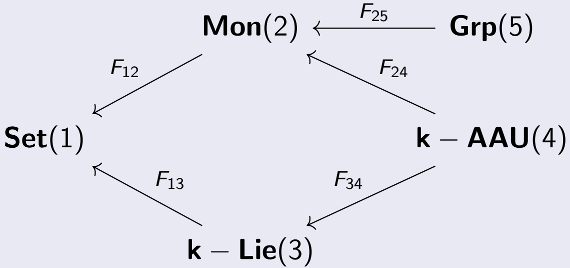


Figure: Rq: Similar lower diagram with algebras and **k** − **Mod** replacing **Set**.

Partially Commutative structures: between commutative and non commutative worlds as first example.

3. As, today, we will consider four categories:

$$\mathbf{Mon},\ \mathbf{Grp},\ \mathbf{k\text{-}Lie},\ \mathbf{k\text{-}AAU} \tag{3}$$

   In each of these categories, there is a notion of "What are two commuting elements"

   - in **Mon**, **Grp**, **k-AAU**, it is $xy = yx$
   - in **k-Lie** it is $[x, y] = 0$

   but, for all of them, this relation is *reflexive* and *symmetric*.
   This leads us to the following questions

4. What is elimination in these categories ?
5. What is the best system or category of formal generators ?

Partially Commutative structures/2

6. By "category of formal generators", we mean that in the noncommutative world we have noncommutative alphabets and words, in the fully commutative world, have indeterminates (commutative alphabets) and monomials (with multiindex power notation). About Partially Commutative Lie algebras [6], Pr. Schützenberger asked us the following questions:
   - Is the free partially commutative Lie algebra torsion free (over $\mathbb{Z}$) ?
   - If yes (in which case it is linearly free over $\mathbb{Z}$), is it possible to construct combinatorial bases of it ?
   - To which extent can it be considered as "free" ? (more than "as a module")
7. What is the combinatorics of these structures ?
8. What is Lazard elimination ?

First remarks

1. As a motivation, we will begin by answering question 8 (the last one), and by very simple examples.
2. Let us first consider the **k**-algebra $\mathbf{k}\langle x, y\rangle = \mathbf{k}[\{x, y\}^*]$ of non-commutative polynomials in the two noncommuting variables x, y over **k**.
3. Consider now the **k**-algebra $\mathbf{k}[x, y] = \mathbf{k}[\{x^p y^q\}_{p,q\in\mathbb{N}}]$ of commutative polynomials in two (commuting) variables x, y over **k**.
4. We remark that these two algebras share a common feature: *they are algebras of monoids*, so we will consider this question in general and see that it covers the celebrated Möbius arithmetic function.
5. We remark also that commutations can be formulated as relations between words. After the list of classical eliminations, we will embark to the notion of *monoidal congruence*.

Free objects and their fine grading.

Category	Abbv.	Free Gen. by X
Monoids	**Mon**	X^*
Groups	**Grp**	$F(X)$ ($\to FG(X)$)
k unital associative algebras	**k** $-$ **AAU**	$\mathbf{k}\langle X\rangle$ ($= \mathbf{k}[X^*]$)
k-Lie algebras	**k** $-$ **Lie**	$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \subset \mathbf{k}\langle X\rangle$

- $X^* = \sqcup_{\alpha\in\mathbb{N}^{(X)}} X^\alpha = \sqcup_{n\in\mathbb{N}} X^n$
- $\mathbf{k}\langle X\rangle = \oplus_{\alpha\in\mathbb{N}^{(X)}} \mathbf{k}\langle X\rangle^\alpha = \oplus_{n\in\mathbb{N}} \mathbf{k}\langle X\rangle^n$
- $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle = \oplus_{\alpha\in\mathbb{N}^{(X)}} \mathcal{L}ie_{\mathbf{k}}\langle X\rangle^\alpha = \oplus_{n\in\mathbb{N}} \mathcal{L}ie_{\mathbf{k}}\langle X\rangle^n$

Classical Lazard elimination theorem

Theorem (Lazard elimination theorem)

Let $X = B \sqcup Z$ be a set partitioned in two blocks. We have an isomorphism of split short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle & \xhookrightarrow{j_{B|Z}} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle & \xrightarrow{p_{B|Z}} & \mathcal{L}ie_{\mathbf{k}}\langle B\rangle & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{rn}} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \overline{j_B}} & & \\ 0 & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} & \xhookrightarrow{j} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle & \xrightarrow{p} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{B} & \longrightarrow & 0 \end{array} \tag{4}$$

Free objects, partition of alphabets and eliminations.

Category	Abbv.	Free Gen. by X
Monoids	**Mon**	X^*
Groups	**Grp**	$F(X)$ $(\to FG(X))$
k unital associative algebras	$\mathbf{k}-\mathbf{AAU}$	$\mathbf{k}\langle X\rangle$ $(=\mathbf{k}[X^*])$
k-Lie algebras	$\mathbf{k}-\mathbf{Lie}$	$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \subset \mathbf{k}\langle X\rangle$

Category	Abbv.	Elimination formula (free case)
Monoids	**Mon**	$X^* = (B^*Z)^*B^*$
Groups	**Grp**	$F(X) = F(C_B(Z)) \rtimes F(B)$
k AAU	$\mathbf{k}-\mathbf{AAU}$	$\mathbf{k}\langle X\rangle = \mathbf{k}\langle B^*Z\rangle \otimes \mathbf{k}\langle B\rangle$
k-Lie algebras	$\mathbf{k}-\mathbf{Lie}$	$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \cong \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B\rangle$

Categorical setting for a presentation

6. For the considered categories, we have a forgetful functor $F : \mathcal{C} \to \mathbf{Set}$, and the following diagram

$$T \underset{v_\bullet}{\overset{u_\bullet}{\rightrightarrows}} Free(X) \tag{5}$$

7. The presented algebra and its arrow $Free(X) \xrightarrow{j} P$ is then a solution of the following universal problem

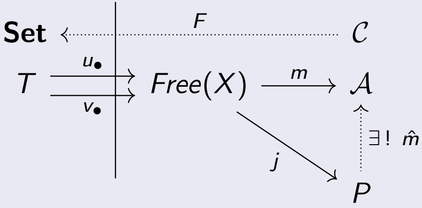


Figure: The arrow m is a morphism within the category $\mathcal{C}$ which equalizes the relators i.e. $F(m \circ u_\bullet) = F(m \circ v_\bullet)$. The arrow m is a coequalizer.

Categorical setting for a presentation: transitivity.

8. If the relator presenting P_2 is a set of a "lower category" The presented algebra and its arrow $Free(X) \xrightarrow{j} \mathcal{A}$ is then a solution of the following universal problem

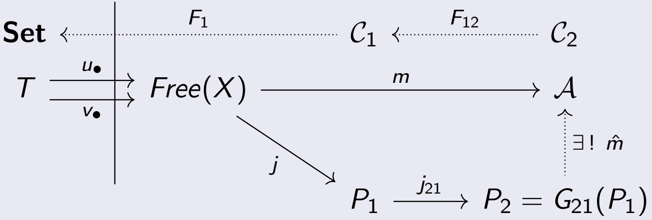


Figure: The arrow m is a morphism within the category $\mathcal{C}$ which equalizes the relators i.e. $F(m \circ u_\bullet) = F(m \circ v_\bullet)$. The arrow m is a coequalizer.

Presentations of monoids

9. A (monoidal) relator is then a set of pairs of words $\mathbf{R} = \{(u_i, v_i)\}_{i \in I}$
10. A congruence in M is an equivalence relation $\equiv$ stable by left and right translations i.e.

$$u \equiv v \implies sut \equiv svt$$

11. The congruence generated by $\mathbf{R}$, is the finest congruence $\equiv_{\mathbf{R}}$ such that, for all $i \in I$ $u_i \equiv v_i$
12. and

$$\langle X; \mathbf{R} \rangle_{\mathbf{Mon}} := X^* / \equiv_{\mathbf{R}} \tag{6}$$

Counting the words

13. Take a total ordering on the alphabet $X = \{x_1, \ldots, x_n\}$ increasingly and X^* by the graded lexicographic order $\prec_{grlex}$ (left to right) defined by

$$u \prec_{grlex} v \iff |u| < |v| \text{ or } u = p{\color{red}x}s_1,\ \ u = p{\color{red}y}s_2 \text{ with } x < y \tag{7}$$

14. Order $\mathbf{R}$ such that $u \prec_{grlex} v$ for all $(u, v) \in \mathbf{R}$.

15. Construct the following sequence

$$\begin{array}{rcl}
P_0 := \{1_{X^*}\} & ; & W_{(0,0)} = \{1_{X^*}\} = X^0; \\
\vdots \quad \vdots \quad \vdots & ; & \vdots \quad \vdots \quad \vdots \\
P_n & ; & W_{(n,0)} = W_{n,max(n-1)} \cdots W_{n,max(n)}, \\
 & ; & W_n = \cup_{0 \le j \le max(n)} W_{(n,j)} \\
\text{consider all } {\color{red}x}W_n \cup W_n{\color{red}x},\ \ x \in X \text{ and} & ; & \\
\text{eliminate all } v \text{ with } (u, v) \in \mathbf{R} & ; & P_{n+1} = P_n \cup W_n
\end{array}$$

Counting the words/2

Example of the symmetric group

16. The symmetric group $\mathfrak{S}_n$ can be defined by the Moore-Coxeter presentation

$$\langle\{t_1, t_2, \cdots, t_{n-1}\};\ t_i^2 = 1,\ t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}\rangle_{\mathbf{Mon}} \tag{8}$$

17. For example $\mathfrak{S}_3 = \langle\{t_1, t_2\};\ t_i^2 = 1,\ t_1 t_2 t_1 = t_2 t_1 t_2\rangle_{\mathbf{Mon}}$
18. The algorithm gives

$$\begin{array}{rcl}
P_0 := \{1_{X^*}\} & ; & W_{(0,0)} = \{1_{X^*}\} = X^0; \\
 & ; & W_{(1,1)} = \{t_1\}\ W_{(1,2)} = \{t_2\} \\
 & ; & W_1 = \{t_1, t_2\} \\
P_1 := \{1_{X^*}, t_1, t_2\} & ; & W_{2,1} = \{t_1t_1, t_1t_2\}, W_{2,2} = \{t_2t_1, t_2t_2 \\
 & ; & W_2 = \{t_1t_2, t_2t_1\} \\
P_2 := \{1_{X^*}, t_1, t_2, t_1t_2, t_2t_1\} & ; & W_{3,1} = \{t_1t_1t_2, t_1t_2t_1\}, \\
W_{3,2} & = & \{t_2t_1t_2, t_2t_2t_1\}, W_3 = \{t_1t_2t_1\} \\
P_3 := \{1_{X^*}, t_1, t_2, t_1t_2, t_2t_1, t_1t_2t_1\} & ; & \text{and then stop because } W_4 = \emptyset
\end{array}$$

Counting the words/3

19. Let us further consider the (square-free) monoid

$$\langle\{a,b\};\ a^2=b^2=1\rangle_{\mathbf{Mon}} \tag{9}$$

20. The algorithm gives

$$\begin{array}{rcl}
P_0 := \{1_{X^*}\} & ; & W_{(0,0)} = \{1_{X^*}\} = X^0; \\
 & ; & W_{(1,1)} = \{a\}\ W_{(1,2)} = \{b\} \\
 & ; & W_1 = \{a,b\} \\
P_1 := \{1_{X^*}, a, b\} & ; & W_{2,1} = \{\color{red}{aa}, ab\}, W_{2,2} = \{ba, \color{red}{bb}\} \\
 & ; & W_2 = \{ab, ba\} \\
P_2 := \{1_{X^*}, a, b, ab, ba\} & ; & W_{3,1} = \{\color{red}{aab}, aba\}, \\
W_{3,2} & = & \{bab, \color{red}{bba}\}, W_3 = \{aba, bab\} \\
 & ; & \text{never stops, normal forms } a(ba)^*, b(ab)^*
\end{array}$$

21. Enumeration $M_0 = 1;\ M_{n+1} = \{a(ba)^n, b(ab)^n\}$. Hilbert series $T = \sum_{n\geq 0} |M_n|.t^n$ is here $T = 1 + \dfrac{2x}{1-x} = \dfrac{1+x}{1-x}$

Counting the words: Hilbert Series

22. When the monoid M is finitely graded (i.e. $M = \uplus_{n\in\mathbb{N}} M_n,\ M_p.M_q \subset M_{p+q}$ and $|M_n| < +\infty$), we have a Hilbert series

$$Hilb(M,t) := \sum_{n\geq 0} |M_n|.t^n \tag{10}$$

for example, for the commutative monoid $M = \{x^{n_1}y^{n_2}u^{n_3}v^{n_4}\}_{n_i\in\mathbb{N}}$ (the one of monomials for the polynomials over the commutative alphabet $X = \{x, y, u, v\}$, graded by the length $|x^{n_1}y^{n_2}u^{n_3}v^{n_4}| = n_1 + n_2 + n_3 + n_4$, the Hilbert series is

$$Hilb(M,t) = \frac{1}{1 - 4t + 6t^2 - 4t^3 + t^4} = \frac{1}{(1-t)^4} \tag{11}$$

Partially Commutative monoids

23. A partially commutative alphabet (X, θ) is a set endowed with a commutation relation $\theta \subset X \times X$, reflexive and symmetric.
24. The partially commutative monoid $M(X, \theta)$ is

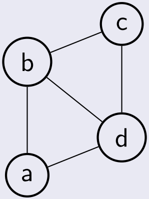
$$M(X, \theta) := \langle X;\ (xy, yx)_{(x,y)\in\theta} \rangle_{\mathbf{Mon}} \tag{12}$$

25. If the alphabet is finite, we have

$$Hilb(M(X, \theta), t) = \frac{1}{\sum_{n\geq 0} (-1)^n c_n t^n} \tag{13}$$

where c_n is the number of n-cliques of θ. This is a consequence of a more general theorem of Cartier and Foata [2].

Commutation graph, cliques and Hilbert series



26. For this graph, all singletons are totally non-commutative, and we only have $\{a, c\}$ as higher non-commutative subalphabet.
27. The Hilbert series is

$$Hilb(M(X,\theta),t) = \frac{1}{1-4t+5t^2-2t^3} \tag{14}$$

Part two :

Partially commutative structures.

Adjunction "A la Samuel".

28 We recall here the mechanism of adjunction w.r.t. a functor.
Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two categories and $F_{12} : \mathcal{C}_2 \to \mathcal{C}_1$ a (covariant) functor between them

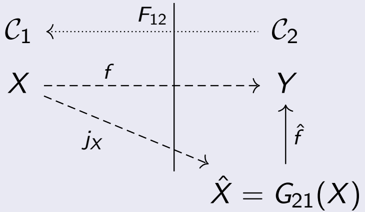


Figure: In natural language, the universal problem reads:
Does it exist a pair $(j_X, \hat{X})$ (where $j_X \in Hom_{\mathcal{C}_1}(X, \hat{X})$ and $\hat{X} \in \mathcal{C}_2$) such that, for every $\mathcal{C}_1$-theoretical arrow f (this means that $f \in Hom_{\mathcal{C}_1}(X, F(Y))$), there is a unique $\hat{f} \in Hom_{\mathcal{C}_2}(\hat{X}, Y)$ such that $f = F(\hat{f}) \circ j_X$. If it is the case for every object $X \in \mathcal{C}_1$, then the correspondence $X \to \hat{X}$, $f \to \hat{f}$ between $\mathcal{C}_1$ and $\mathcal{C}_2$ turns out to be a (covariant) functor G_{21}.

Mon, **Grp**, **k − Lie**, **k − AAU** and their eliminations.

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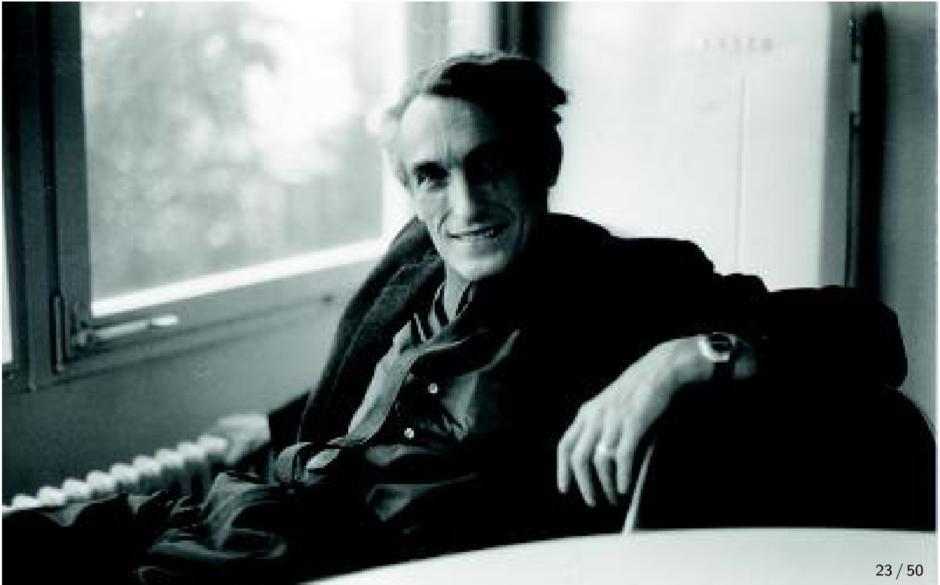


Where the (forgetful) functor comes: Monoids.

29. Def **CAlph** be the category of alphabets with commutation i.e. reflexive and symmetric graphs (X, θ) with $f: (X_1, \theta_1) \to (X_2, \theta_2)$ such that $f: X_1 \to X_2$, set-theoretical such that $(u, v) \in \theta_1 \Longrightarrow (f(u), f(v)) \in \theta_2$ and **Mon** the category of monoids. Now a monoid M being given $\theta_M = F(M) = \{(u, v) \in M \mid uv = vu\}$ can be checked to be a functor $F: \mathbf{Mon} \to \mathbf{CAlph}$

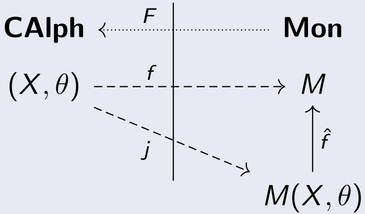


Figure: $M(X, \theta)$ is the monoid freely generated by (X, θ) w.r.t. F. To say that $f \in Het_F((X, \theta), M)$ amounts to say that $f: X \to M$ set-theoretically and $(u, v) \in \theta \Longrightarrow f(u)f(v) = f(v)f(u)$

Functor/2: Groups.

30 Let **Grp** the category of groups. Now a monoid G being given $\theta_G = F(G) = \{(u,v) \in G \mid uv = vu\}$ can be checked to be a functor $F: \ \mathbf{Grp} \to \mathbf{CAlph}$

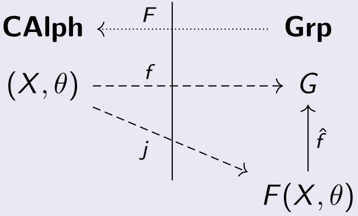


Figure: $F(X,\theta)$ is the group freely generated by (X,θ) w.r.t. F. To say that $f \in Het_F((X,\theta), G)$ amounts to say that $f: \ X \to G$ set-theoretically and $(u,v) \in \theta \Longrightarrow f(u)f(v) = f(v)f(u)$.

Functor/3: **k**-Lie algebras.

31 Let **k-Lie** be the category of **k**-Lie algebras (**k** is a ring). Now $L \in$ **k-Lie** being given $\theta_L = F(L) = \{(u, v) \in L \mid [u, v] = 0\}$ can be checked to be a functor $F :$ **k-Lie** $\rightarrow$ **CAlph**

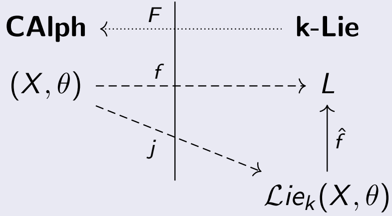


Figure: $\mathcal{L}ie_k(X, \theta)$ is the **k**-Lie algebra freely generated by (X, θ) w.r.t. F.
To say that $f \in Het_F((X, \theta), L)$ amounts to say that $f : X \rightarrow L$ set-theoretically and $(u, v) \in \theta \Longrightarrow [f(u), f(v)] = 0$

Functor/4: **k**-AAU.

32 Let **k-AAU** be the category of **k**-algebras (associative with unit) (**k** is a ring). Now $\mathcal{A} \in$ **k-AAU** being given
$\theta_{\mathcal{A}} = F(\mathcal{A}) = \{(u, v) \in \mathcal{A} \mid [u, v] = 0\}$ can be checked to be a functor
$F :$ **k-AAU** $\to$ **CAlph**

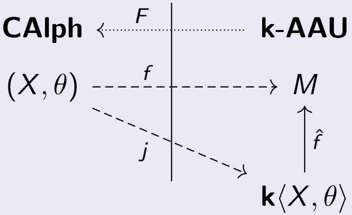
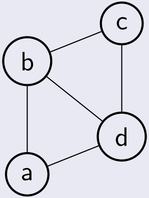


Figure: $\mathbf{k}\langle X, \theta\rangle$ is the **k**-AAU freely generated by (X, θ) w.r.t. F. To say that $f \in Het_F((X, \theta), \mathcal{A})$ amounts to say that $f : X \to \mathcal{A}$ set-theoretically and $(u, v) \in \theta \Longrightarrow f(u)f(v) = f(v)f(u)$.

Total non-commutativity



33 For this graph, all singletons are totally non-commutative, and we only have $\{a, c\}$ as higher non-commutative subalphabet.

Partition of alphabets, free and partially commutative eliminations.

Category	Abbv.	Elimination formula (free)
Monoids	**Mon**	$X^* = (B^*Z)^*B^*$
Groups	**Grp**	$F(X) = F(C_B(Z)) \rtimes F(B)$
k AAU	**k − AAU**	$\mathbf{k}\langle X\rangle = \mathbf{k}\langle B^*Z\rangle \otimes \mathbf{k}\langle B\rangle$
k-Lie algebras	**k − Lie**	$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \cong \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B\rangle$

With free partially commutative structures (Z totally non-commutative and $X = B + Z$).

Category	Abbv.	Elim. formula (part. comm.)
Monoids	**Mon**	$M(X,\theta) = C_B(Z)^*M(B,\theta_B)$
Groups	**Grp**	$F(X,\theta) = F(C_B(Z)) \rtimes F(B,\theta_B)$
k AAU	**k − AAU**	$\mathbf{k}\langle X,\theta\rangle = \mathbf{k}\langle C_B(Z)\rangle \otimes \mathbf{k}\langle B,\theta_B\rangle$
k-Lie algebras	**k − Lie**	$\mathcal{L}ie_{\mathbf{k}}\langle X,\theta\rangle \cong \mathcal{L}ie_{\mathbf{k}}\langle C_B(Z)\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B,\theta_B\rangle$

Part three :

General case.

Main result: Elimination for presented Lie algebras/1.

34 Let $\mathbf{k}$ be a ring. Let $X = B + Z$ be a set partitioned in two blocks. We suppose given a relator $\mathbf{r} = \{r_j\}_{j \in J} \subset \mathcal{L}ie_{\mathbf{k}}\langle X \rangle$ (cf. [3] Ch II §2.3[a]) which is compatible with the alphabet partition i.e. there exists a partition of the set of indices $J = J_Z \sqcup J_B$ such that

- $\mathbf{r}_B = \{r_j\}_{j \in J_B} = \mathbf{r} \cap \mathcal{L}ie_{\mathbf{k}}\langle X \rangle_B$ and $\mathbf{r}_Z = \{r_j\}_{j \in J_Z} = \mathbf{r} \cap \mathcal{L}ie_{\mathbf{k}}\langle X \rangle_{BZ}$.

The notations being as above, we construct the ideals

- $\mathcal{J}_B$ is the Lie ideal of $\mathcal{L}ie_{\mathbf{k}}\langle X \rangle_B$ generated by $\{r_j\}_{j \in J_B}$
- $\mathcal{J}, \mathcal{J}_Z$ and $\mathcal{J}_{BZ}$ are the Lie ideals of $\mathcal{L}ie_{\mathbf{k}}\langle X \rangle$ generated respectively by $\mathbf{r}, \mathbf{r}_Z$ and $\mathbf{r}_{BZ} := \{\mathrm{ad}_Q\, z\}_{Q \in \mathcal{J}_B, z \in Z}$.

[a] The set I there being replaced by X.

Elimination for presented Lie algebras/2

When we have such a type of relator, we can state the following theorem.

Theorem (Th 2)

With our constructions above, we get the following properties:

i) *we have* $(\mathcal{J}_Z + \mathcal{J}_{BZ}) \subset \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ}$ *(and then* $(\mathcal{J}_Z + \mathcal{J}_{BZ}) \cap \mathcal{J}_B = \{0\}$*). Moreover,* $(\mathcal{J}_Z + \mathcal{J}_{BZ})$ *is a Lie ideal of* $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ}$ *(and even, by definition, of* $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle$*).*

ii) *the action of* $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B$ *on* $\mathfrak{Der}(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ}$ *(by internal* ad*) passes to quotients as an action*

$$\alpha : \ \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \to \mathfrak{Der}\left(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ})\right) \tag{15}$$

such that $\mathbf{r}_B \subset \ker(\alpha)$ *and then, we get an action*

$$\overline{\alpha} : \ \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \big/ \mathcal{J}_B \to \mathfrak{Der}\left(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ})\right) \tag{16}$$

Elimination for presented Lie algebras/3

Th 2 cont'd

iii) We can construct an isomorphism (and its inverse) from presented Lie algebra $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \big/ \mathcal{J}$ by the set $\mathbf{r} = \{r_j\}_{j\in J}$ of relators onto the semidirect product of Lie algebras $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) \rtimes \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \big/ \mathcal{J}_B$ which will be denoted by

$$\beta_{25} : \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \big/ \mathcal{J} \xrightarrow{\simeq} \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) \rtimes \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \big/ \mathcal{J}_B \tag{17}$$

iv) In fact, one has a commutative diagram of Lie algebras with split short exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} & \xhookrightarrow{j} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle & \xrightarrow{p} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B & \longrightarrow & 0 \\
 & & \downarrow s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} & & \downarrow s_{\mathcal{J}} & & \downarrow s_{\mathcal{J}_B} & & \\
0 & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \big/ \mathcal{J} & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \big/ \mathcal{J}_B & \longrightarrow & 0
\end{array}$$

Example: Infinitesimal Pure Braids Relations.

35. We consider the alphabet $\mathcal{T}_n = \{t_{ij}\}_{1\leq i<j\leq n}$ and the infinitesimal pure braid relator $\mathbf{R[n]}$ in the free Lie algebra

$$\mathbf{R[n]} = \begin{cases} \mathbf{R_1[n]} & [t_{i,j}, t_{i,k} + t_{j,k}] & \text{for } 1 \leq i < j < k \leq n, \\ \mathbf{R_2[n]} & [t_{i,j} + t_{i,k}, t_{j,k}] & \text{for } 1 \leq i < j < k \leq n, \\ \mathbf{R_3[n]} & [t_{i,j}, t_{k,l}] & \text{for } {}^{1\leq i<j\leq n,}_{1\leq k<l\leq n,} \text{ and } |\{i,j,k,l\}| = 4 \end{cases}$$

36. This is a typical example of relator compatible with the partition

$$X := \mathcal{T}_n = \mathcal{T}_{n-1} \sqcup T_n := B \sqcup Z$$

where $T_n = \{t_{i,n}\}_{1\leq i\leq n-1}$ and the infinitesimal pure braid relator $\mathbf{r} := \mathbf{R[n]} \subset \mathcal{L}ie_{\mathbf{k}}\langle \mathcal{T}_n \rangle = \mathrm{DK}_{\mathbf{k},n}$ the Drinfel'd-Kohno Lie algebra.

37. Applying the theorem, we get a semi-direct decomposition. In order to prove that the first (i.e. "acted") factor is free, we need an extra criterium.

Elimination of the subalphabet $Z/1$

38. In certain cases (which is that of the Lie algebras $\mathrm{DK}_{\mathbf{k},n}$), it can happen that the left factor of the semidirect product (17) be isomorphic to $\mathcal{L}ie_{\mathbf{k}}\langle Z\rangle$. We start from the commutative diagram (33) with an additional arrow

$$\begin{array}{ccccccccc}
 & & \mathcal{L}ie_{\mathbf{k}}\langle Z\rangle & & & & & & \\
 & & \big\downarrow j_Z & & & & & & \\
0 & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} & \xrightarrow{j} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle & \xrightarrow{p} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B & \longrightarrow & 0 \\
 & & \big\downarrow s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} & & \big\downarrow s_{\mathcal{J}} & & \big\downarrow s_{\mathcal{J}_B} & & \\
0 & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ}\Big/(\mathcal{J}_Z+\mathcal{J}_{BZ}) & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle\big/\mathcal{J} & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B\big/\mathcal{J}_B & \longrightarrow & 0
\end{array} \tag{18}$$

where j_Z is the subalphabet embedding such that

$$Im(j_Z) = \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_Z = \bigoplus_{\substack{\alpha\in\mathbb{N}^{(X)}\\ |\alpha|_B=0}} \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_\alpha . \tag{19}$$

Elimination of the subalphabet $Z/2$

39 We are now in the position to state the following

Proposition

With the notations as in slide 35, let us consider the composite map $\beta = s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} \circ j_Z$, then

a. *In order that β be injective, it is necessary and sufficient that $(\mathcal{J}_Z + \mathcal{J}_{BZ}) \cap \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_Z = \{0\}$.*

b. *In order that β be surjective, it is necessary and sufficient that, for all $(b,z) \in B \times Z$, we had*

$$s_{\mathcal{J}_Z+\mathcal{J}_{BZ}}([b,z]) \in s_{\mathcal{J}_Z+\mathcal{J}_{BZ}}(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_Z). \tag{20}$$

Case of the partially commutative Free Lie algebra.

Proposition

40. Here, the code C below must be extended. We consider the code $C_B(Z) = \{s_\theta(uz) | u \in B^*, z \in Z, \mathrm{TAlph}(s_\theta(uz)) = \{z\}\}$ Let $C = j_\theta(C_B(Z))$ and j_C be the subalphabet embedding, we have the diagram.

$$
\begin{array}{ccccccccc}
 & & \mathcal{L}ie_{\mathbf{k}}\langle C\rangle & & & & & & \\
 & & \big\downarrow{\scriptstyle j_C} & & & & & & \\
0 & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} & \overset{j}{\hookrightarrow} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle & \overset{p}{\longrightarrow} & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{B} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle s_{\mathcal{J}_Z+\mathcal{J}_{BZ}}} & & \big\downarrow{\scriptstyle s_{\mathcal{J}}} & & \big\downarrow{\scriptstyle s_{\mathcal{J}_B}} & & \\
0 & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \big/ \mathcal{J} & \longrightarrow & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{B} \big/ \mathcal{J}_B & \longrightarrow & 0
\end{array}
$$

Then, with $C = j_\theta(C_B(Z)))$, $s_{\mathcal{J}_Z+\mathcal{J}_{BZ}} \circ j_C$ is an isomorphism. In particular, the left factor of the semi-direct product (17), here $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \big/ (\mathcal{J}_Z + \mathcal{J}_{BZ})$ is a free Lie algebra.

Strange Gradings

Structure	Grading support	Formula	Row
Set	Set	$X = \sqcup_{i \in I} X_i$	1
Modules	Set	$M = \oplus_{i \in I} M_i$	2
$\mathbf{k} - \mathbf{AA}$	Semigroup	$\mathcal{A} = \oplus_{s \in S} \mathcal{A}_s$	3
$\mathbf{k} - \mathbf{AAU}$	Monoid	$\mathcal{A} = \oplus_{s \in S} \mathcal{A}_s$	4

Comments. –

1. Rows R1 and R2 imply no multiplication whereas, in rows R3-R4, condition
$$\mathcal{A}_s \mathcal{A}_t \subset \mathcal{A}_{s.t} \tag{21}$$
2. If the semigroup (or monoid) is commutative (which is the classical case), RHS of Eq. (21) is replaced by $\mathcal{A}_{s+t}$.
3. For new tensor categories offered by gradings, see color tensor product [5].

Short Exact Sequences revisited.

41. The prototype of a short exact sequence (SES) is of the form

$$0 \longrightarrow \mathcal{J} \xrightarrow{j} \mathcal{A} \xrightarrow{s} \mathcal{A}/\mathcal{J} \longrightarrow 0$$

42. Now, taking **k**-Lie algebras, let us remark that, saying $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$ amounts to say that the SES of Lie algebras

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \xrightarrow{\alpha_{23}} \mathfrak{b} \longrightarrow 0$$

is split (i.e. α_{23} admits a section σ). Then, it reads

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \underset{\alpha_{23}}{\overset{\sigma}{\leftrightarrows}} \mathfrak{b} \longrightarrow 0$$

and, in fact, $\mathfrak{g} \simeq ker(\alpha_{23}) \rtimes Im(\sigma)$ and one has

$[.\,,.]$	$\mathfrak{b}$	$\mathfrak{h}$
$\mathfrak{b}$	$\mathfrak{b}$	$\mathfrak{h}$
$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$

(22)

SES and strange gradings.

43. Such a (complemented) nesting amounts to have a $\mathbb{B}$-grading. Where $(\mathbb{B}, +)$ is the additive part of the boolean semiring whose law reads

+	0	1
0	0	1
1	1	1

(23)

44. Indeed, if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$, we can set $\mathfrak{g}_0 = \mathfrak{b}$ and $\mathfrak{g}_1 = \mathfrak{h}$ and check that, in this way, $\mathfrak{g}$ is $\mathbb{B}$-graded.
45. Of course all classical properties about graded generators hold, in particular those with homogeneous generators. This sheds some light on our results and Th 2 could be rephrased in the light of $\mathbb{B}$-gradings.

Semidirect products as colimits/1

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Are semi-direct products categorical (co)limits?

Ask Question

Asked 10 years, 6 months ago Modified 6 years, 4 months ago Viewed 4k times

31

Products, are very elementary forms of categorical limits. My question is whether in the category of groups, semi-direct products are categorical limits.

As was pointed in: http://unapologetic.wordpress.com/2007/03/08/split-exact-sequences-and-semidirect-products/

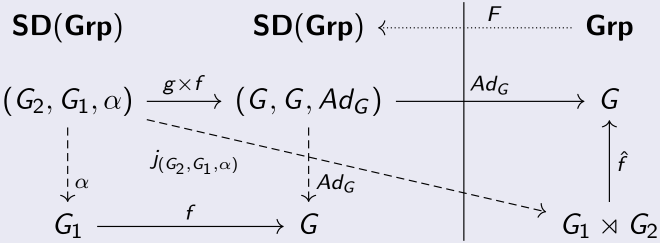
Bourbaki (General Topology, Prop. 27) gives a universal property:

Let $f\colon N \to G$, $g\colon H \to G$ be two homomorphisms into a group G, such that $f(\phi_h(n)) = g(h)f(n)g(h^{-1})$ for all $n \in N$, $h \in H$. Then there is a unique homomorphism $k\colon N \rtimes H \to G$ extending f and g in the usual sense.

However, I remain unsatisfied. The condition $f(\phi_h(n)) = g(h)f(n)g(h^{-1})$ is a condition on elements of groups, rather than a condition that says that some diagram is commutative.

So the question remains: are semi-direct products in the category of groups categorical limits?

ct.category-theory gr.group-theory



SD(Grp)
SD(Grp)
F
Grp
(G_2, G_1, α)
$g \times f$
(G, G, Ad_G)
Ad_G
G
$j_{(G_2, G_1, \alpha)}$
α
Ad_G
$\hat{f}$
G_1
f
G
$G_1 \rtimes G_2$

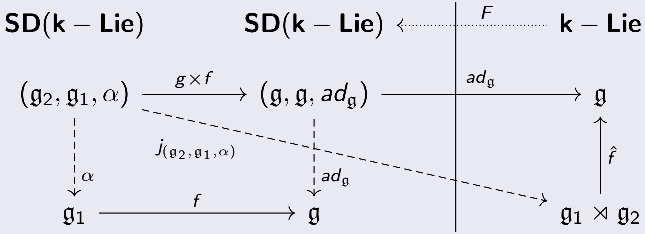
Semidirect products as adjuncts (Lie algebras).

46 We first rephrase Bourbaki in the context of Lie algebras,
Proposition B. – Let $\alpha \mapsto \alpha(h,-)$ a morphism of **k**-Lie algebras $\mathfrak{g}_2 \to \mathfrak{Der}(\mathfrak{g}_1)$ and $f\colon \mathfrak{g}_1 \to \mathfrak{g}$, $g\colon \mathfrak{g}_2 \to \mathfrak{g}$ be two homomorphisms into a **k**-Lie algebra $\mathfrak{g}$, such that

$$f(\alpha(h,n)) = [g(h), f(n)] \; (= ad_{\mathfrak{g}}(g(h), f(n))) \tag{24}$$

for all $n \in \mathfrak{g}_1$, $h \in \mathfrak{g}_2$. Then there is a unique homomorphism $\hat{f}\colon \mathfrak{g}_1 \rtimes \mathfrak{g}_2 \to \mathfrak{g}$ extending f and g in the usual sense.

47 This situation can be set in the following diagram



Concluding remarks

48. We have seen semi-direct products of Lie algebras as a universal problem.
49. Many presentations considered in combinatorial group theory and combinatorial Lie algebra theory (in particular arising from topology and graph theory) have a lot of commutations and provide naturally semidirect products (e.g. from fibered spaces).
50. The natural structure to compute with them is to use a presentation with “generators and relations”.
51. We have seen the general Lazard's elimitation for these structures and the category of Lie algebras.
52. This Lazard elimitation generalizes the classical one and provides a semi-direct product. Every semidirect product is the image of some Lazard elimination.

Concluding remarks

53. Strange gradings allow not only to manage semidirect products but, more complex elimination schemes like iterated decompositions. Indeed, suppose we had an elimination scheme

$$\mathfrak{g}(x_1, x_2, \cdots x_n) \cong \mathfrak{s}(n) \rtimes \mathfrak{g}(x_1, x_2, \cdots x_{n-1}) \qquad (25)$$

iterating it, we get

$$\mathfrak{g}(x_1, x_2, \cdots x_n) \cong (\mathfrak{s}(n) \rtimes (\mathfrak{s}(n-1) \rtimes \cdots \rtimes \mathfrak{s}(1)) \cdots) \qquad (26)$$

in particular $\mathfrak{g}(x_1, x_2, \cdots x_n) = \bigoplus_{0 \le j \le n-1} \mathfrak{g}_j$ is $([0, \cdots, n-1], sup)$-graded (with $\mathfrak{g}_j = \mathfrak{s}(j-1)$). We can even manage infinite decompositions with $(\mathbb{N}, sup)$ or non-linear eliminations with other semigroups.

Thank you for your presence (close or remote) ...
and your attention.

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators

   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra

   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module

    https://ncatlab.org/nlab/show/Hilbert+module

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[2] P. Cartier and D. Foata, *Problèmes combinatoires de commutation et réarrangements* Lecture Notes in Mathematics, **85**, Berlin, Springer-Verlag, (1969)

[3] N. Bourbaki, *Lie groups and Lie algebras, Chapters 1-3*, Springer-Verlag; (1989).

[4] G. Duchamp, *Orthogonal projection onto the free Lie Algebra*, Theorerical Computer Science, 79 , 227-239 (1991)

[5] Gérard H. E. Duchamp, Christophe Tollu, Karol A. Penson and Gleb A. Koshevoy, Deformations of Algebras: Twisting and Perturbations, Séminaire Lotharingien de Combinatoire, B62e (2010).

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[6] G. Duchamp, D.Krob, *Free partially commutative structures*, Journal of Algebra, 156 , 318-359 (1993)

[7] Gérard Duchamp, Jean-Gabriel Luque, *Lazard's Elimination (in traces) is finite-state recognizable*, International Journal of Algebra and Computation, **17**, No. 01, pp. 53-60 (2007).
https://arxiv.org/abs/math/0607280

[8] Kernels in nlab
https://ncatlab.org/nlab/show/kernel