Random Lattices as Sphere Packings

Presentation by

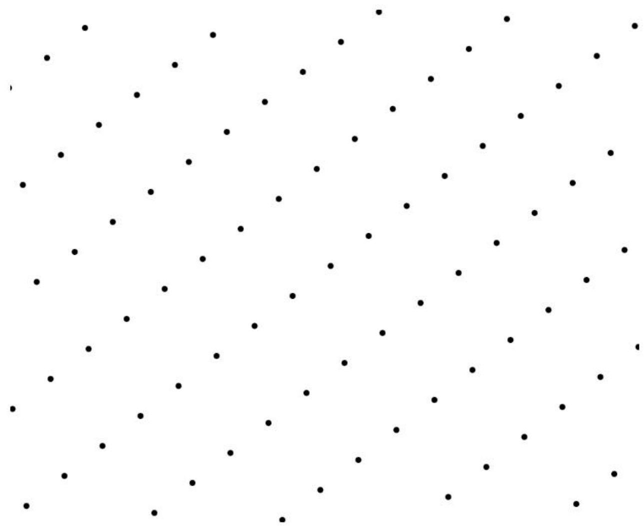
Nihar P. Gargava

Doctoral student,
Chair of Number Theory,
Section of Mathematics,
École Polytechnique Fédérale de Lausanne

29th November 2022

What is a lattice packing?

Visualizing in $\mathbb{R}^2$



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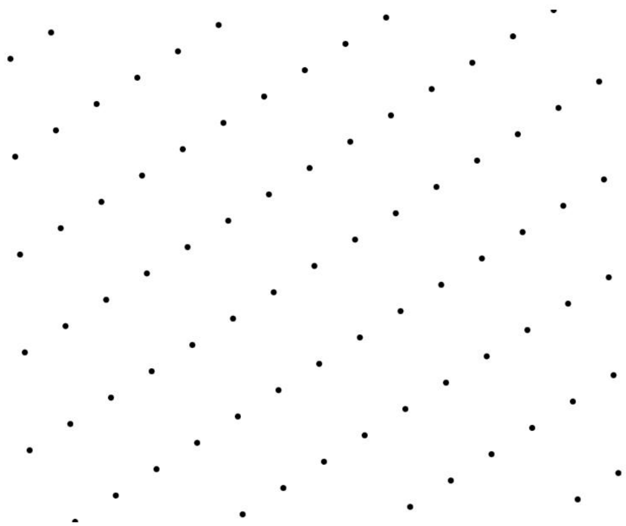

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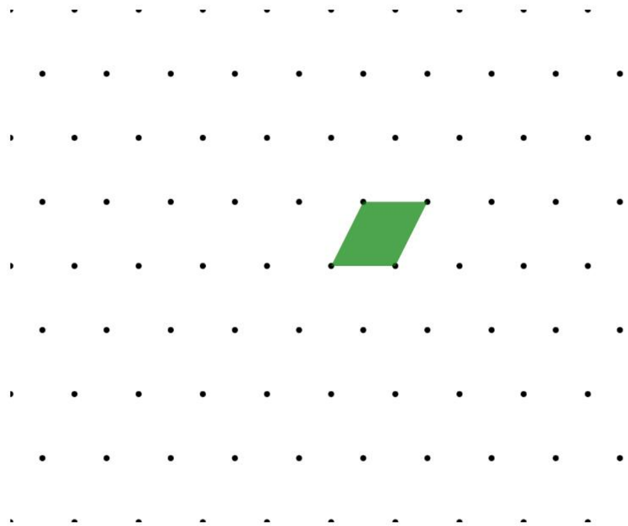

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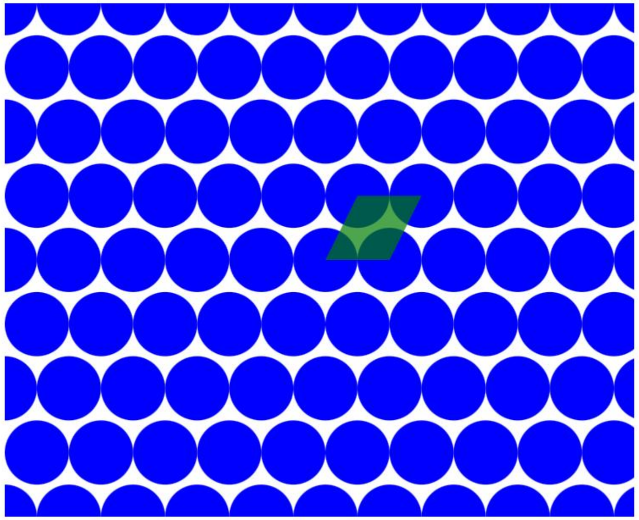

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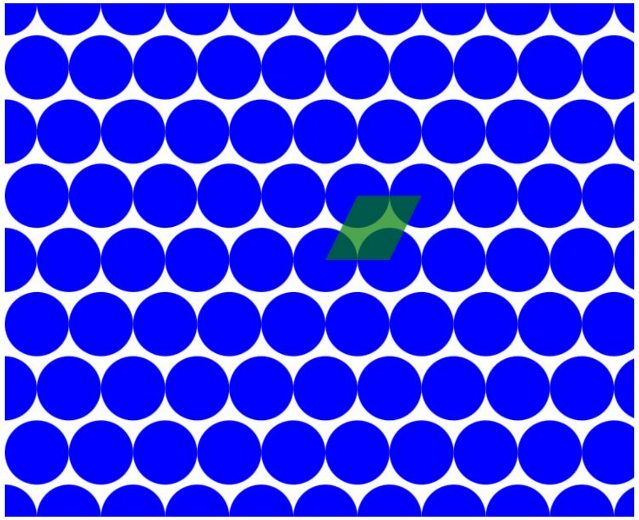

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Packing density of a lattice packing is:

$$\lim_{R \to \infty} \frac{\mu\left(B_R(0) \cap \left(\bigsqcup_{v \in \Lambda} B_r(v)\right)\right)}{\mu\left(B_R(0)\right)} = \frac{\mu(B_r(0))}{\mu(\mathbb{R}^d / \Lambda)}$$

It is always in the interval $[0, 1]$.

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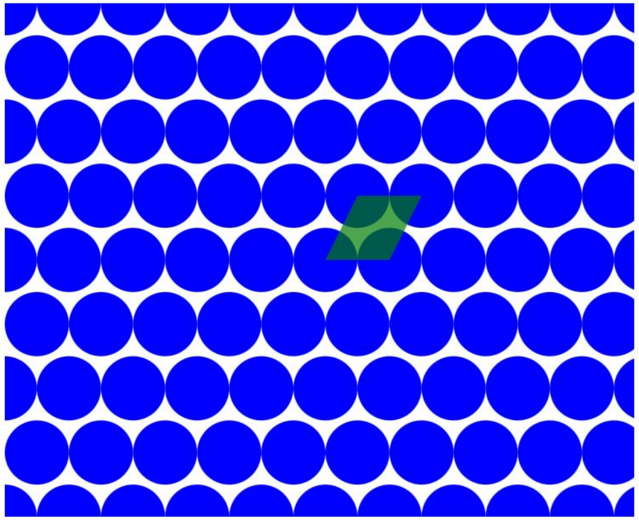

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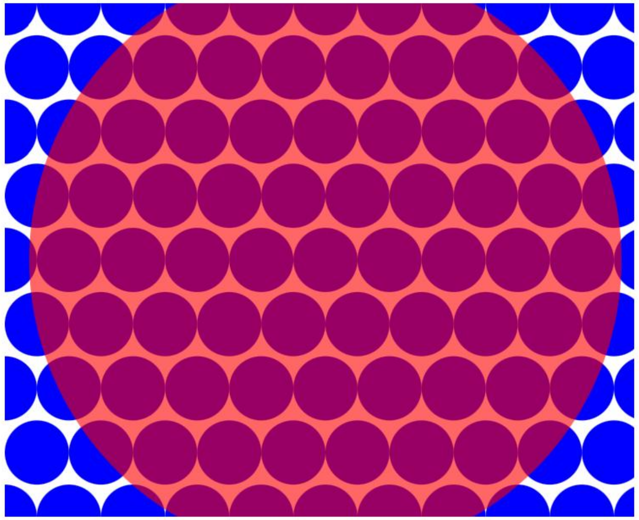

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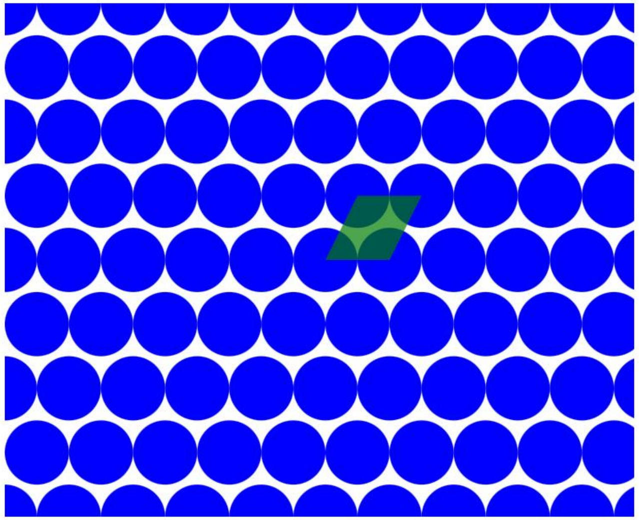

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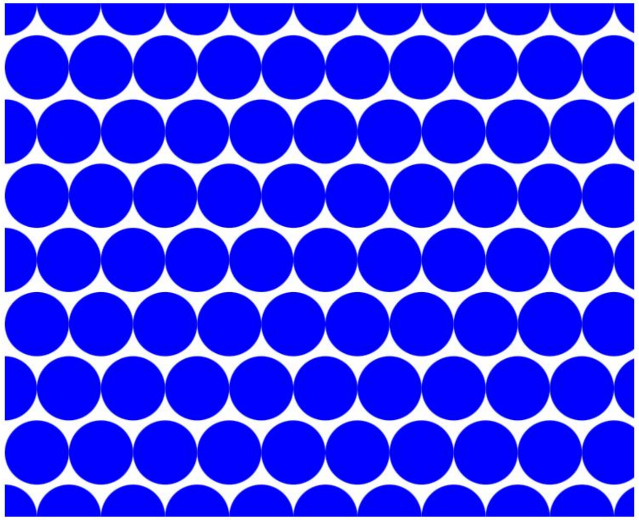

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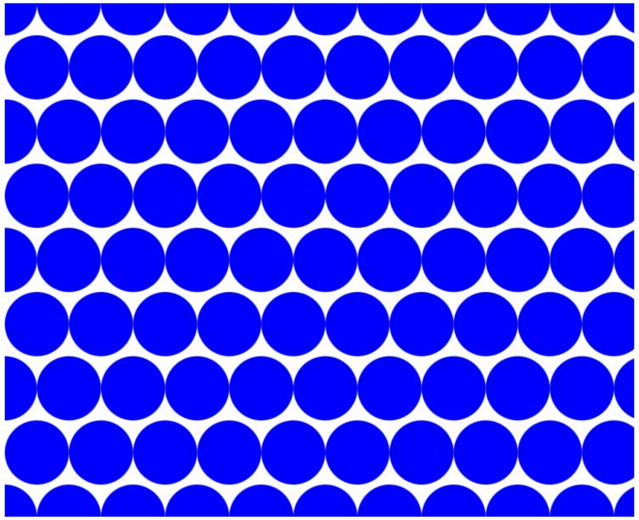

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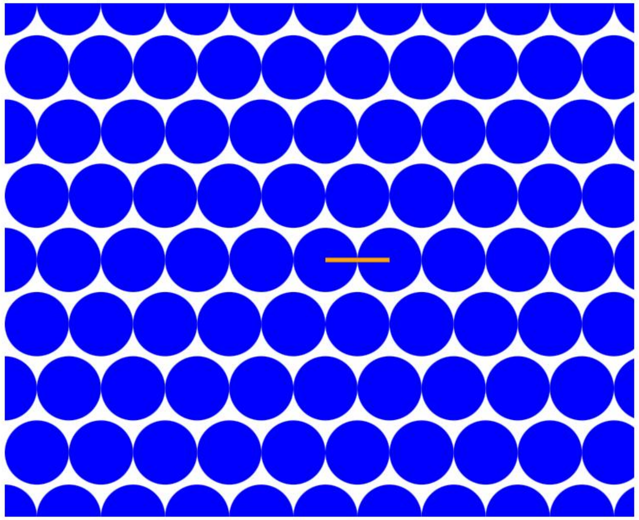

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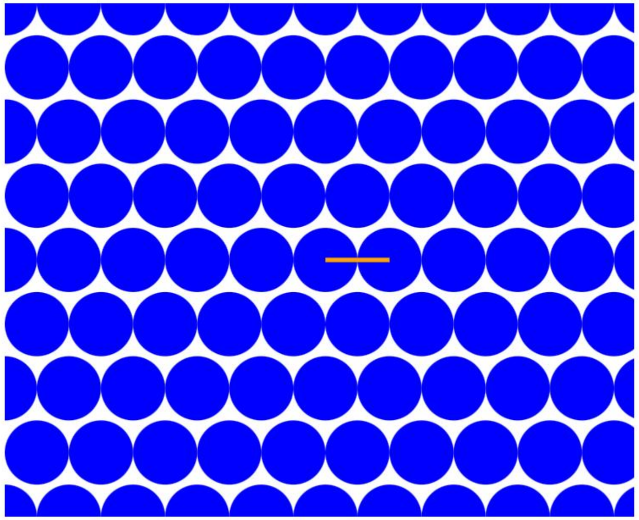

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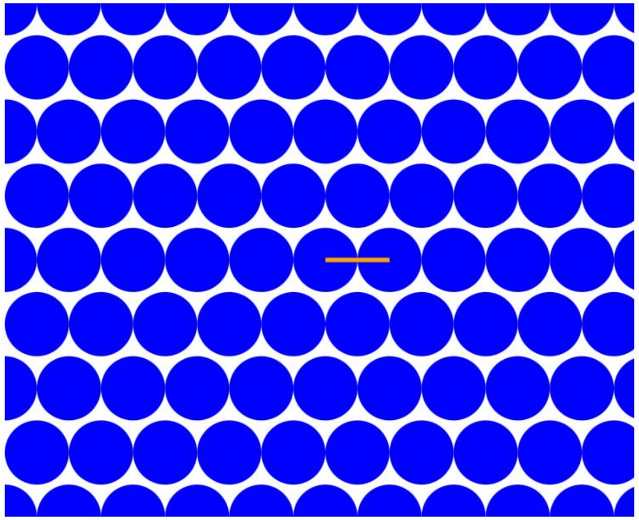

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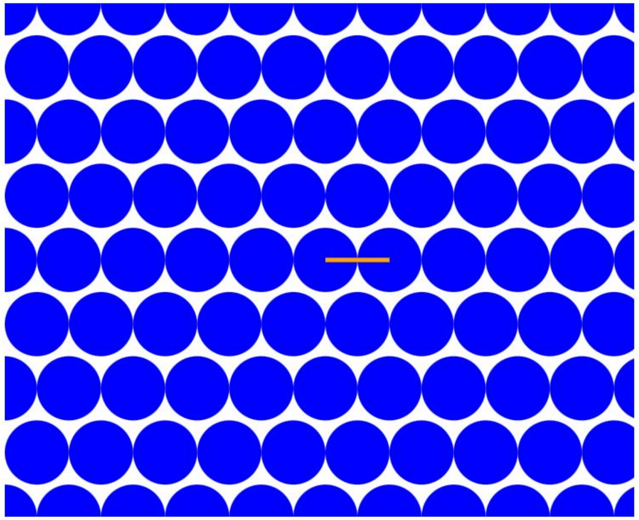

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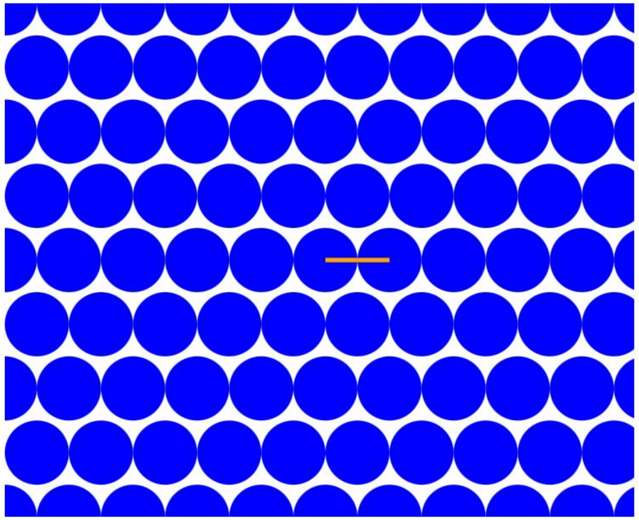

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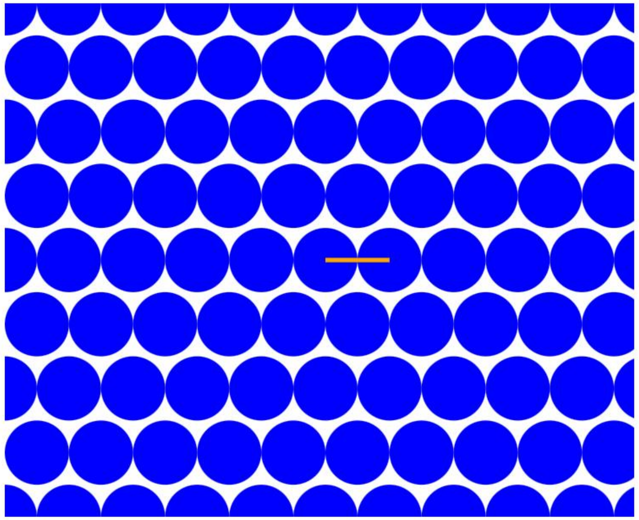

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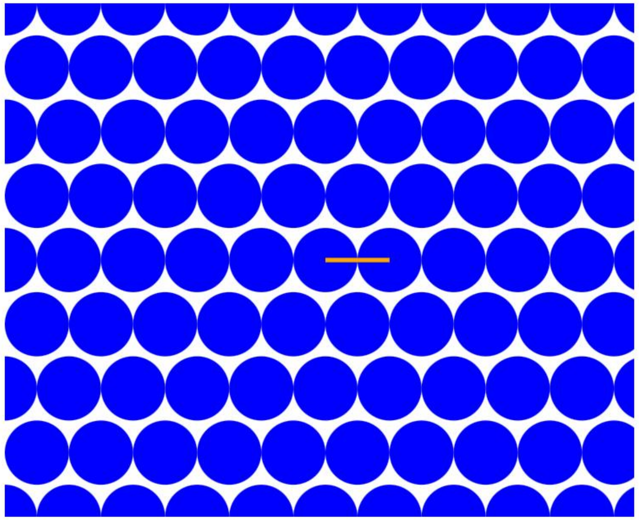

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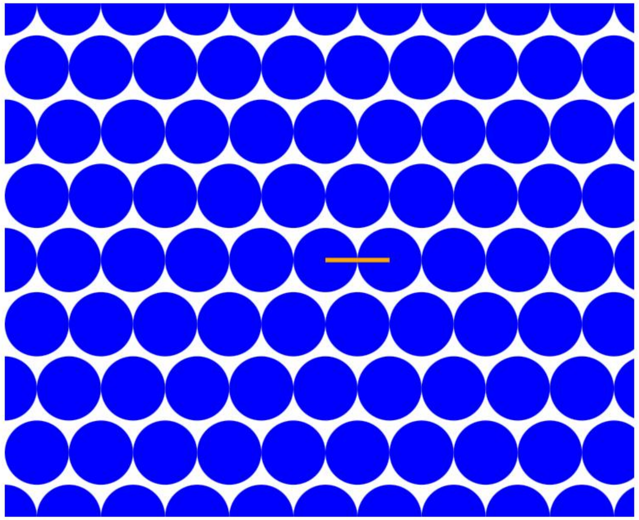

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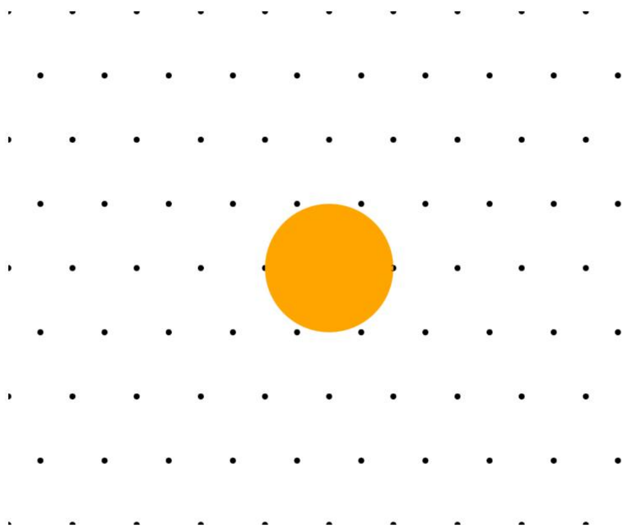

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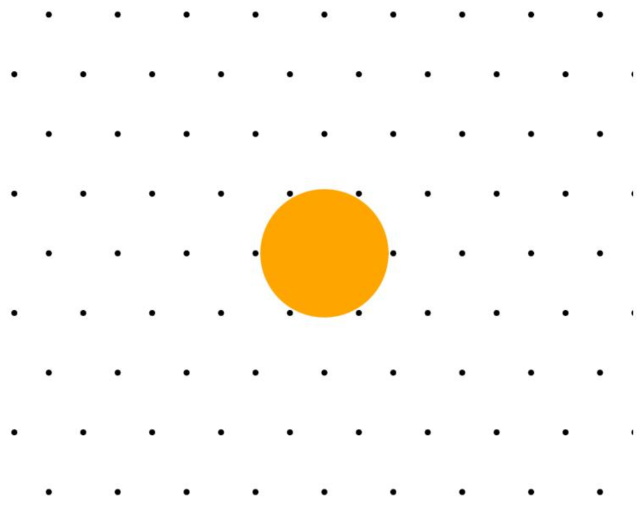

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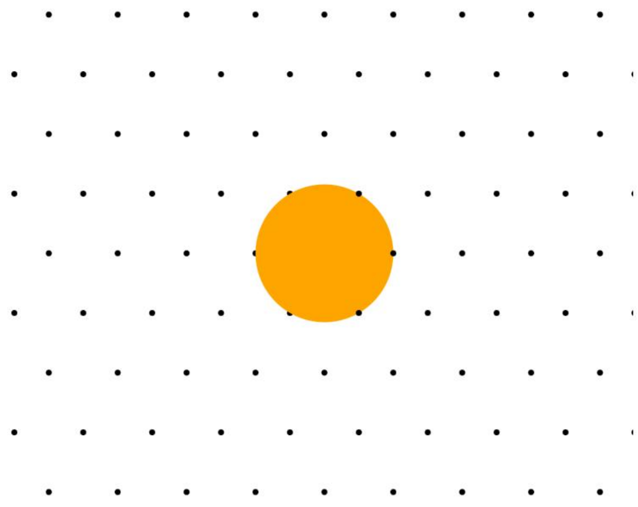

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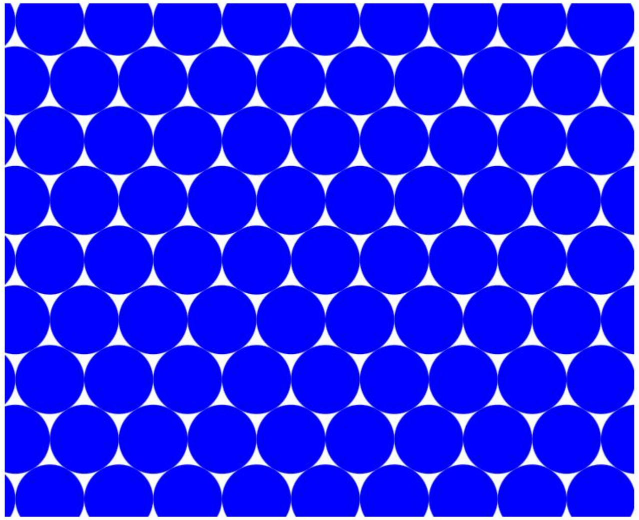

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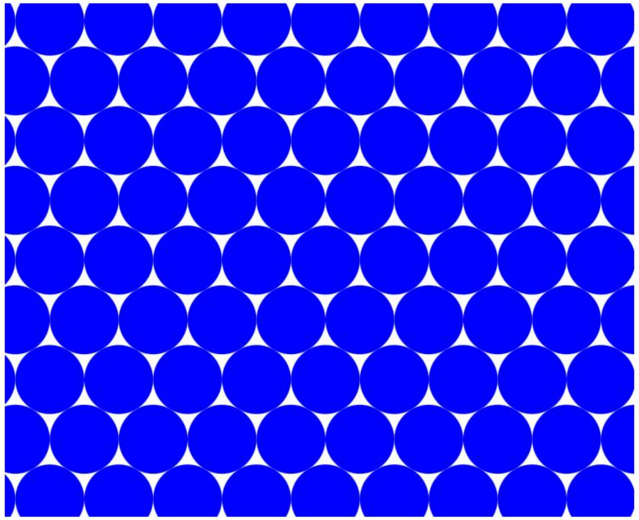

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Clearly, $c_d \in [0, 2^d]$, so supremum exists!

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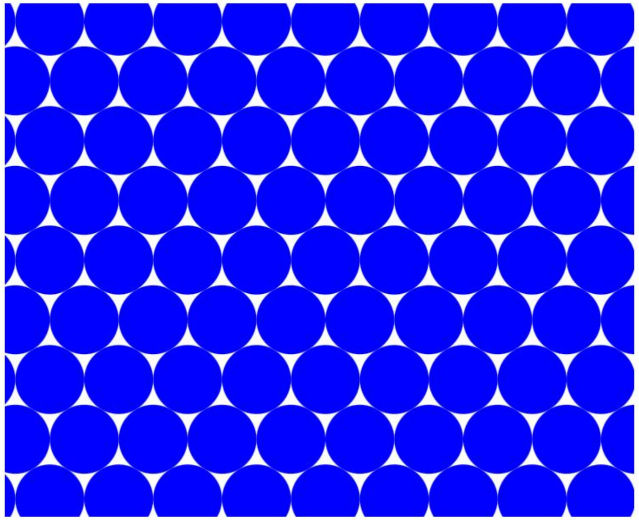

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Some asymptotic lower bounds for large dimensions

Lower bound	Contribution of	Dimensions covered
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$c_d \geq 2(d-1)$	Ball (1992)	$\forall\, d \geq 1$
$c_{4n} \geq 8.8n$	Vance (2011)	$d = 4n, n \geq 1$
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Since $\liminf \left(\frac{\varphi(n)}{n} \log \log n \right) = e^{-\gamma}$, the last bound is the best lower bound (among these, and overall) on c_d in infinitely many dimensions. The first dimension where it outperforms all others in this list is $d = 960$.

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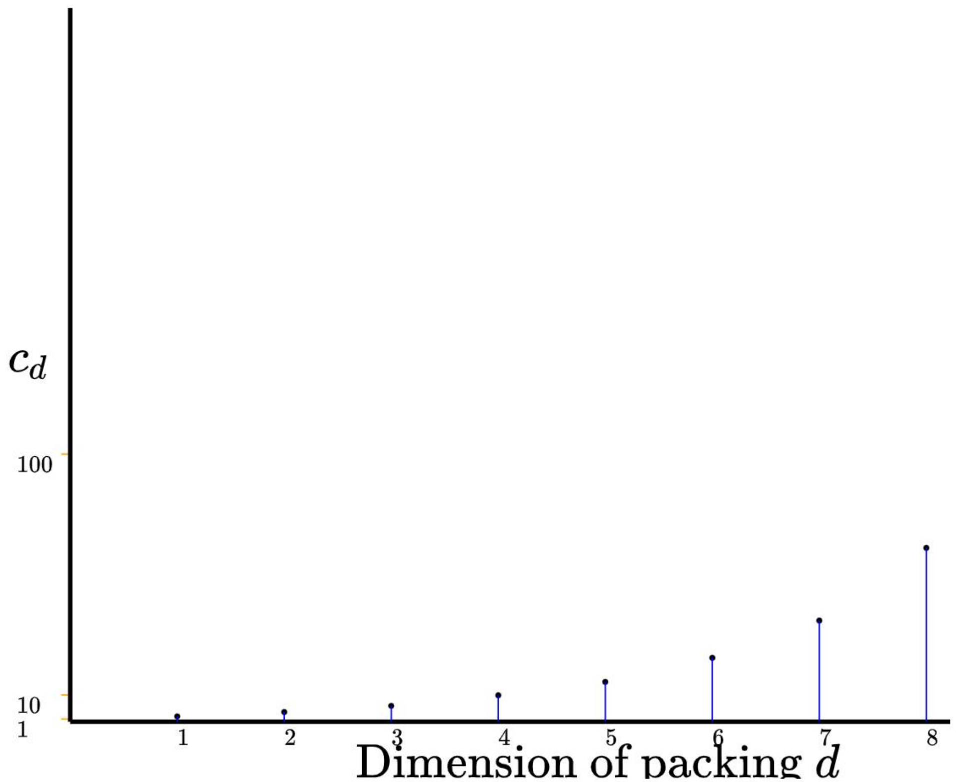
Comparison of bounds



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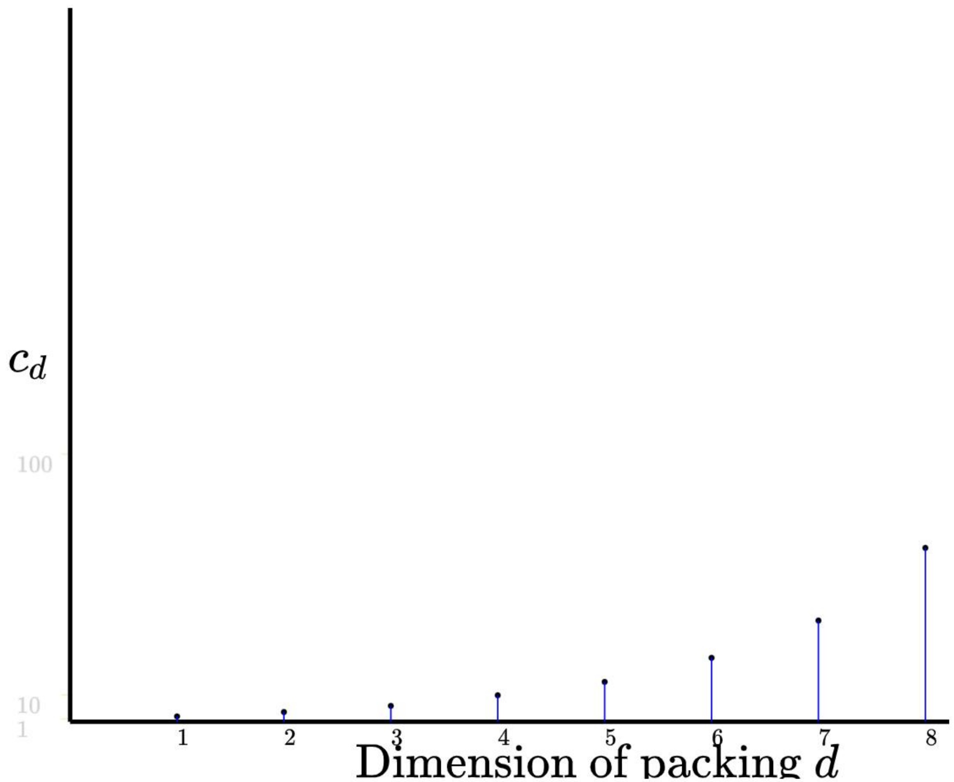

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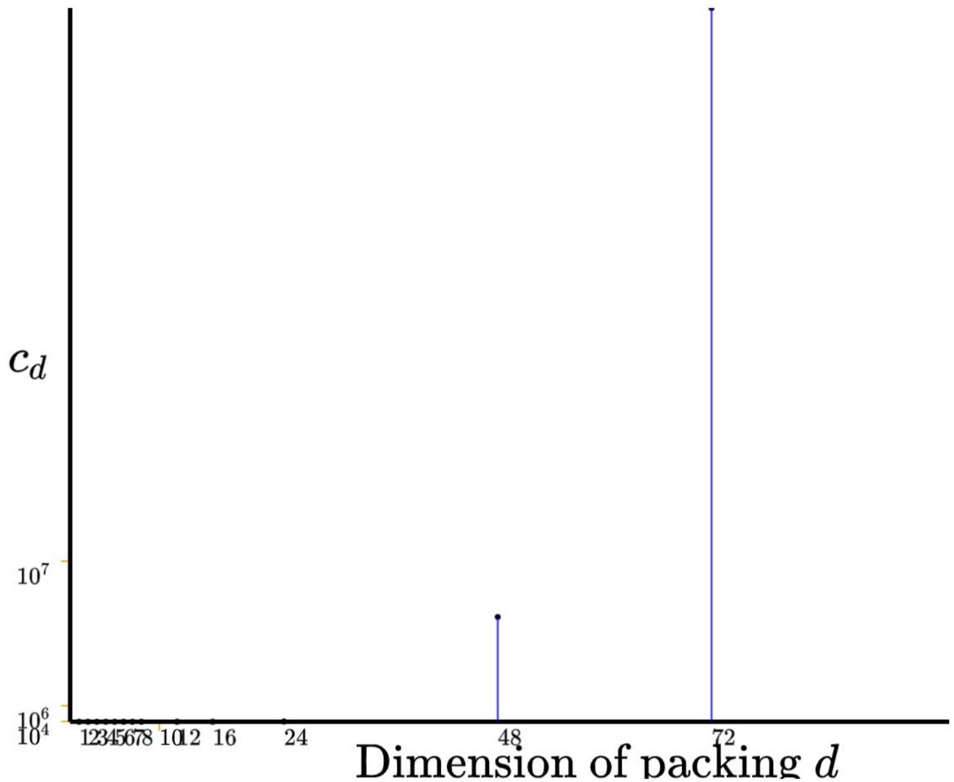

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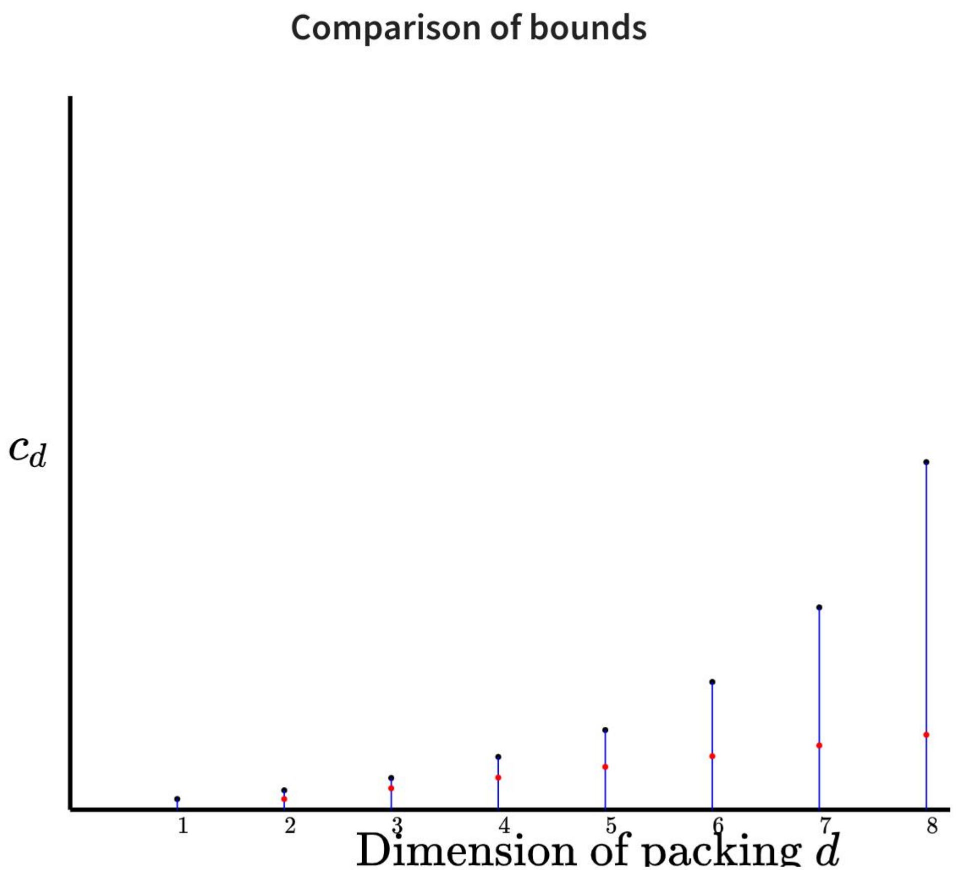

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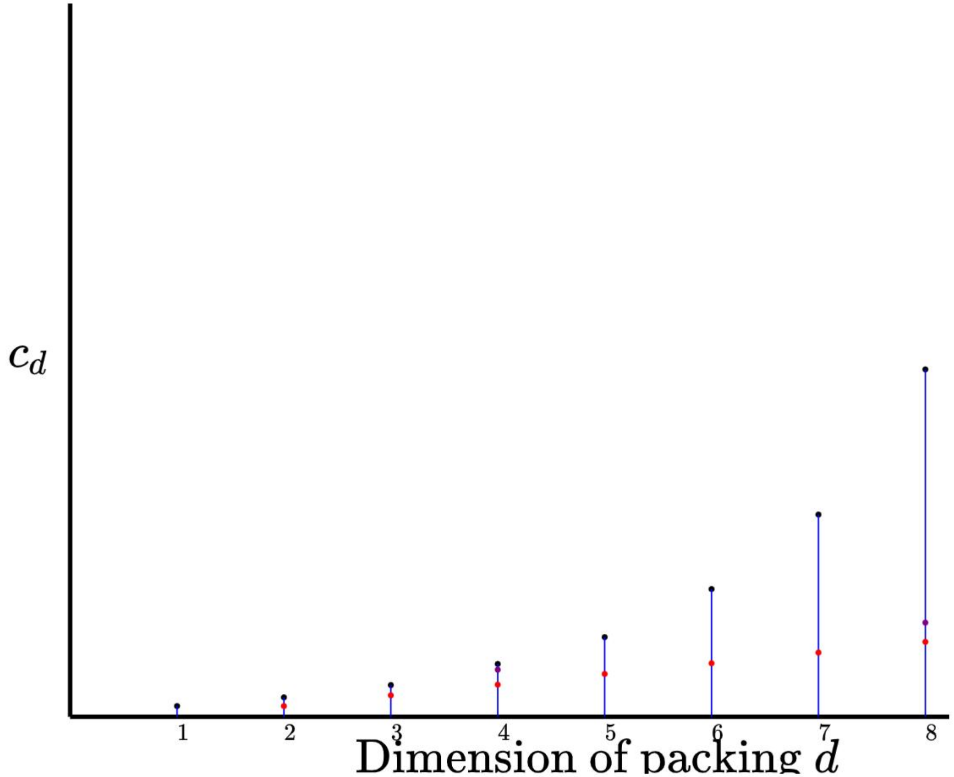

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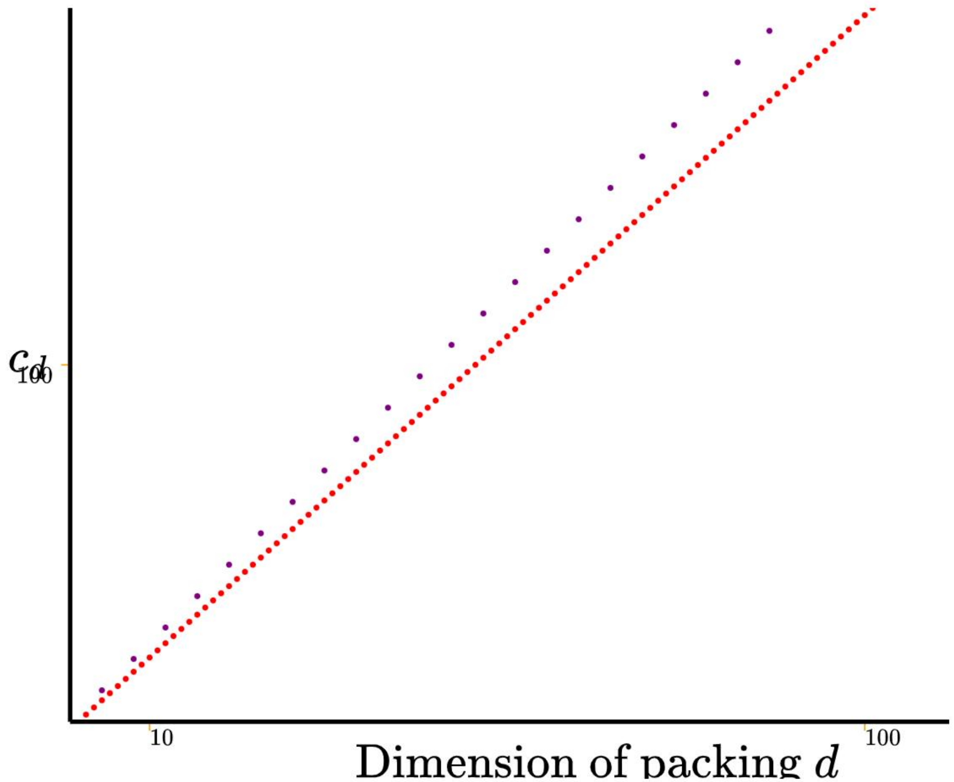

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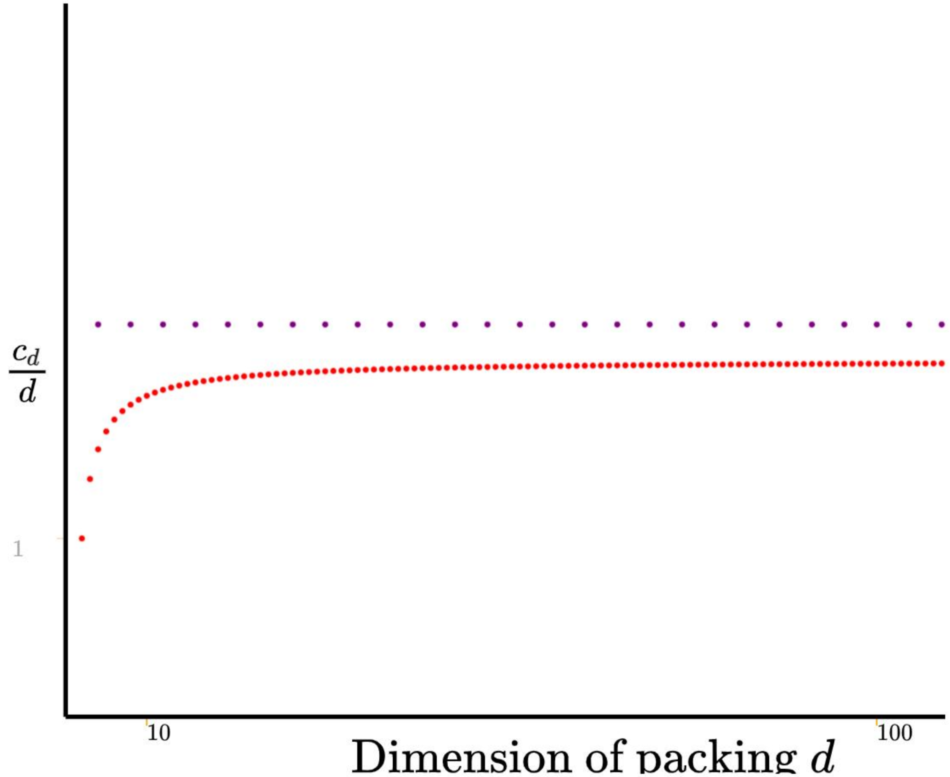

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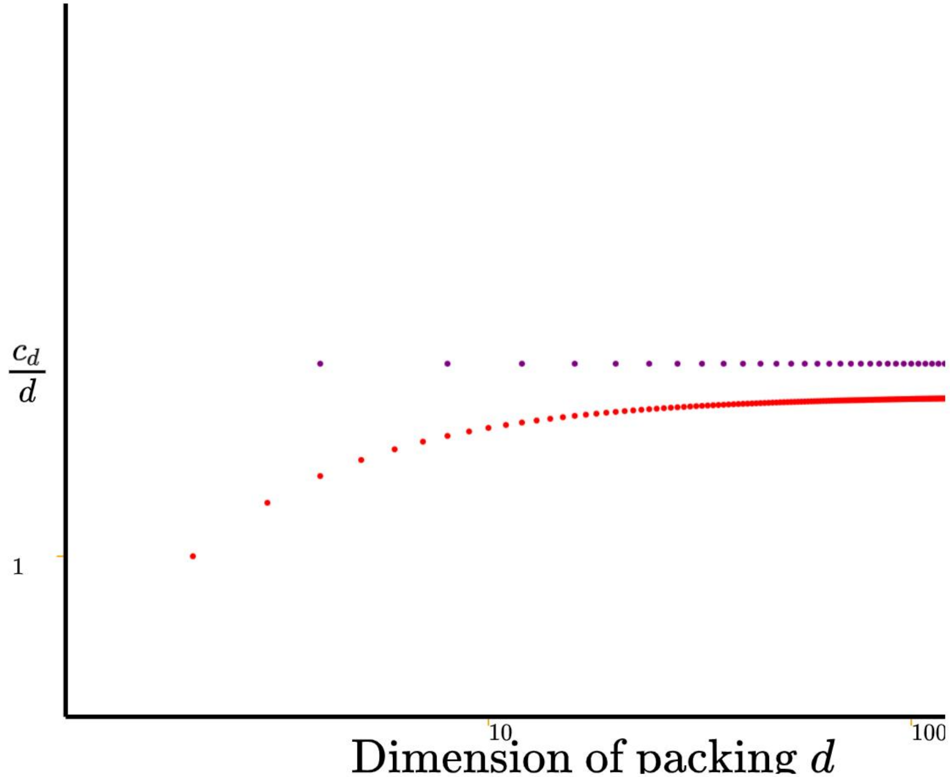

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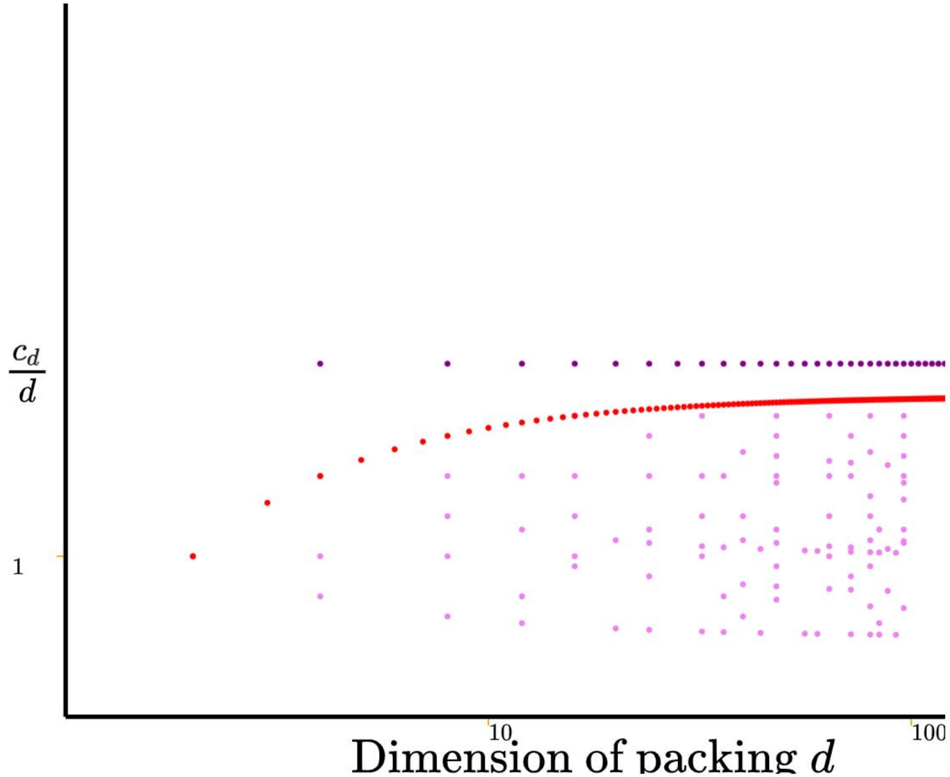

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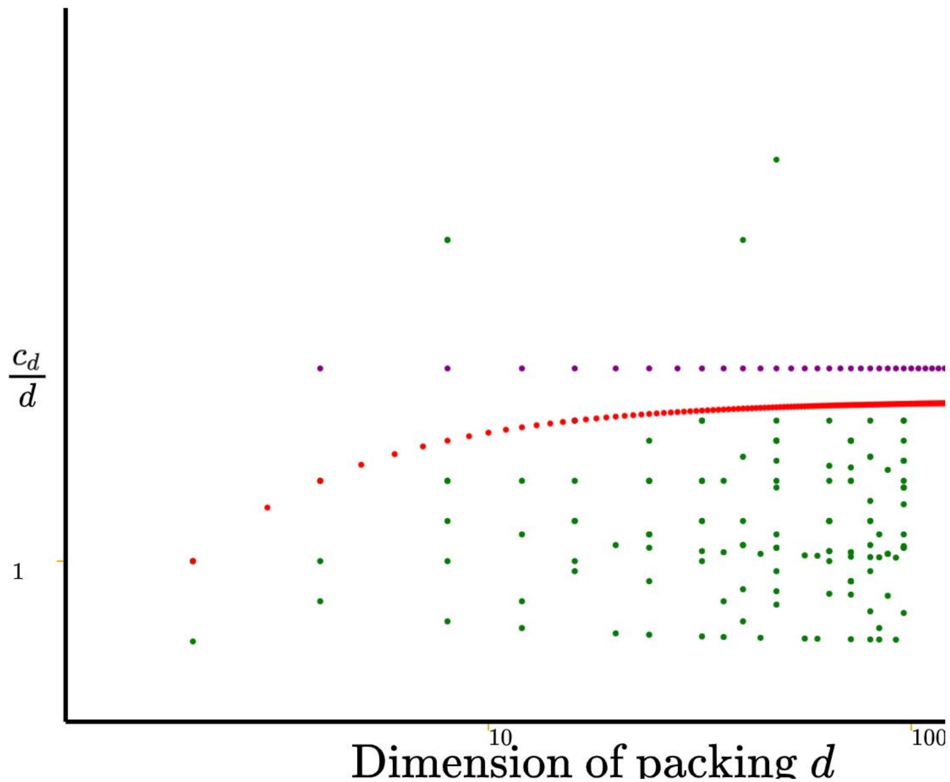

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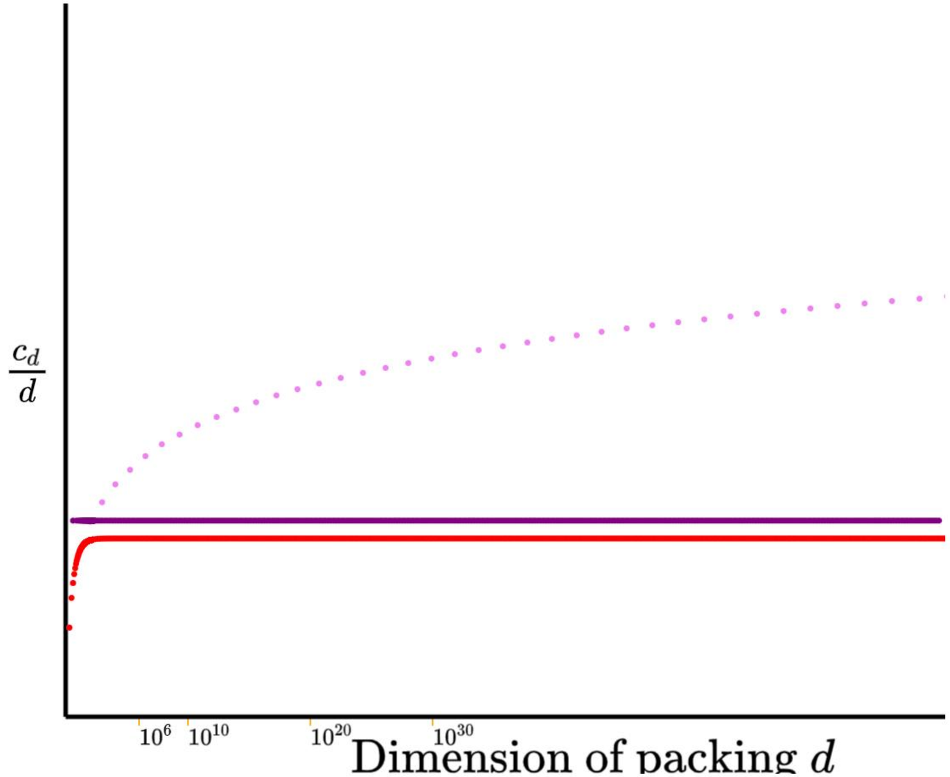

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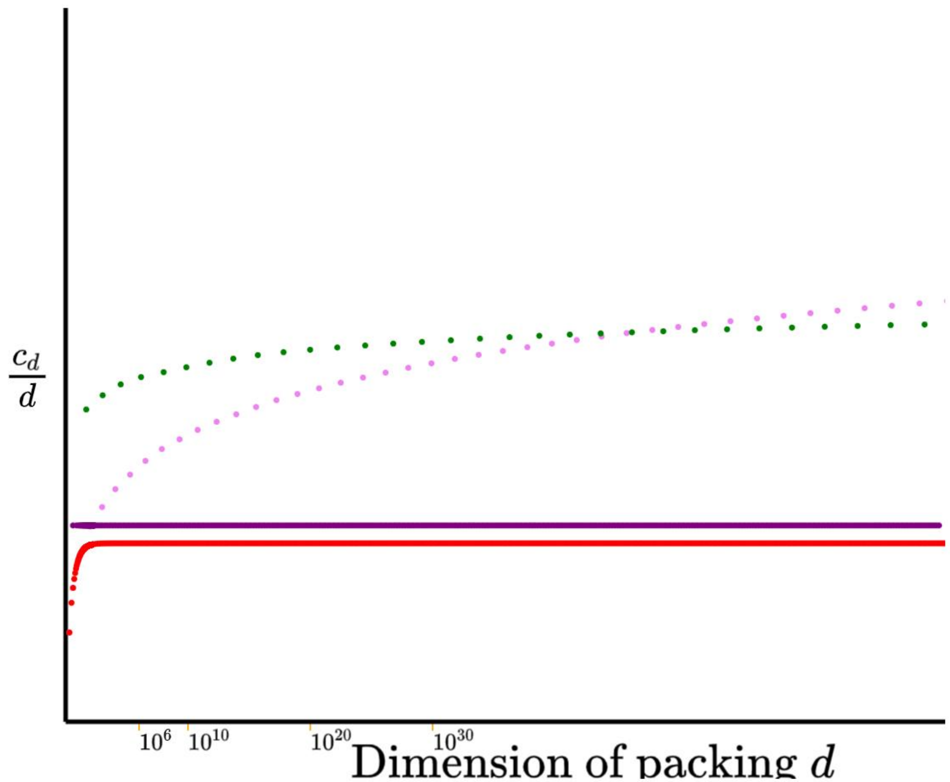

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Let D be a finite-dimensional division algebra over $\mathbb{Q}$. Let $\mathcal{O} \subseteq D$ be an order in D and $G_0 \subseteq \mathcal{O}$ be a finite group embedded in the multiplicative group of D. Then if $d = 2\dim_{\mathbb{Q}} D$, then

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The division algbera construction gives more freedom to cherrypick sequences. Instead of choosing a sequence of cyclotomic fields, we can now choose sequences of $\mathbb{Q}$-divison algebras. However, no such sequence will be able to give an asymptotic result strictly better than $O(d \log\log d)$. Improvements in individual dimensions is still possible, as shown before.

The probabilistic method

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To show $c_d \geq K$, we must prove the existence of $g \in SL_d(\mathbb{R})$ such that the origin centered ball B with $\mu(B) = K$ has $g\mathbb{Z}^d \cap B = \{0\}$

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$SL_d(\mathbb{R})$ is unimodular. $SL_d(\mathbb{Z})$ is a discrete subgroup inside $SL_d(\mathbb{R})$ and therefore there is a unique left $SL_d(\mathbb{R})$-invariant measure on $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$.

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Proposition

There exists a unique (upto scaling) natural measure on $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$, left-invariant under $SL_d(\mathbb{R})$ action on cosets.
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This gives us a probability space. Hence we can talk about random unit covolume lattices.

Lattice-sum function

Visualizing in $\mathbb{R}^2$

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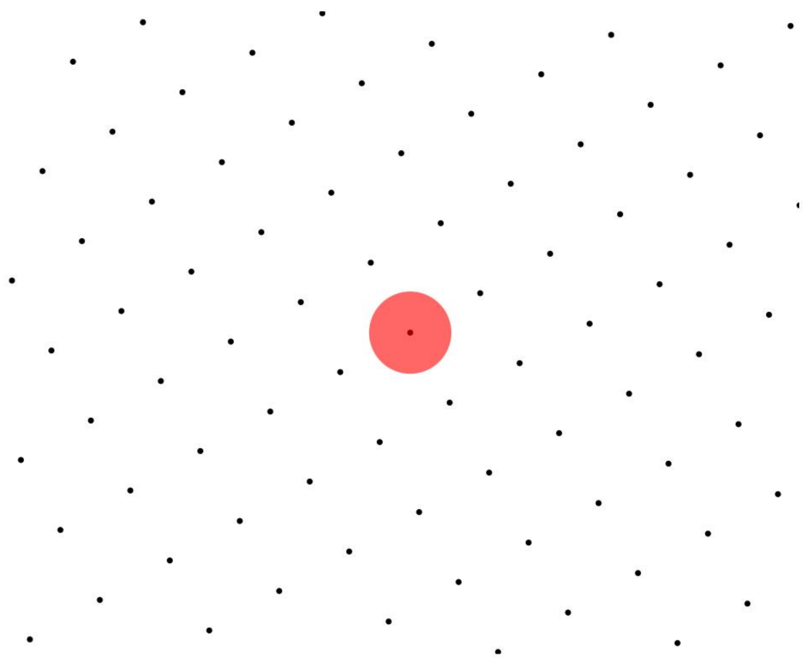

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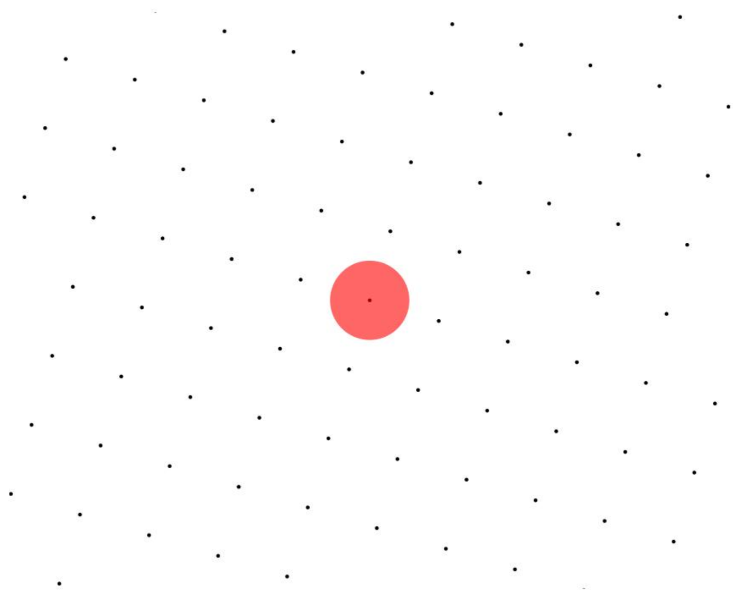

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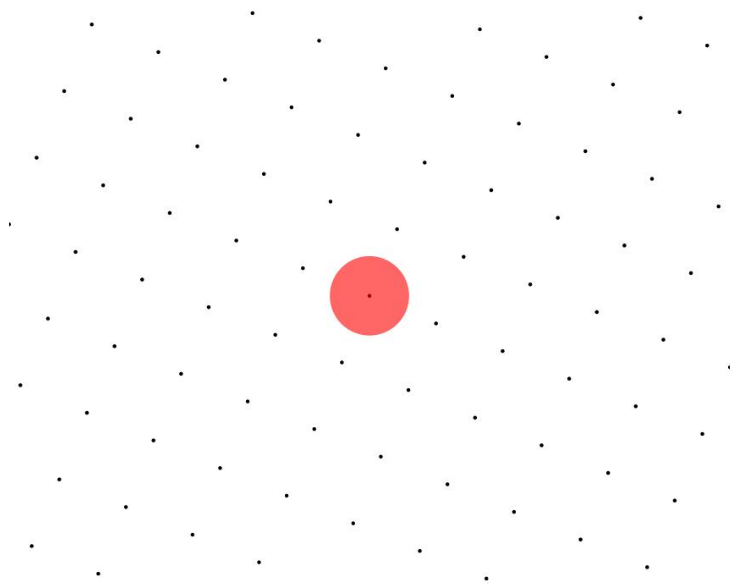

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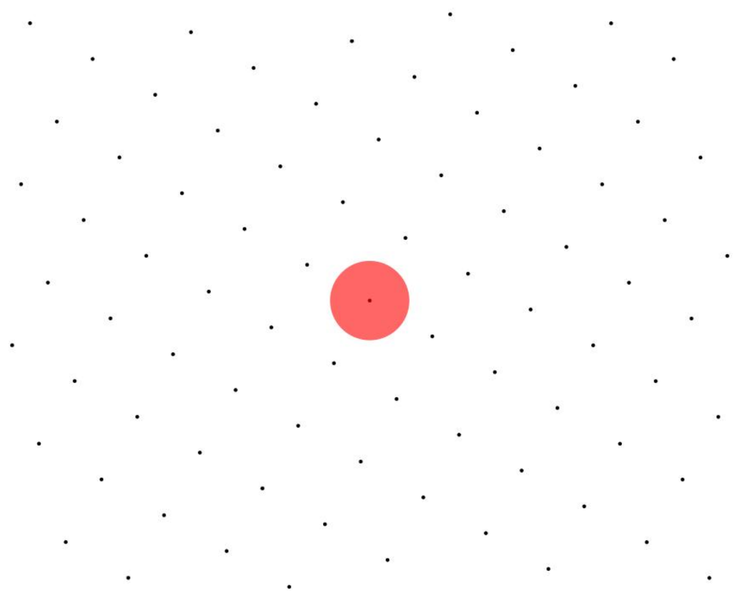

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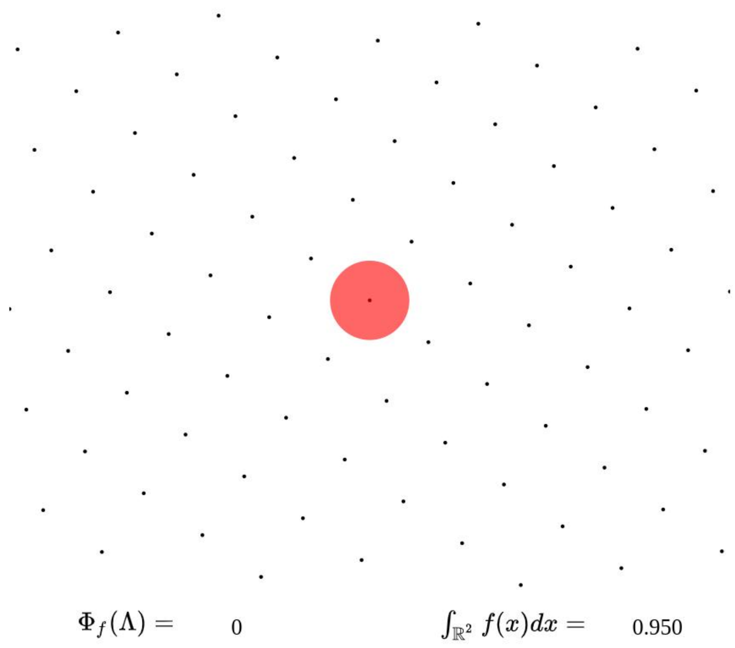

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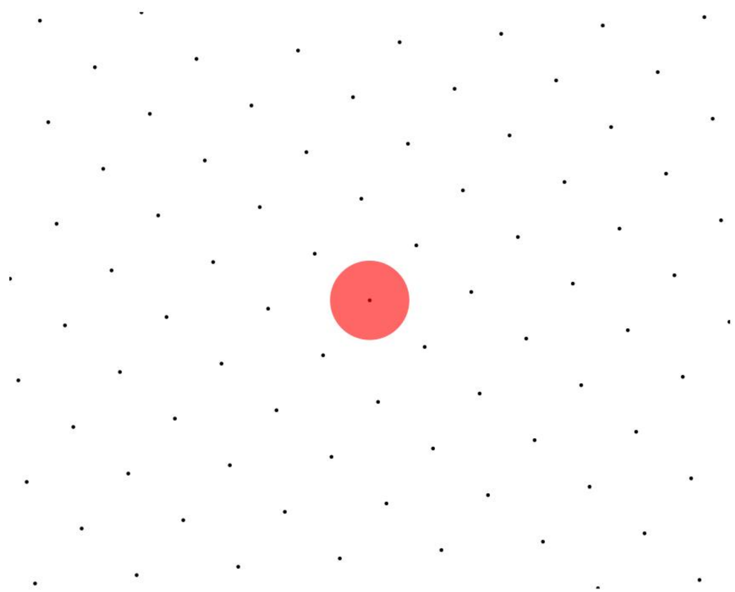

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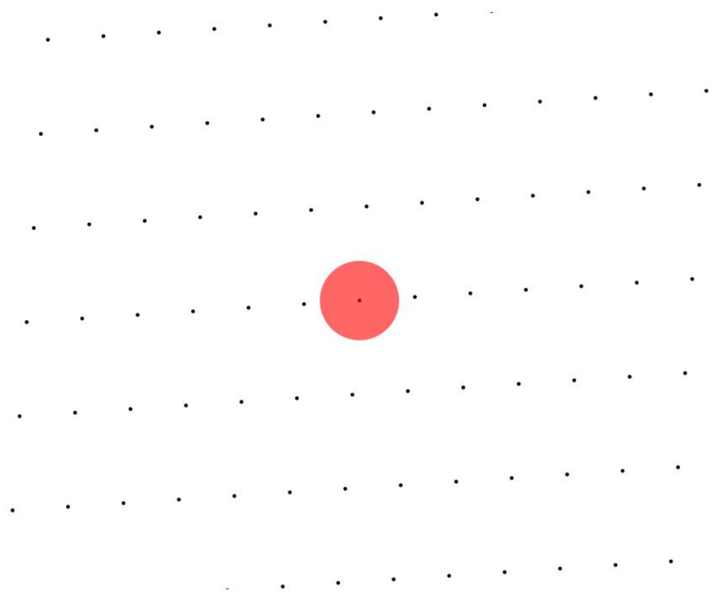

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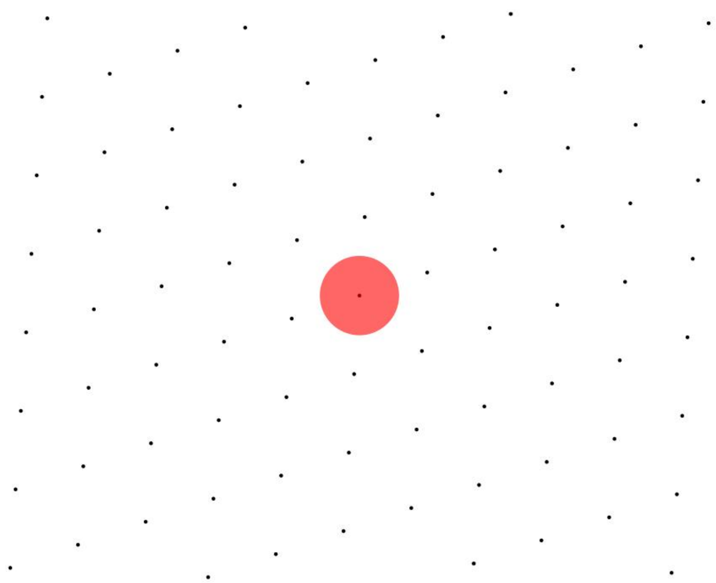

$\Phi_f(\Lambda) =$ 0 $\qquad \int_{\mathbb{R}^2} f(x)dx =$ 0.950

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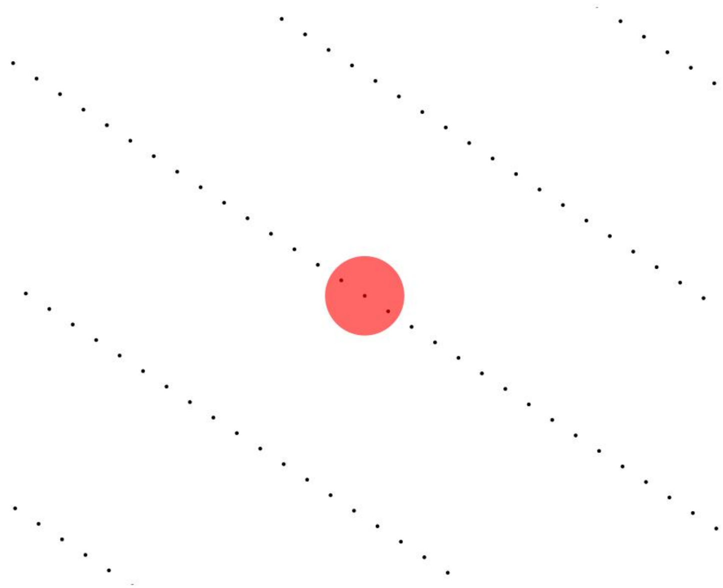

$\Phi_f(\Lambda) =$ 2 $\quad$ $\int_{\mathbb{R}^2} f(x)dx =$ 0.950

$\frac{\sum_{\Lambda \in S} \Phi_f(\Lambda)}{\#S} = \frac{2}{3} =$ 0.667

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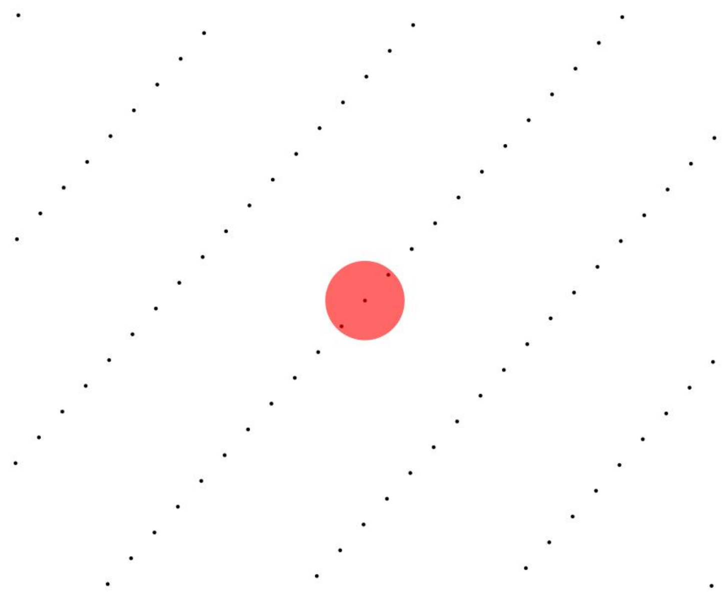

$\Phi_f(\Lambda) =$ 2 $\int_{\mathbb{R}^2} f(x)dx =$ 0.950

$\frac{\sum_{\Lambda \in S} \Phi_f(\Lambda)}{\# S} = \frac{4}{4} =$ 1.00

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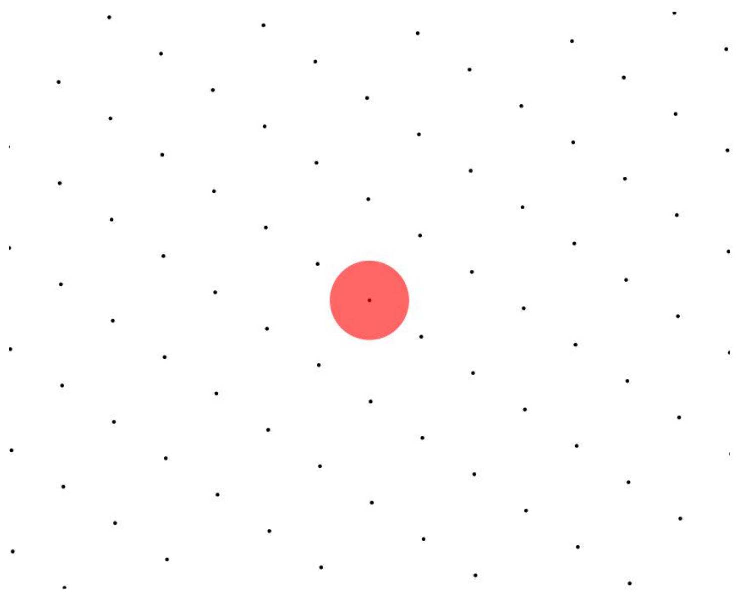

$\Phi_f(\Lambda) =$ 0 $\int_{\mathbb{R}^2} f(x)dx =$ 0.950

$\frac{\sum_{\Lambda \in S} \Phi_f(\Lambda)}{\# S} = \frac{4}{5} =$ 0.800

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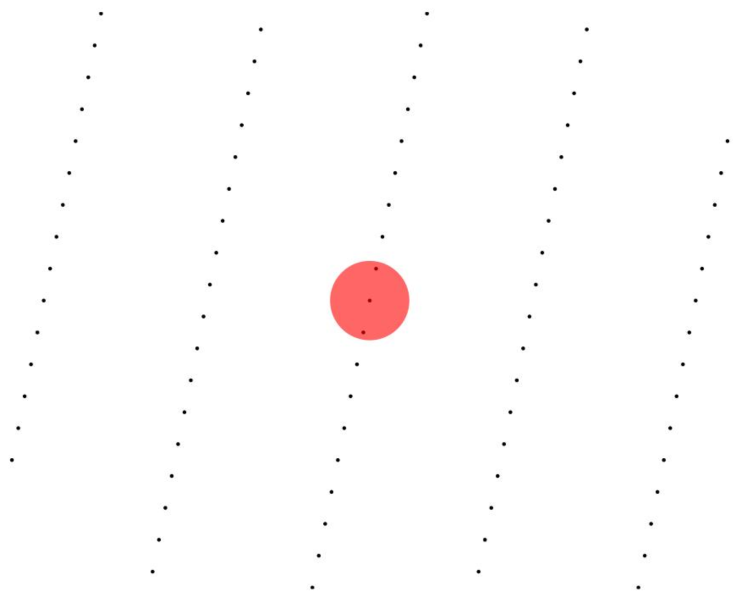

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$\frac{\sum_{\Lambda \in S} \Phi_f(\Lambda)}{\# S} = \frac{68}{100} =$ 0.680

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So we see that it is almost the integral.

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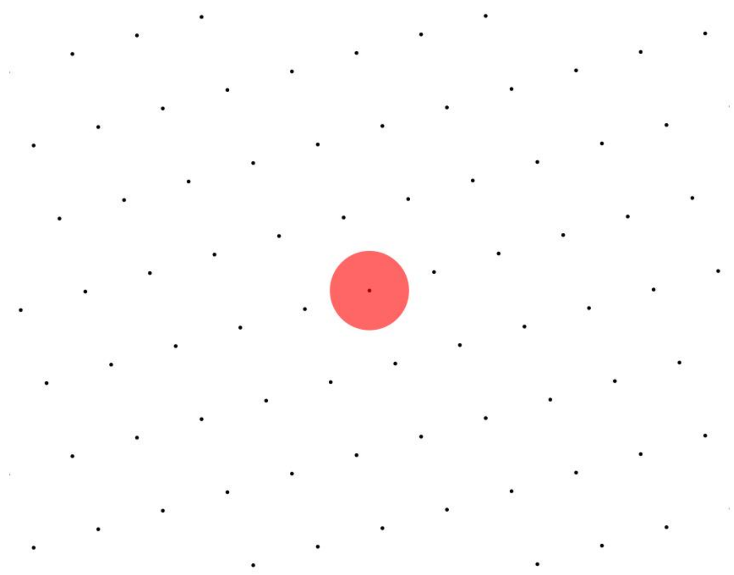

$\Phi_f(\Lambda) =$ 0 $\qquad \int_{\mathbb{R}^2} f(x)dx =$ 0.950

$\frac{\sum_{\Lambda \in S} \Phi_f(\Lambda)}{\#S} = \frac{242}{257} =$ 0.942

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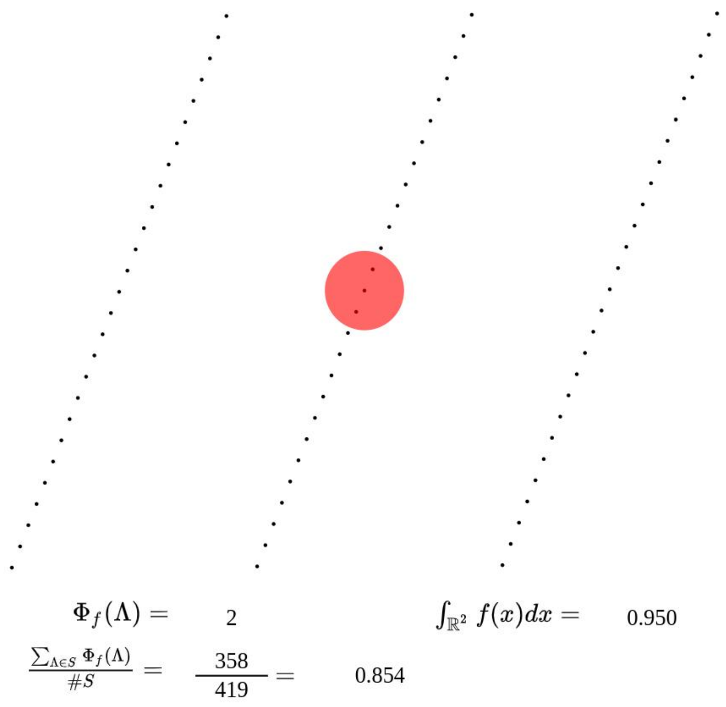

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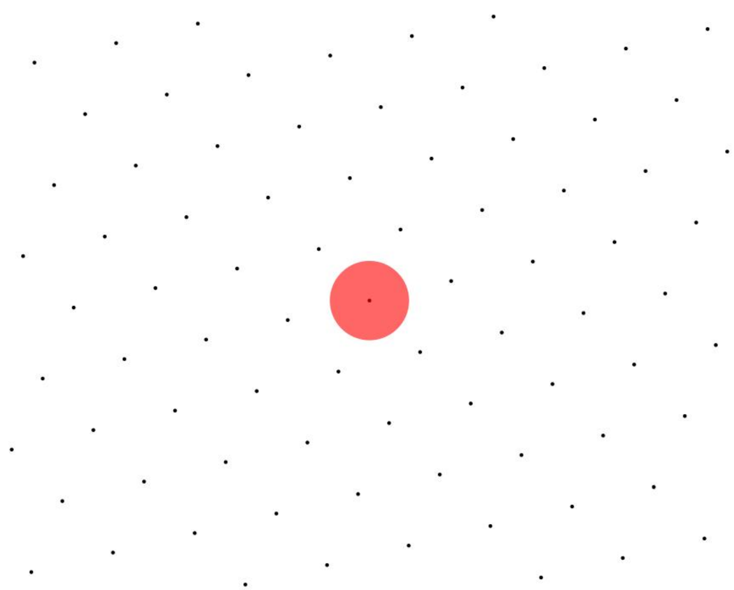

$\Phi_f(\Lambda) =$ 0

$\int_{\mathbb{R}^2} f(x)dx =$ 0.950

$\frac{\sum_{\Lambda \in S} \Phi_f(\Lambda)}{\#S} = \frac{462}{500} =$ 0.924

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What we are empirically confirming is the following.

Lattice-sum function

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Theorem *(Siegel, 1945)*

Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is a compactly supported bounded measurable function. Then, the following holds.

$$\int_{X_d} \Phi_f = \int_{SL_d(\mathbb{R})/SL_d(\mathbb{Z})} \left(\sum_{v \in g\mathbb{Z}^d \setminus \{0\}} f(v) \right) dg = \int_{\mathbb{R}^d} f(x) dx,$$

where the dx on the right-hand side is the usual Lebesgue measure on $\mathbb{R}^d$ and dg is the unique $SL_d(\mathbb{R})$-invariant probability measure on $SL_d(\mathbb{R})/SL_d(\mathbb{Z})$.

Lattice-sum function

But as you saw that for any $\Lambda \in X_d$, when f is the indicator of a ball, we must have $\Phi_f(\Lambda) \in \{0, 2, 4, 6, \dots\}$. That's because balls are symmetric.

$$v \in \operatorname{supp}(f) \cap (\Lambda \setminus \{0\}) \Rightarrow -v \in \operatorname{supp}(f) \cap (\Lambda \setminus \{0\}).$$

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If f is the indicator function of a ball of volume $2 - \varepsilon$, then this tells us that for any dimension d

$$\int_{X_d} \Phi_f = 2 - \varepsilon$$

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$$\int_{X_d} \Phi_f = 2 - \varepsilon$$

Conclusion: There exists some lattice $\Lambda \in X_d$ such that $\Phi_f(\Lambda) = 0$. That is, there is some lattice of unit covolume that intersects trivially with an origin centered ball of volume $2 - \varepsilon$.

Lattice-sum function

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For Venkatesh, the group of symmetries is always a cyclic group. For the new result, the symmetries are non-commutative.

Venkatesh's lower bound

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Define the set Y_d as

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So if f is the indicator function of an origin-centered ball with respect to a quadratic form that is invariant under $\langle \mu_n \rangle$, we have that $\Phi_f(\Lambda) \in \{0, n, 2n, 3n, 4n, \ldots\}$ for $\Lambda \in Y_d$. Such a quadratic form always exists by averaging!

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Theorem *(Venkatesh 2013)*

Let $d = 2\varphi(n)$ Suppose $f : K_{\mathbb{R}}^2 \to \mathbb{R}$ is a compactly supported bounded measurable function. Then, the following holds.

$$\int_{Y_d} \Phi_f = \int_{SL_2(K_{\mathbb{R}})/SL_2(\mathcal{O}_K)} \left(\sum_{v \in g\mathcal{O}_K^{\oplus 2} \setminus \{0\}} f(v) \right) dg = \int_{\mathbb{R}^d} f(x)dx,$$

where the dx on the right-hand side is that lebesgue measure on $\mathbb{R}^d$ that makes $\mathcal{O}_K^{\oplus 2}$ of unit covolume and dg is the unique $SL_2(K_{\mathbb{R}})$-invariant probability measure on Y_d.

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Conclusion: By setting f as the indicator function of a ball in a suitable quadratic form, we conclude that there exists some lattice $\Lambda \in Y_d$ such that $\Phi_f(\Lambda) = 0$. That is, there is some lattice of unit covolume that intersects trivially with an origin centered ball of volume $n - \varepsilon$.

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Another conclusion: $c_{2\varphi(n)} \geq n$ for all n!

Towards division algebra

The division algebra case is also very similar. Let D be a finite-dimensional division algebra over $\mathbb{Q}$. Let $\mathcal{O} \subseteq D$ be an order in D. We work in $d = 2\dim_{\mathbb{Q}} D$ dimensions. Define $D_{\mathbb{R}} = D \otimes_{\mathbb{Q}} \mathbb{R}$.

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$$SL_2(D_{\mathbb{R}}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} \text{ is a measure preserving map on } D_{\mathbb{R}}^{\oplus 2} \right\}$$

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Y_d consists of lattices that are invariant under diagonal right-multiplication by units in $\mathcal{O}$

$$g(\mathcal{O}^{\oplus 2}) = g(\mathcal{O}^{\oplus 2}) \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}, \text{ for any } \mu \in \mathcal{O}^*$$

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Theorem *(G. 2021)*

Let $d = 2[D : \mathbb{Q}]$. Suppose $f : D_{\mathbb{R}}^2 \to \mathbb{R}$ is a compactly supported bounded measurable function. Then, the following holds.

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To get packing bounds, fix a finite subgroup $G_0 \subseteq \mathcal{O}^*$ to act diagonally on the right of $\mathcal{O}^{\oplus 2}$. We get the bounds

$$c_{2 \dim_{\mathbb{Q}} D} \geq \# G_0.$$

Towards division algebra

In fact we only need to find finite subgroups that live in $\mathbb{Q}$-division algebras. The order $\mathcal{O}$ can be aligned according to the finite group. Fortunately, there exists a complete classification of such finite subgroups due to Amitsur, 1955.

Towards division algebra

FINITE SUBGROUPS OF DIVISION RINGS

BY

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1. **Introduction.** The problem of determining all finite groups which can be embedded in the multiplicative group of the nonzero elements of division rings was first proposed and partially solved in [6] by I. N. Herstein. It was shown there that the only finite subgroup of division rings of finite characteristic are cyclic, and that the subgroups of odd order of division rings of characteristic zero are of a very special type [6, Theorem 5]. In particular, the odd subgroups of the real quaternions are all cyclic. This brought I. N. Herstein to the conjecture that all odd subgroups of division rings are cyclic.

The purpose of the present paper is to determine completely all subgroups (of even and odd order) of division rings. These groups are classified in five classes connected in some way to the finite groups of rotations of the 3-Euclidean sphere. Among others we disprove the conjecture of Herstein and exhibit infinitely many finite subgroups of division rings of odd order. In particular the minimal order of an odd noncyclic group contained in a division ring is 63.

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LEMMA 7. *Let x, y be two integers and let $\beta=\beta(q,\ x-1)\geqq 1$* (i.e., $x\equiv 1 \pmod q$)) *and $\beta_\nu=(q,\ y)\geqq 0$ for a prime q. Then:* (1) *if* $q\neq 2$ or $\beta\geqq 2$ (*i.e.,* $x\equiv 1 \pmod 4$) *in case* $q=2$) *then* $\beta(q,\ x^\nu-1)=\beta+\beta_\nu$. (2) *If $q=2$ and $\beta=1$ then: $\beta_\nu=0$ implies that $\beta(2,\ x^\nu-1)=1$, and $\beta_\nu\geqq 1$ implies that $\beta(2,\ x^\nu-1)=\beta_\nu+i+1$ where $x=1+2+\cdots+2^i+2^{i+2}x_1$, $i\geqq 1$.*

The proof is by induction on β_ν. If $\beta_\nu=0$, let $x=1+q^\beta z$, $(z,\ q)=1$. Then, $(1+q^\beta z)^\nu=1+q^\beta yz+$terms with higher powers of q, and this case is proved since $(yz,\ q)=1$. Let $y=qy'=q^{\beta_\nu}y''$, $(y'',\ q)=1$, and $\beta_\nu\geqq 1$. By induction it follows that $x^{\nu'}=1+q^{\beta+\beta_\nu-1}u$, $(u,\ q)=1$. Hence,

$$x^\nu=(1+q^{\beta+\beta_\nu-1}u)^q=1+q^{\beta+\beta_\nu}u+\cdots+C_{q,\nu}\,q^{\nu(\beta+\beta_\nu-1)}u^{q-\nu}+\cdots.$$

The highest power of q dividing

$$C_{q,\nu}\,q^{\nu(\beta+\beta_\nu-1)}\ u^{q-\nu}$$

is $\nu(\beta+\beta_\nu-1)+1$ if $1\leqq\nu<q$ and it is $q(\beta+\beta_\nu-1)$ if $\nu=q$. Hence, the exceptional case to the proof of this lemma may occur if $q(\beta+\beta_\nu-1)=1\cdot(\beta+\beta_\nu-1)+1$. Equivalently, $(q-1)(\beta+\beta_\nu)=q$. This may happen only if $q=2$ and $\beta+\beta_\nu=2$. This proves the first part of the lemma.

To prove the second part it suffices to show it only for $y=2$. For, if x is

THEOREM 5. *A necessary and sufficient condition that $\mathfrak{A}_{m,r}$ is a division algebra is that* (3C) *or* (3D) *holds and either:*

(1) $n=s=2$ *and* $r\equiv -1 \pmod m$ *or,*

(2) *For every prime $q\mid n$ there exists a prime $p\mid m$ such that $q\nmid n_p$ and that one of the following holds:*

(2a) $p\equiv 1 \pmod 4$ *or* $q\neq 2$ *and* $\beta(q,\ s)\geqq\beta(q,\ p-1)+\mathrm{Max}_i\,\beta(q,\ \gamma_i)$.

(2b) $p\equiv 1+2+\cdots+2^i \pmod{2^{i+2}}$, $i\geqq 1$ *and* $q=2$, (3C) *holds; and* $\beta(2,\ s)\geqq i+1+\mathrm{Max}\,\{1,\ \beta(2,\ \gamma_i)\}$ *if* $s\equiv 0 \pmod 4$, *but if* $s\not\equiv 0 \pmod 4$ *then all* $\beta(2,\ \gamma_i)=0$; i.e., *all γ_i are odd integers.*

(2c) $p=q=2$, (3D) *holds, $m/4$ and all γ_i are odd integers.*

Proof. Evidently, (1) of Theorem 4 and condition (1) of the present theorem are equivalent. The proof of this theorem will be achieved by showing that the condition (2a) is equivalent to (I_1), (2b) is equivalent to (II_2) and that (2c) and (b) of Theorem 4 are equivalent. This will prove the theorem since it was shown that (I_1) and (I_2) together are equivalent to (a) of Theorem 4.

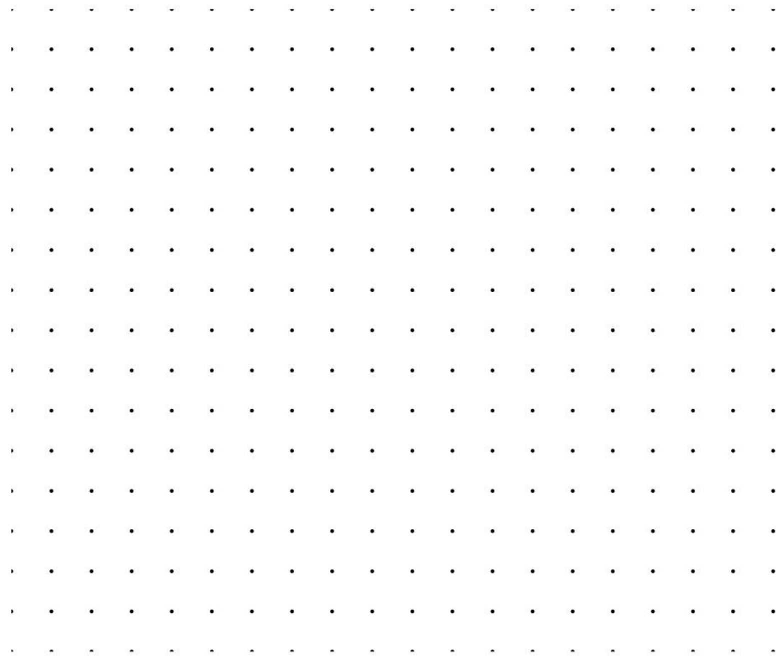
Substituting (III_1) in (II) we obtain by (IV) that (I_1) is equivalent to the

Towards division algebra

For brewity, we will not discuss classification of of finite subgroups ofdivision rings. Instead, let us talk about effectiveness of these results.

Controlled randomness

Visualizing in $\mathbb{R}^2$



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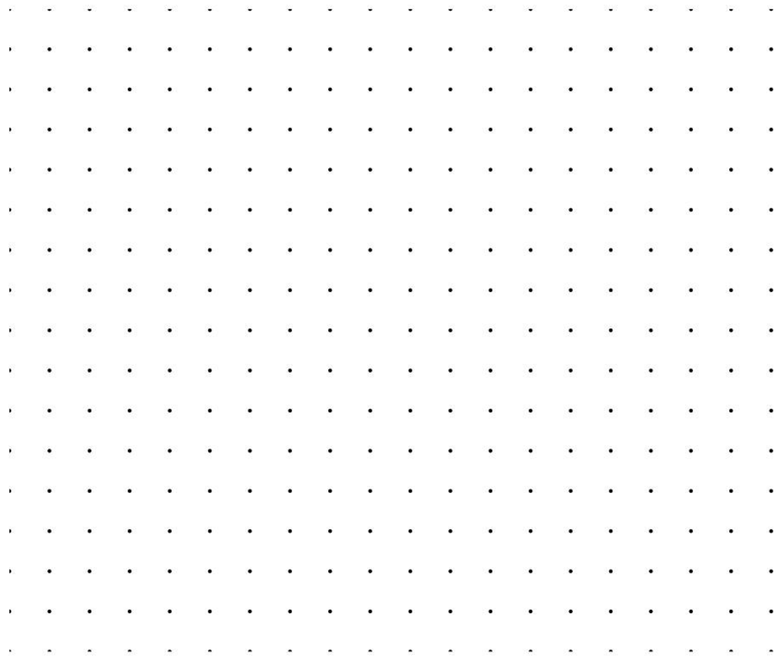

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$\mathbb{F}_p^d \setminus \{0\}$ is a disjoint union of lines.

$$\mathbb{F}_p^d \setminus \{0\} = \bigsqcup_{v \in (\mathbb{F}_p^d \setminus \{0\})/\mathbb{F}_p^*} (\mathbb{F}_p \, v \setminus \{0\})$$

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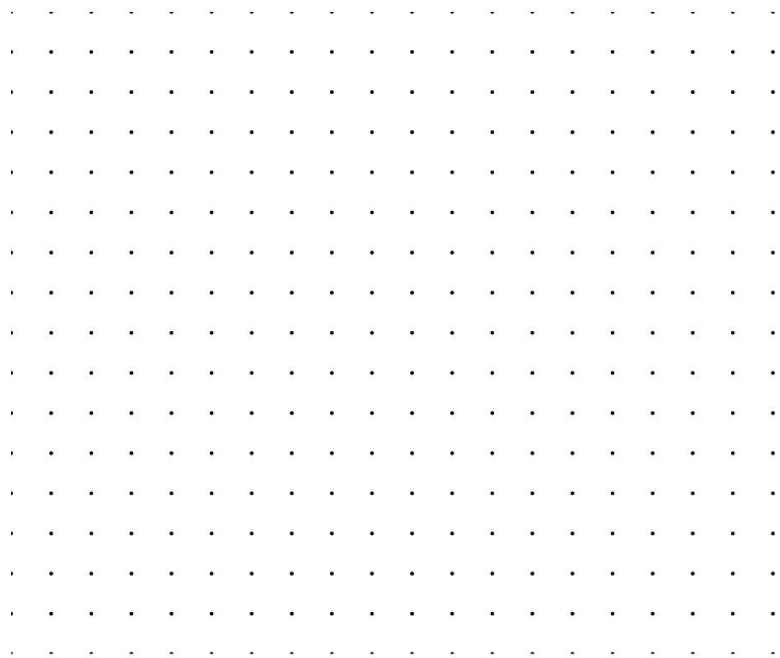

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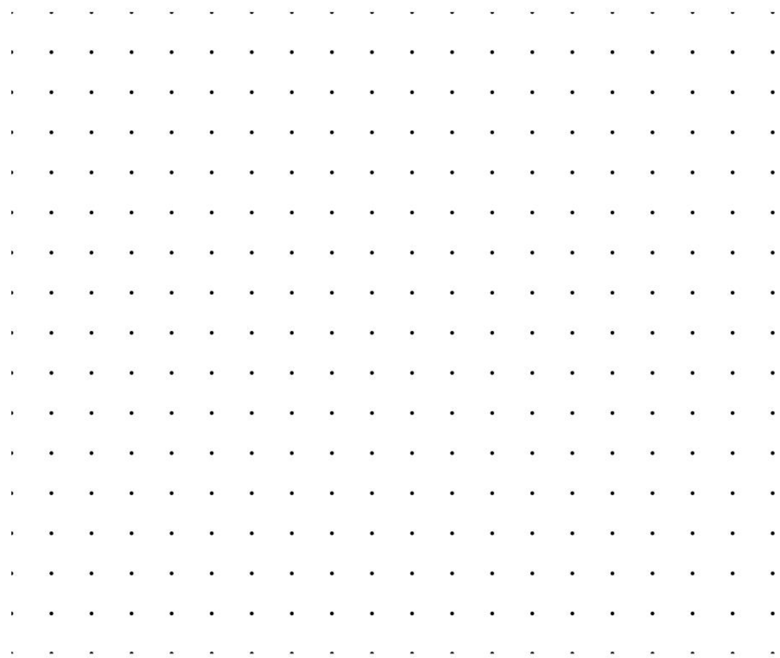

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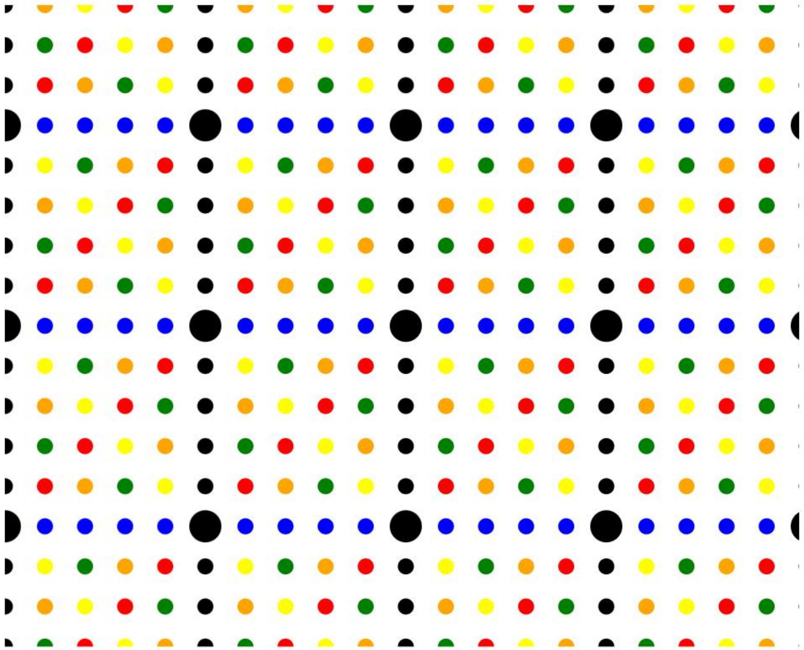

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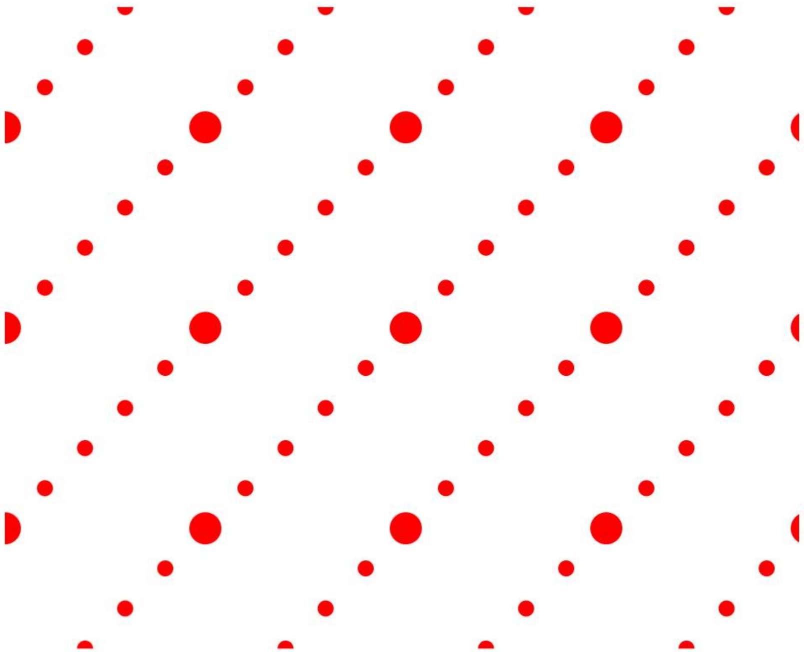

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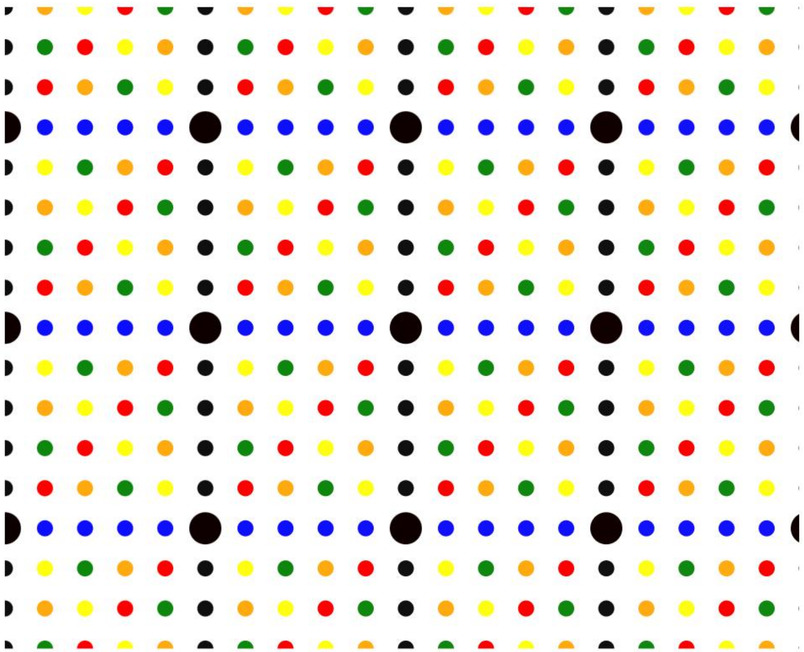

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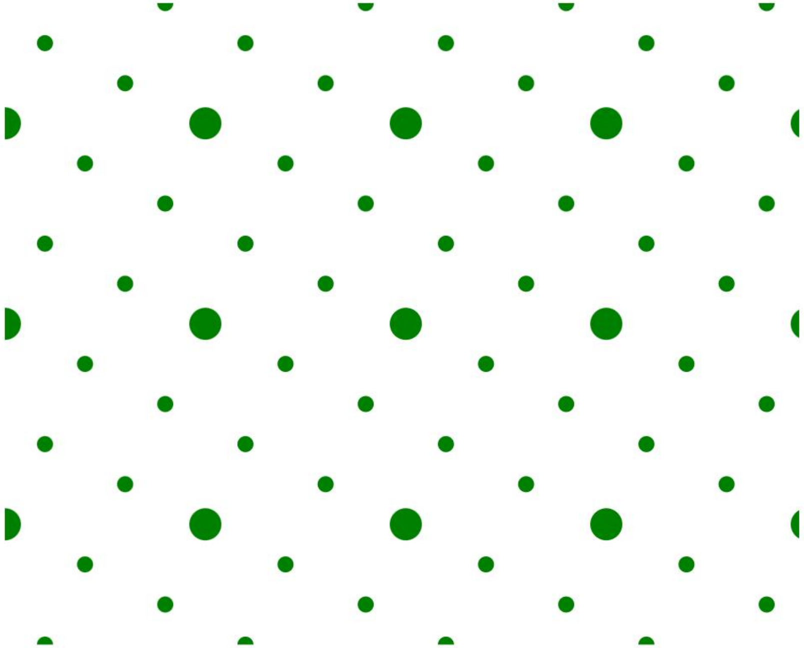

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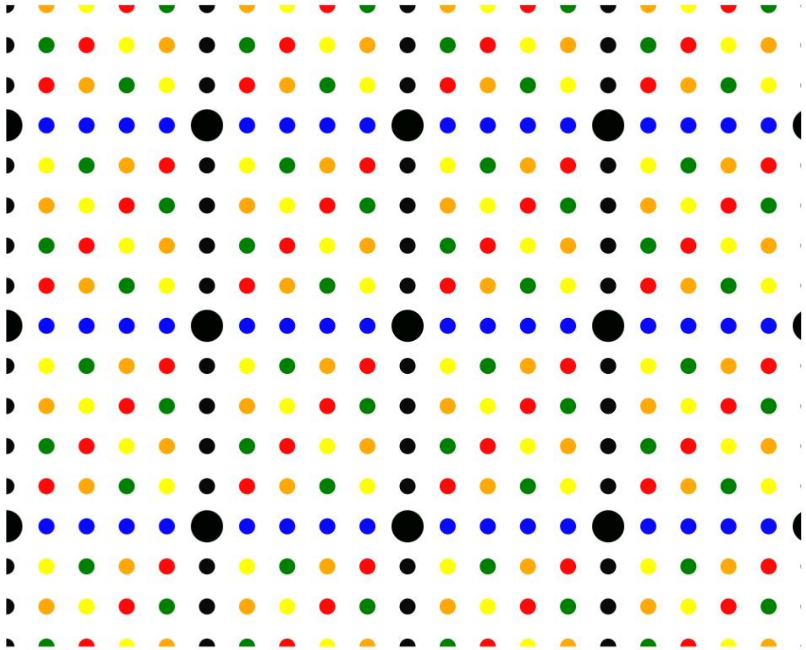

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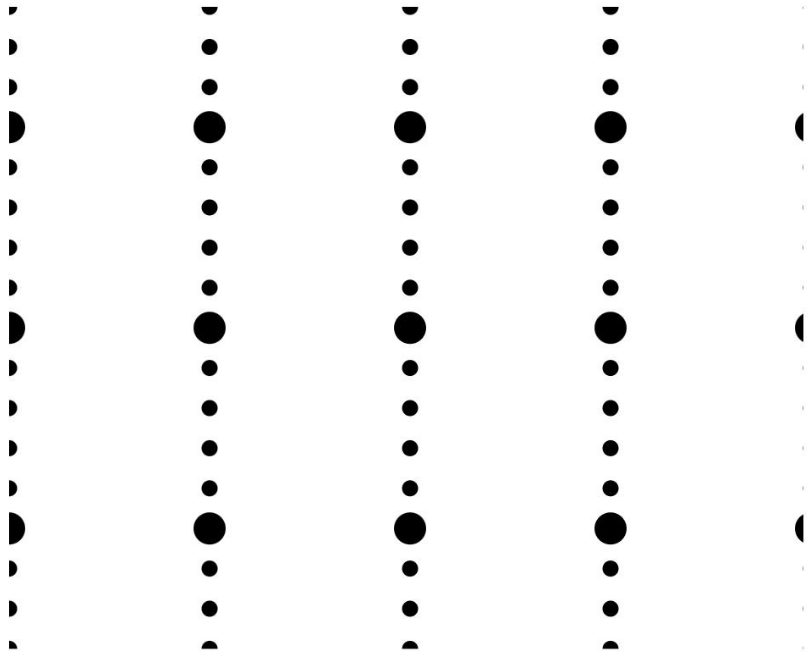

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Controlled randomness

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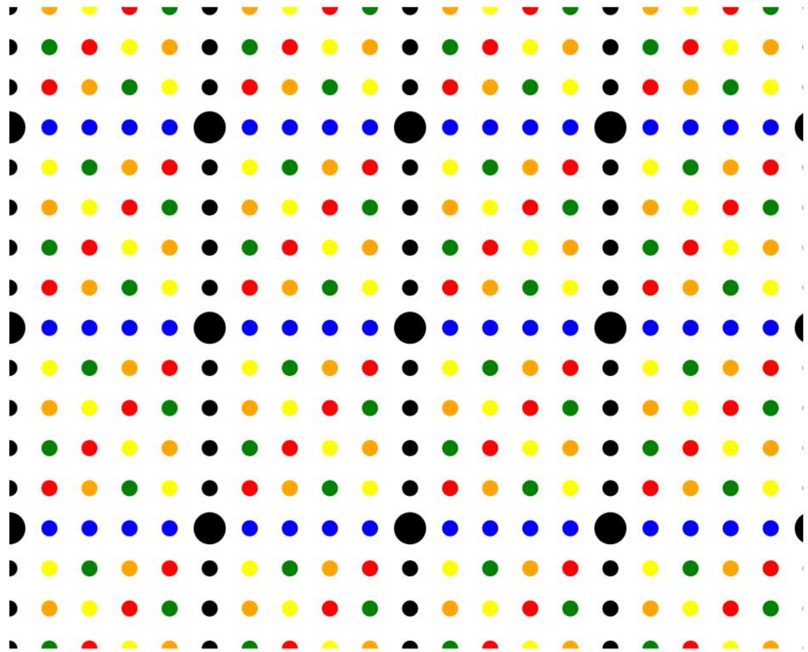

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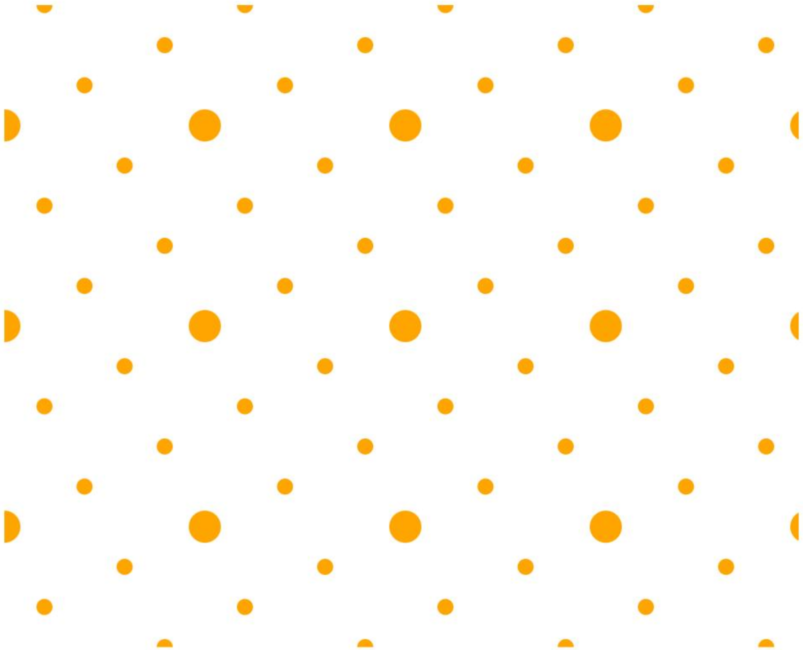

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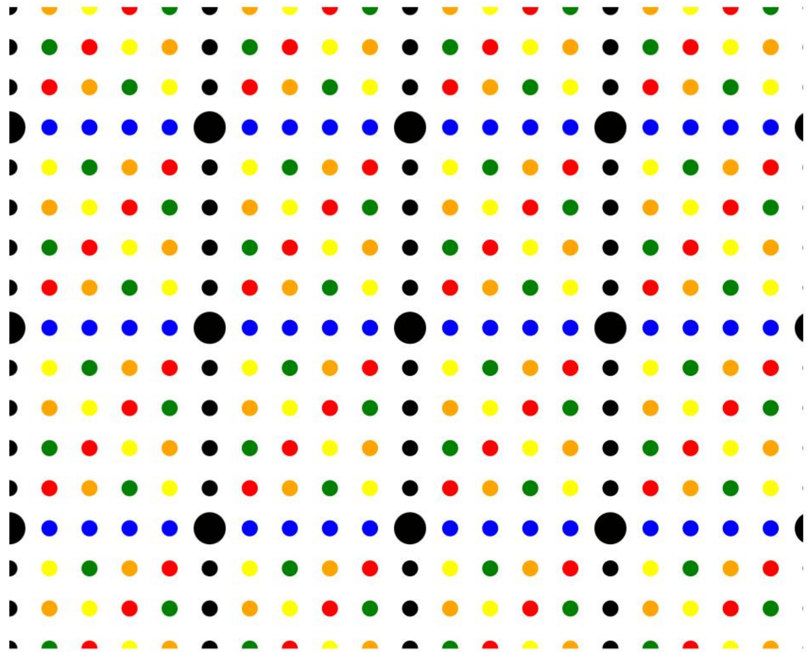

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What do you expect this quantity to be as $p \to \infty$?

$$\frac{1}{\#\mathcal{L}_p} \sum_{\Lambda \in \mathcal{L}_p} \Phi_f(\Lambda) = \frac{1}{\#\mathcal{L}_p} \sum_{\Lambda \in \mathcal{L}_p} \left(\sum_{v \in \Lambda \setminus \{0\}} f(v) \right)$$

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Theorem *(Rogers, 1947)*

Let p be an arbitrary prime, $\mathbb{F}_p$ be the field with p elements and let $\pi_p : \mathbb{Z}^d \to \mathbb{F}_p^d$ be the natural coordinate-wise projection map. Let $\mathcal{L}_p$ be the set of sub-lattices of $\mathbb{Z}^d$ that are pre-images of one-dimensional subspaces in this projection map scaled to become unit covolume, i.e.

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Consider a compactly supported continuous function $f : \mathbb{R}^d \to \mathbb{R}$ and the lattice-sum function $\Phi_f : X_d \to \mathbb{R}$. Then the following holds.

$$\lim_{p\to\infty} \left[\frac{1}{\#\mathcal{L}_p} \sum_{\Lambda\in\mathcal{L}_p} \Phi_f(\Lambda)\right] = \int_{\mathbb{R}^d} f(x)dx.$$

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After using the integrality gap lemma, this is a constructive proof of $c_d \geq 2$.

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Suppose D is a $\mathbb{Q}$-division ring, K be the center of the ring and $\mathcal{O}$ be an $\mathcal{O}_K$-order in D. Let $[D:K]=n^2$.

A prime ideal of an $\mathcal{O}$ is a proper two-sided ideal $\mathfrak{p}$ in $\mathcal{O}$ such that $K \cdot \mathfrak{p} = D$ and such that for every pair of two sided ideals S, T in $\mathcal{O}$, we have that $S \cdot T \subset \mathfrak{p}$ implies $S \subset \mathfrak{p}$ or $T \subset \mathfrak{p}$.

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Important property: For all but finitely many primes $\mathfrak{p}$ of $\mathcal{O}$, the quotient $\mathcal{O}/\mathfrak{p}\mathcal{O}$ is isomorphic to $M_n(\mathbb{F}_q)$, where $\mathcal{O}_K/\mathcal{O}_K \cap \mathfrak{p} \cong \mathbb{F}_q$.

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Hence we get countably many projection maps $\pi_{\mathfrak{p}} : \mathcal{O} \to M_n(\mathbb{F}_q)$.

Controlled randomness

Theorem *(G., Serban, 2021)*

Let $d = 2[D:\mathbb{Q}]$. Let $\mathfrak{p} \subseteq \mathcal{O}$ be a prime as above and let $\pi_{\mathfrak{p}} : \mathcal{O}^t \to M_n(\mathbb{F}_q)^2$ be the projection map as above (on two copies of $\mathcal{O}$). Consider the set of sub-lattices of $\mathcal{O}^2$ that are pre-images of $M_n(\mathbb{F}_q)$-submodules of $\mathbb{F}_q$-dimension $n(2n-1)$, i.e.

$$\begin{aligned} \mathcal{C}_{\mathfrak{p}} &= \{C \subseteq M_n(\mathbb{F}_q)^2 \mid C \text{ is a } M_n(\mathbb{F}_q)\text{-submodule } \simeq (\mathbb{F}_q^n)^{\oplus(2n-1)}\}, \\ \mathcal{L}_{\mathfrak{p}} &= \{\beta_{\mathfrak{p}} \pi_{\mathfrak{p}}^{-1}(C) \mid C \in \mathcal{C}_{\mathfrak{p}}\}, \quad \beta_{\mathfrak{p}} = q^{-1/nmt} \end{aligned}$$

Consider a compactly supported continuous function $f : \mathbb{R}^d \to \mathbb{R}$ and the lattice-sum function $\Phi_f : X_d \to \mathbb{R}$. Then the following holds.

$$\lim_{\#\mathbb{F}_q \to \infty} \left[\frac{1}{\#\mathcal{L}_{\mathfrak{p}}} \sum_{\Lambda \in \mathcal{L}_{\mathfrak{p}}} \Phi_f(\Lambda) \right] = \int_{\mathbb{R}^d} f(x) dx.$$

where the dx on the right-hand side is that Lebesgue measure on $\mathbb{R}^d$ that makes $\mathcal{O}^2 \subseteq (D \otimes \mathbb{R})^2 \simeq \mathbb{R}^d$ of unit covolume.

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We can therefore prove the following from above.

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Let $m_k = \prod_{\substack{p \leq k \text{ prime} \\ 2 \nmid \text{ord}_2 p}} p$ and set $n_k := 8\varphi(m_k)$. Then for any $\varepsilon > 0$ there is an effective constant c_ε such that for $k > c_\varepsilon$ a lattice Λ in dimension n_k with density

$$\Delta(\Lambda) \geq (1 - \varepsilon)\frac{24 \cdot m_k}{2^{n_k}}$$

can be constructed in $e^{4.5 \cdot n_k \log(n_k)(1+o(1))}$ binary operations. This construction leads to the asymptotic density of $\Delta(\Lambda) \geq (1 - e^{-n_k})\frac{3 \cdot n_k (\log\log n_k)^{7/24}}{2^{n_k}}$.

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The condition of $2 \nmid \mathrm{ord}_p 2$ has to do with division ring contructions. Details can be given on request!

Open questions and ongoing work

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Open problem:

What are explicit descriptions of lattices that prove at least Minkowski's lower bounds as $d \to \infty$? That is, what are the lattices that have the most optimal packing density in large dimensions?

In terms of coding theory, this problem is to find explicitly lattices that achieve "goodness"

$$\Delta(\Lambda^{(d)})^{\frac{1}{d}} = \frac{r_{\text{pack}}(\Lambda^{(d)})}{r_{\text{eff}}(\Lambda^{(d)})} \geq \frac{1}{2},$$

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Need to decrease the search space to get smaller running times.

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Explicitly describe the higher moments of these random lattices.
That is, give a mean value formula for $\left(\sum_{v\in\Lambda} f(v)\right)^2, \left(\sum_{v\in\Lambda} f(v)\right)^3, \ldots$ for any of these random sets of lattices.

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Such work has not been satisfactorily generalized to $SL_t(K)/SL_t(\mathcal{O}_K)$. This is an ongoing project jointly with V. Serban and M. Viazovska.

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Question:

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For division rings, this is still open.

Thank you for your attention!

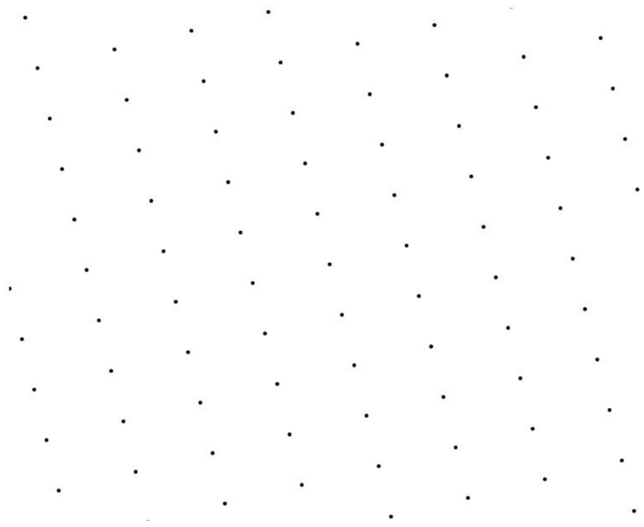
Email:
`nihar.gargava@epfl.ch`

Based on arXiv preprints:
`2107.04844, 2111.03684`

Slides:
`nihargargava.com/cap_2022`

How does it work?:
This presentation and all the animations are written in `html` and `javascript` using reveal.js and d3.js.

Feel free to contact for questions/comments.



Appendix: How to generate random 2-dimensional lattices

To the map $\psi : [\pi/3, 2\pi/3] \times]0, 1] \to \mathbb{H}$ given by $\psi(a, b) = \cos(a) + i \sin(a)/b$ is a measure preserving map!

It maps the rectangle bijectively to a fundamental domain of $\mathbb{H}/SL_2(\mathbb{Z})$.

Using this, the following map randomly generates a lattice.

$$\psi_1 : [0, 2\pi] \times [\pi/3, 2\pi/3] \times]0, 1] \to SL_2(\mathbb{R})$$

$$\psi_1(x, y, z) = \begin{bmatrix} 1 & \cos(y) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{\sin(y)}{z}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(\frac{\sin(y)}{z}\right)^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

This only works for $d = 2$. It is not known how to generalize this to higher dimensions!