Statistical systems on RRG

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Plan of the talk

- Motivation #1
- Derivation of the matrix model.
- Solution to the matrix model via elliptic curve
- Check of the limits
- New double scaling and Lambert function
- Disc partition functions
- Open questions and new directions
- Motivation #2

The model in continuum

Mass of the fermion

$$Z(\Lambda_0, \eta, m_0) = \int Dg(\sigma) \int D\psi(\sigma) D\bar{\psi}(\sigma) \exp\Big[-\int_D d^2\sigma \sqrt{g} \Big(g^{\alpha\beta} \partial_\alpha \bar{\psi} \partial_\beta \psi + m_0^2 \bar{\psi}\psi + \Lambda_0 + \kappa R \Big) - \eta_0 \int_{\partial D} ds \, [g(\sigma(s))]^{1/4} \Big] \qquad (1.1)$$

Boundary cosmological constant

Cosmological constant

2d gravity description.Reminder

- Consider the non-critical string . According to Polyakov we have to integrate over the worldsheet metrics which is taken into account via Liouville action

- Dimension of the target space D for string enters via power of determinant upon the integration over matter field.

- D=0 target space (c=0 theory) — no determinant

- D=-2 ,(c= -2 theory) — «fermionic determinant» positive power

--- Instead of the integration over the metrics — summation over the graphs
Kazakov 85, David 85

– Summation over the graphs(RRG) gets substituted by the large N matrix model
Whose perturbation theory reproduces the combinatorics of the summation over
The graphs. Leading terms in N — planar graphs

Partition functions

$$Z \equiv \log\zeta = \sum_G N^{2-2g} \lambda^{|G|} \det[m^2 + \Delta(G)]$$

--Sum is over random regular graphs (RRG) q=3

--pure gravity c=0 limit — no determinant at all

– c=-2 limit —massless determinant with zero mode removed

– finite mass- interpolation between two limits

Useful formula for the c=-2 limit

$$Z_{c=-2}(\lambda) = \frac{d}{dm^2} Z(\lambda, m^2 = 0)$$

Matrix-forest theorem (Chelnokov,Kelmanis 74)

$$\det[m^2 + \Delta(G)] = \sum_{F=(F_1 \ldots F_l)\in G} \prod_{i=1}^{l} m^2 V(F_i)$$

Number of nodes
In the tree

Mass is the generating parameter for the counting of the number of trees
In the forest. Hence the mass dependence of the partition function governs
the number of the components the surface is «broken at» due to the back reaction of
fermions.

Matrix models for 2d gravity coupled to matter

– Pure 2d gravity. Cubic potential Brezin-Itzykson-Parisi-Zuber 78' Kazakov 85, David 85

- 2d gravity coupled to D=-2 matter David 86, Kostov- Mehta 87, Boulatov-Kostov-Migdal-Kazakov 86

– 2d gravity interacting with Ising and Potts models. Multimatrix models
Kazakov 86, Boulatov -Kazakov 86, Kazakov -87

– Double scaling limit Douglas-Shenker, Kazakov-Brezin, Gross-Migdal 90'

- Minimal strings. C=1 string Seiberg-Shih, Douglas, Klebanov, Maldacena, Seiberg, Kutasov, Martinec 2001-2004

-2d gravity interacting with massive spilness fermions + four-fermion interaction at special coupling . Non-analytic potential strongly depends on the degree of nodes. The model is equivalent to the ensemble of unrooted trees in the forest.
Caracciolo, Jacobsen, Saleur, Sokal -04'
Caracciolo, Sportiello -09'
Bondesan, Caracciolo, Sportiello — 17'

Matrix model for massive fermions coupled to 2d gravity. Derivation via Parisi-Sourlas trick

Let us use the Parisi-Sourlas trick applied for massless case by David and Kostov-Mehta. Reduce the dimension from D=0 to D=-2 by adding the anticommuting coordinates.

$$\zeta = \int D^{N^2}\Phi(\theta)\, e^{NS(\Phi)} = \int d^{N^2}\phi\, d^{2N^2}\psi\, d^{2N^2}\epsilon\, e^{NS(\Phi)}$$

$$\Phi(\theta) = \phi + \bar{\theta}\psi + \theta\bar{\psi} + \bar{\theta}\,\theta\epsilon$$

Action for superfield depending on mass and cosmological constant

$$S(\Phi) = \mathrm{tr}\int d^2\theta\left(-\frac{1}{2}\Phi^2(\theta) - \frac{1}{2}\partial_\theta\Phi(\theta)\partial_{\bar{\theta}}\Phi(\theta) + \frac{\lambda}{3}e^{-\frac{3}{2}M^2\bar{\theta}\theta}\left[\Phi(\theta)\right]^3\right)$$

Potential for the matrix model

$$S(\Phi) = \mathrm{tr}\left[\frac{1}{2}\epsilon^2 - (\phi - \lambda\phi^2)\epsilon - \frac{1}{2}\lambda M^2\phi^3 + \bar{\psi}\psi - \lambda(\bar{\psi}\phi\psi + \bar{\psi}\psi\phi)\right].$$

$$\zeta = \int d^{N^2}\phi \ \det(1 - 2\lambda\phi) \ e^{N\,\mathrm{tr}\,[-\frac{1}{2}(\phi-\lambda\phi^2)^2 + \frac{1}{2}\lambda M^2\phi^3]}$$

$$X = \phi - \lambda\phi^2 \qquad\qquad *$$

Introduce the change of variables to cancel determinant

$$\zeta = \int d^{N^2}X \ e^{N\,\mathrm{tr}\,[-\frac{1}{2}X^2 + \frac{1}{2}\lambda M^2\phi^3(X)]}$$

We select only one root for the quadratic equation

$$S_X = \mathrm{tr}\left[-\frac{1}{2}X^2 + \frac{\lambda M^2}{2}\left(\frac{1-\sqrt{1-4\lambda X}}{2\lambda}\right)^3\right].$$

Shape of potential in the matrix model.
Two extrema and wall.

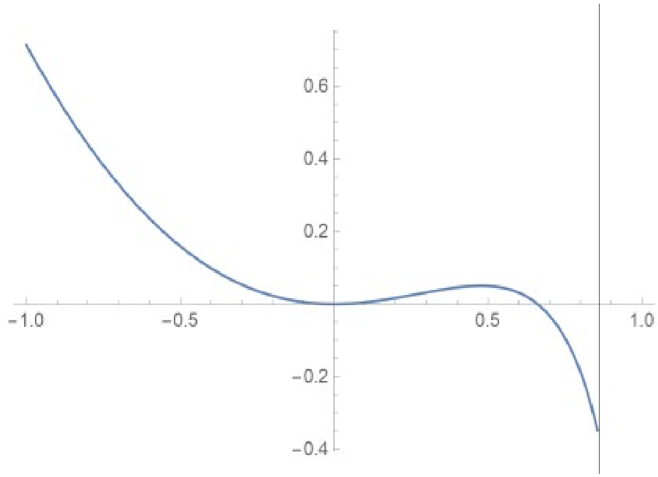


Figure 1: Potential for eigenvalue for $M = 1$ (without Vandermonde term). The eigenvalues should be confined in this well, below the branchpoint $x = 1/4$).

Matrix model for massive fermions coupled to 2d gravity. Derivation via matrix-forest theorem

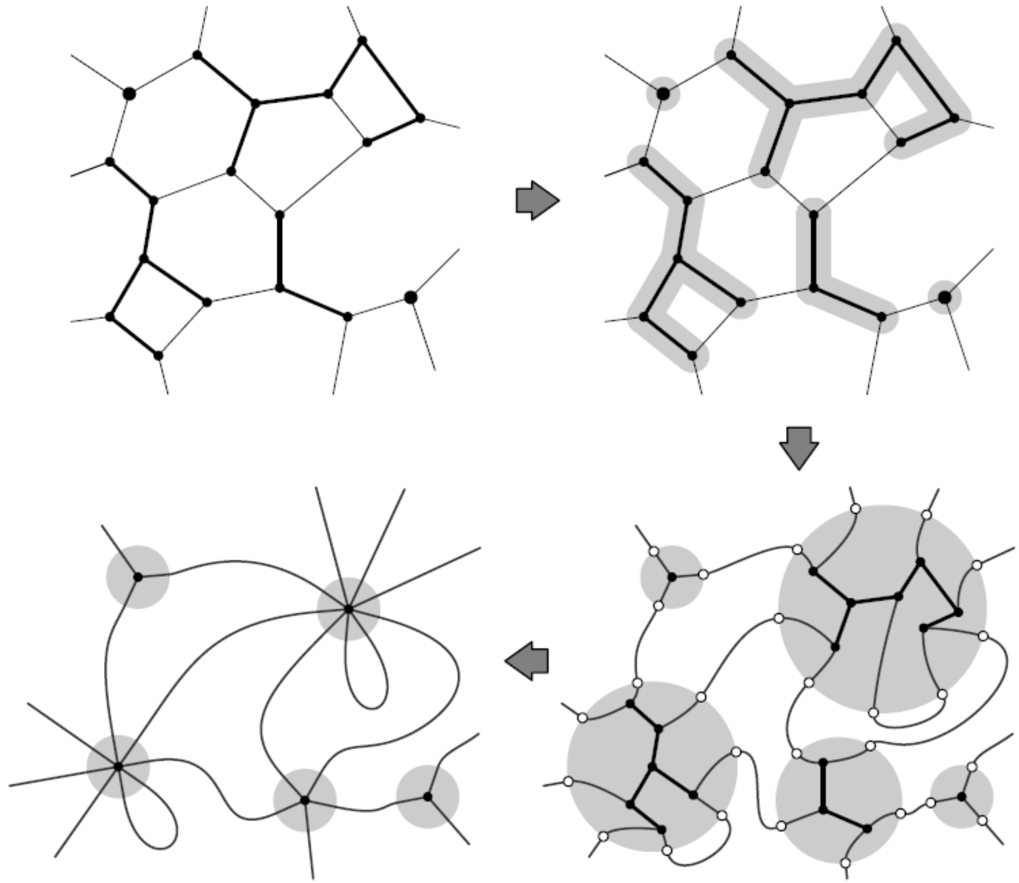
Matrix-tree theorem of Kirchoff det' L =# (spanning trees) for any graph G where a zero-mode is removed at lhs

Generalization to the matrix-forest theorem Chelnokov-Kelmans 74
David-Duplantier 88

All trees are rooted

$$\det[\frac{1}{2}\hat{M}^2 + \frac{1}{2}\Delta(G)] = \sum_{F=(F_1 \ldots F_l) \in G} \prod_{i=1}^{l} M^2 V(F_i) \qquad (3)$$

where $V(F_i)$ is the number of nodes in the tree F_i and M^2 counts the number of trees in the forest.



Step 1 — combinatorial brute force derivation of potential for the unrooted trees in a forest from matrix-forest theorem   Bondesan, Caracciolo,Sportiello 17'

Step 2- derivation of the potential for the rooted trees in the forest

$$\tilde{V}(z) = \frac{1}{12z^2}\left(-6z^2 + (1-4z)^{3/2} + 6z - 1\right) = \sum_{n=1}^{\infty} \tilde{c}_n z^{n+2}, \qquad \text{where } \tilde{c}_n = \frac{(2n)!}{n!(n+2)!}.$$

$$\longrightarrow \frac{\lambda}{3}\left(\frac{1-\sqrt{1-4\lambda X}}{2\lambda}\right)^3 = \lambda\partial_\lambda\left(X^2\tilde{V}(\lambda X)\right)$$

Critical curve

At the critical curve the large area surfaces dominate. For c=0 and C=-2 there are critical points

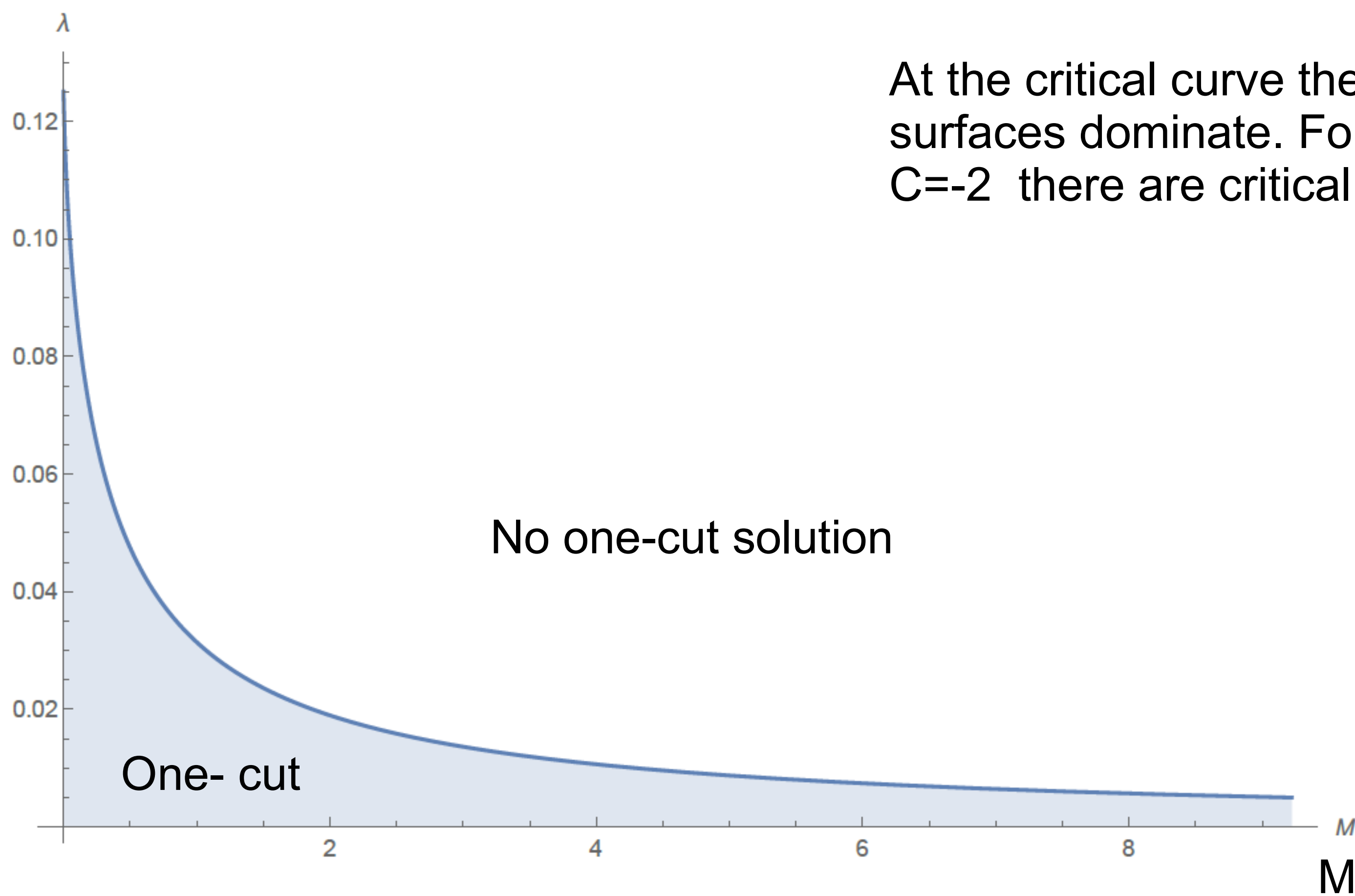
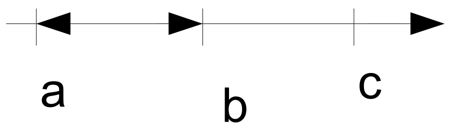


Figure 3. The critical curve $\lambda_c(M)$. The shaded area below the curve shows the allowed physical region on the (M, λ) plane where the density is real and positive.

One-cut solution



Resolvent

$$G(x) = \frac{1}{2}\left(x - \sqrt{(x-a)(x-b)}\right)$$
$$+ \frac{9}{2}cM^2\left(1 + \frac{2\left(a - \frac{2c}{3}\right)K}{\pi\sqrt{c}\sqrt{c-b}}\sqrt{\frac{x-b}{x-a}} - \frac{\left(x - \frac{2c}{3}\right)\vartheta_4'\left(\frac{\pi u}{2K}\right)}{\sqrt{c}\sqrt{x-c}\,\vartheta_4\left(\frac{\pi u}{2K}\right)}\right)$$

$$\mathrm{sn}^2 u = \frac{x-c}{x-a}.$$

Argument of elliptic K(m) $\quad m = \frac{a-b}{c-b}$

$$B^2 = c - b\ , \ \ m = \frac{a-b}{c-b}$$

Equations providing the correct asymptotics of G(x)

$$\pi B^3(m-2) - 18B^2\sqrt{c}M^2E + 2\pi Bc\left(9M^2+1\right) - 18c^{3/2}M^2K = 0$$

$$3\pi B^4m^2 - 36B^3\sqrt{c}M^2((m-2)E - 2(m-1)K) + 108Bc^{3/2}M^2((m-2)K + 2E) - 48\pi = 0$$

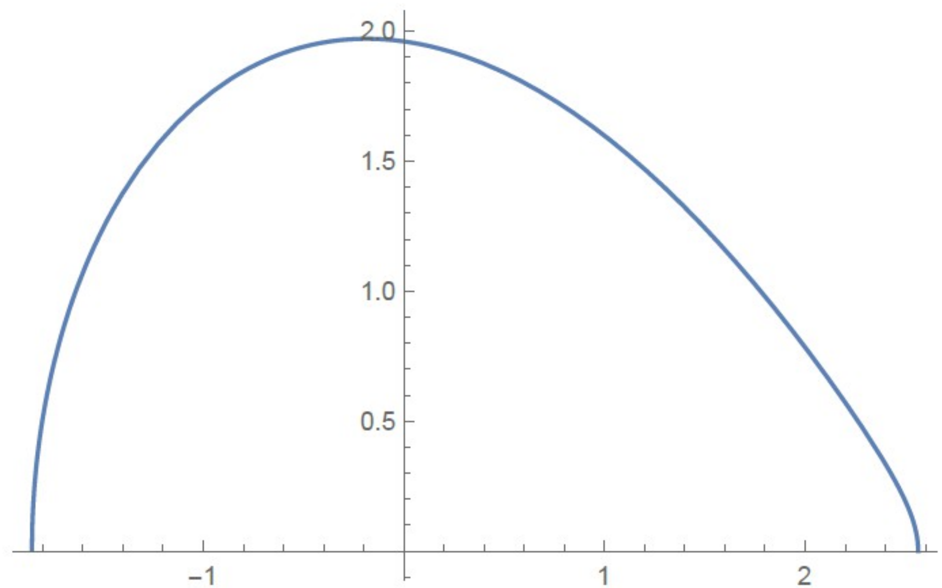


Figure 2. The density $\rho(x)$ given by (4.10) on the $[b, a]$ cut for $\lambda = 5/106 \approx 0.047$, $M = 1156/25 \approx 46$. We chose the small value of λ and large M so that the density has a nontrivial profile substantially different from a semicircle.

Matrix model. Limit to pure 2d gravity

$M \to \infty$ Heavy fermions should decouple and pure gravity emerges

$M^2\lambda = \lambda_{eff}$ Effective cosmological constant is finite

$\lambda = \exp(-\beta_0)$ $\lambda_{eff}(M^2) = \exp(-\beta(M^2))$

Renormalization of cosmological constant
by massive fermions

$\beta(M^2) = \beta_0 - \log M^2$

$(\lambda_{\mathrm{eff,crit}})^2 = \frac{1}{12\sqrt{3}}$

Critical value of the cosmological constant in
pure gravity is reproduced correctly

Matrix model. Limit to 2d gravity coupled to D=-2 matter

$M \to 0$ $\longrightarrow$ $G \sim \sqrt{x^2 - 4}$ The resolvent in Gaussian model

$$< det'L >= \frac{d}{dM^2} < det(L - M^2) >_{M^2=0} \longrightarrow Z_{D=-2}(\lambda) = \frac{d}{dM^2} Z(\lambda, M^2 = 0) = \langle \phi_0^3(\lambda) \rangle$$

$$\langle \phi_0^3 \rangle \sim \int \sqrt{x^2 - 4}(1 - \sqrt{1 - \frac{x}{c}})^3$$

Criticality in D= - 2 emerges when the branchpoint in the resolvent and the branchpoint in the observable collide.

The critical value of the cosmological constant is reproduced correctly

New double scaling.1

$\lambda_c = 1/8.$ $\qquad g = 256\pi\lambda^2\,.$

$$M_c(p) = Ap + [B + C\log(p)]\,p^2 + [D + E\log(p) + F\log^2(p)]\,p^3 + \mathcal{O}\left(p^4\log^3(p)\right) \tag{5.8}$$

where the coefficients read

$$A = \frac{2\sqrt{2}\pi}{9}\,, \qquad B = \frac{1}{18}\pi\left(24\pi - \sqrt{2}(41+\log(16))\right)\,, \qquad C = \frac{\pi}{9\sqrt{2}}\,, \tag{5.9}$$
$$D = \frac{1}{288}\pi\left(64\pi\left(21\pi\sqrt{2} - 155 + \log(16)\right) + \sqrt{2}(8385 + 8\log(2)(173 - 168\log(2)))\right)\,,$$
$$E = -\frac{1}{144}\pi\left(32\pi + \sqrt{2}(173 - 336\log(2))\right)\,, \qquad F = -\frac{7\pi}{12\sqrt{2}}\,.$$

$$g_c(p) = A + [B + C\log(p)]\,p + [D + E\log(p) + F\log^2(p)]\,p^2 + \mathcal{O}\left(p^3\log^3(p)\right) \qquad ($$

Limit of small M and small p (near degeneration of the elliptic curve)

New double scaling. 2

$$z = \frac{p}{16M} = \text{finite}\ , \qquad M \sim p \to 0$$

Z-scaling variable

$$\Phi_s \sim 2\mathcal{J}(t^2 - 2t) + 3t^2 - 2t^3$$

Leading term in J

$$J = 512\pi^2 z - 72\sqrt{2}\pi\log(Mz) +$$
$$+ M\Big[- 81\log^2(Mz) + 864\pi\sqrt{2}z\log(Mz) - 324(\sqrt{2}\pi - 1)\log(Mz)$$
$$-16384\pi^2 z^2 + 9216\pi(\pi - \sqrt{2})z\Big]\ .$$

$$\mathcal{J} = t - \log(m^2 t)$$

t- Lambert function

On the origin of Lambert function

$$p = 16Mz = MW(x)$$

$$z = W(M^{-1}e^{c_0\frac{g-g_c}{M}})$$

Shift from the degeneration point of the elliptic curve

$$W(x) = \sum_{n=1} n^{n-1}\frac{x^n}{n!} \ .$$

Finite radius of convergence

$$x_0 = \frac{1}{e}$$

$$A > \frac{1}{M \log M}$$

Condition for applicability

More examples for Lambert function

Nekrasov,Marshakov — 2007
Okuyama- 2021

1. N=2 Super Yang-Mills coupled to gravity

$$\mathcal{F}_{UV} = t_1 \mathrm{Tr}\,\Phi^2 + t_2 \mathrm{Tr}\,\Phi^3$$

$$Z(t_1, t_2, R) = \langle R | e^{-\frac{J_1}{\hbar}} e^{\hbar^{-1}(t_2 W_3 + t_1 W_2)} e^{-\frac{J_{-1}}{\hbar}} | R \rangle$$

$$v - a = W(t_2 e^{t_1 + a t_2})$$

Lambert provides the same shift of closed moduli

$$\log \Lambda = t_1 + 2t_2 a - 1/2 W(-16 t_2^2 e^{2(t_1 + 2t_2 a)})$$

More on Lambert

2. JT gravity with account of replica wormholes Okuyama(2021)

$$\langle \log Z \rangle - \log\langle Z \rangle = -\int_0^\infty \frac{dx}{x}[e^{-\tilde{Z}x} - e^{-\langle Z \rangle x}]$$

$$\hat{Z} = g_s\sqrt{\frac{\beta}{2\pi}}\sum_k \beta^k \partial_k$$

In the genus zero this operator serves as the generating Function for the boundary creating operator

$$Z_{closed}(\vec{t'}) = Z_{brane} Z_{closed}(\vec{t})$$

$$\tilde{Z}(x) = \mathcal{F}(t_k) - e^{-\hat{Z}x}\mathcal{F}(t_k) = \mathcal{F}(t_k) - \mathcal{F}(t'_k)$$

Lambert function emerges here again as the shift of closed moduli if only two first Times are taken into account. It represent the so called « time-like brane»

Disc partition function.1

$$G(x) = \sum_{k=0}^{\infty} \frac{1}{x^{k+1}} \frac{1}{N} \langle \mathrm{tr}\ X^k \rangle = \int_b^a dy \frac{\rho(y)}{x-y}$$

$$H(r) = \sum_{k=0}^{\infty} \frac{1}{r^{k+1}} \frac{1}{N} \langle \mathrm{tr}\ \phi^k \rangle = \int_b^a dy \frac{\rho(y)}{r-\phi(y)}$$

Double scaling limit

$$G_{sing} = \sqrt{M} \left[\frac{-9\sqrt{2}\cosh^{-1}\left(\sqrt{\frac{\chi-\alpha}{64z}}\right)}{\pi\sqrt{\chi-\alpha-64z}} - \sqrt{\chi-\alpha} \right] + O(M) \qquad x = 2 + M\chi$$

Disc partition function

Generating function for correlators

$$\frac{1}{M}H_{sing}(R) = \frac{\left(\pi R\sqrt{2}\sqrt{128z - R^2} - 18\arccos\left(\frac{R}{\sqrt{128z}}\right)\right)\arccos\left(\frac{R}{\sqrt{128z}}\right)}{\pi^2}.$$

$$R = -\sqrt{128z}.$$

Disc partition function.Limits.

$$\frac{1}{\sqrt{M}} G_{sing} = c_1 + c_2 \frac{\delta}{\xi} + c_3 \frac{\delta^{3/2}}{\xi^{3/2}} (\xi - 2)\sqrt{\xi + 1} + \mathcal{O}(\delta^{\epsilon})$$

Perfect agreement with Pure gravity c=0

$$z = z_c + \delta \,, \quad \chi = \alpha_c + 64\delta/\xi \,, \quad \delta \to 0$$

$$\frac{1}{M} H^{(c=0)}_{sing} \simeq c_1 + c_2 \frac{\delta(X+1)}{X} + c_3 \frac{\delta^{3/2}}{X^{3/2}} (X-2)\sqrt{X+1} + \ldots$$

$$H^{(c=-2)}_{sing} \simeq -\frac{\sqrt{2}}{2\pi^2} (4\pi - g)\zeta\sqrt{\zeta^2 - 1} \log\left(\sqrt{\zeta^2 - 1} + \zeta\right) .$$

$$z \simeq \frac{4\pi - g}{256\pi M} \sim R^2 \to \infty$$

$$\zeta = \frac{R}{\sqrt{128z}}$$

Perfect agreement with c=-2 result

Comment on ZZ instantons

Let us make a comment concerning the instanton effects and consider the single eigenvalue ZZ-brane instanton for our one-cut solution. The instanton action evaluated along the spectral curve reads as follows

$$S_{inst} = \int_a^{x_0} Y(x)dx \tag{9.2}$$

where the spectral curve in terms of the resolvent reads

$$Y(x) = V'_{eff}(x) = V'(x) - 2G(x) = M(x)\sqrt{(x-a)(x-b)}\ . \tag{9.3}$$

The critical point x_0 of the effective potential $V_{eff}(x)$ is defined by condition $M(x_0) = 0$ and corresponds to the pinch point of the spectral curve. The effective potential is constant on the cut and its derivative obeys the useful relation [5]

$$\partial_\lambda V_{eff}(a) = 2\log\frac{(a-b)}{4} \tag{9.4}$$

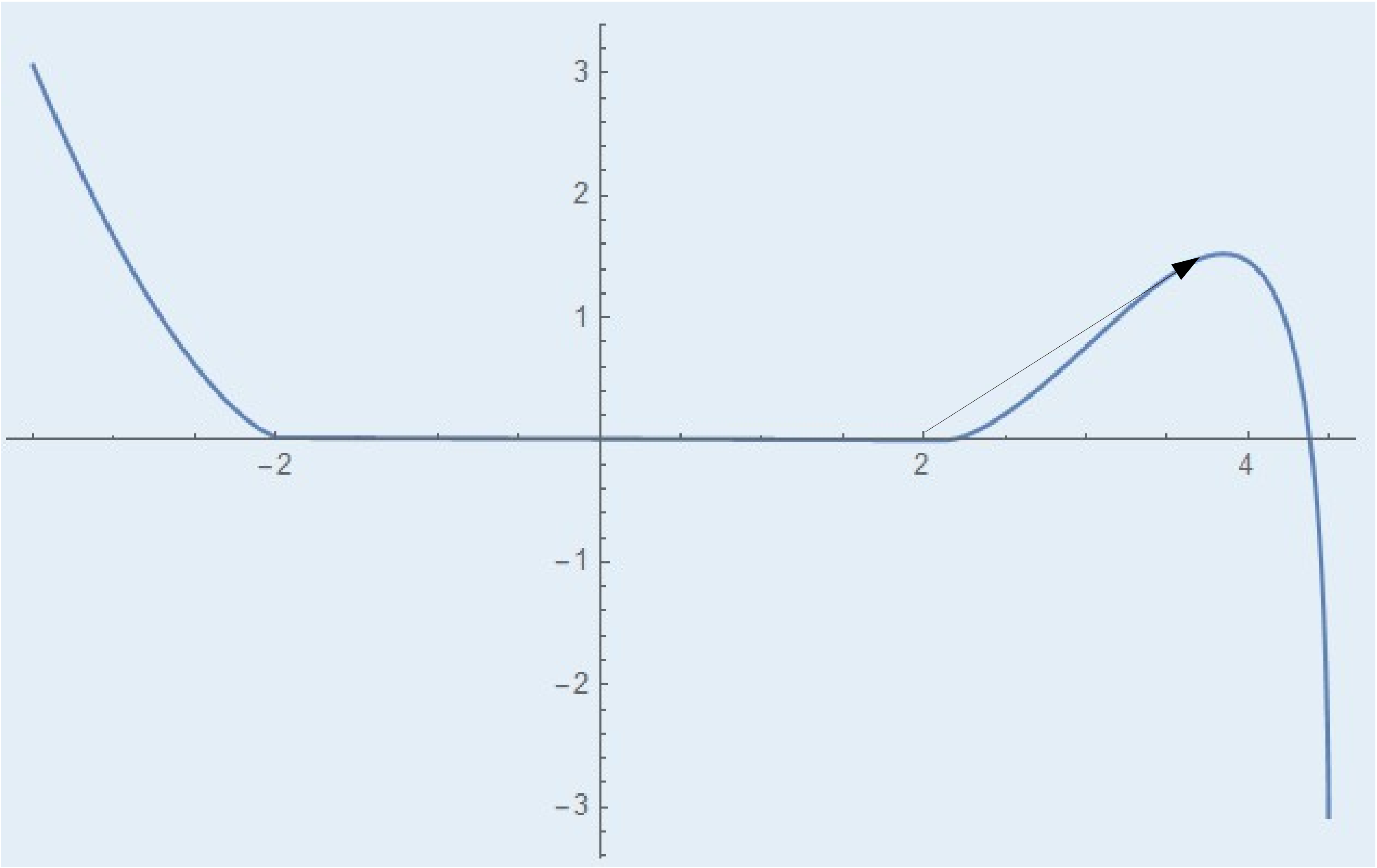
The instanton action can be written as

$$S_{inst} = V_{eff}(a) - V_{eff}(x_0)\ . \tag{9.5}$$

$$S_{inst} \propto \partial_a V_{eff}(a)(x_0 - a) \propto \frac{\partial c}{\partial a}\frac{\partial V_{eff}}{\partial c}(a)(x_0 - a) \propto \log(b-a)(x_0 - a)\ .$$

At small M instanton action becomes unsuppressed

Effective potential for eigenvalues



Open questions concerning solution

- Study of the whole variety of flows in the vicinity of $c = 0$ and $c = -2$ critical points. Comparison to another flow found in [33]. Generalization of such flows to all central charges of matter $c \leq 1$.
- Analysis of the rest of the parameter plane and of the multi-cut solutions to the matrix model.
- Clarification of the role of the second solution to the quadratic equation in the Parisi-Sourlas derivation of the matrix potential.
- Computation of instanton contributions of different kinds, including ZZ branes.
- Establishing the double scaling limit $N \to \infty, \Lambda \to 0$ of our model along the critical line and deriving the universal scaling function in this limit.
- Derivation of this and other critical flows from the continuous 2d QG (Liouville formalism).

Directions for research

- «Standard» double scaling taking account the mass
- Integrability structure behind the solution. Toda lattice expected
- Relation with Kesten-McKay and non-planarity
- Analogy with instanton-antiinstanton matrix model in QCD

RRG as the model for Hilbert space

- Consider very similar model but ask very different questions
- RRG is the model for the Hilbert space of some interacting many-body system
- Nodes=states in the Hilbert space, links=resonances between states
- Idea invented by Altshuler,Gefen,Kamenev, Levitov(97') . Originally- Bethe tree, but does not work. The RRG is good enough toy model

Tools for diagnostics of localization transition

– Two-point spectral correlator and spectral formfactor

$$\rho(\lambda) = Im < Tr\frac{1}{H-\lambda} > \qquad R(\lambda_1,\lambda_2) =< \rho(\lambda_1)\rho(\lambda_2) > - < \rho(\lambda_1 >< \rho(\lambda_2) >$$

$$K(E,t) = \frac{1}{2\pi\hbar}\int d\lambda R(E+\lambda, E-\lambda)e^{\frac{i\lambda t}{\hbar}}$$

– Level spacing distribution P(s)

$$\begin{cases} P_{deloc}(s) = A\, s\, e^{-Bs^2} & \text{below mobility edge, } \lambda_m \\ P_{loc}(s) = e^{-s} & \text{above mobility edge, } \lambda_m \end{cases}$$

Ergodic delocalized

Localized

Also r-statistics and IPR_q as «ergodicity versus localization» measures

RRG perturbed by 3-cycles

Statistical model of exponential random graphs

$$Z(\mu_2, \mu_3 \ldots \mu_n) = \sum_{graphs} \exp(-\sum_k^n \mu_k Tr A^k)$$

We consider ensemble of random regular graphs . In terms of statistical mechanics it is mixed ensemble. Number of nodes is fixed while the chemical potentials, say, for 3-cycles and 4-cycles are introduced

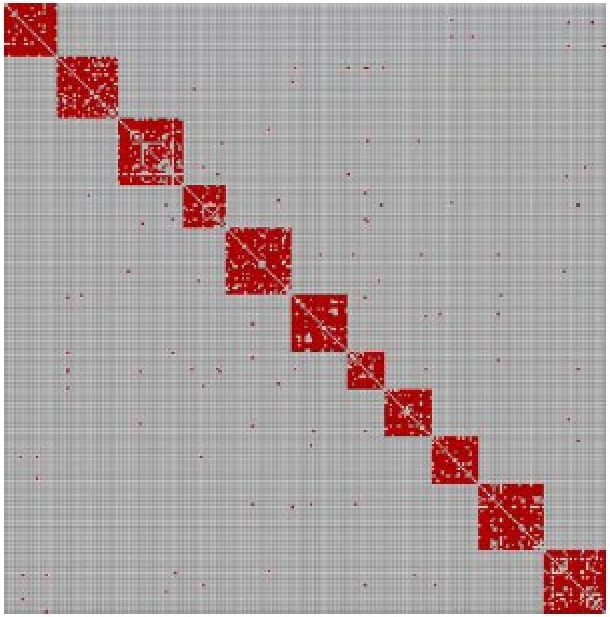
Clusterization transition in RRG

Avetisov,Hovhannesyan, Nechaev
Tamm,Valba A.G. 16'

$\mu > \mu_{crit}$

Chemical potential
for 3-cycles

Clusterized phase



N/q clusters
for RRG

q - degree of node

The cluster sizes in RRG are the same. One-step replica symmetry breaking.
If there is some node degree distribution → distribution for cluster sizes

The clusters are the eigenvalue instantons in the spectrum.

RRG perturbed by 4-cycles and Thermofield double

$$Z(\mu_4) = \sum_{\{\text{states}\}}{}' e^{-\mu_4 N_4}$$

$$\mu_4 > \mu_4^{cRRG}$$

Above the critical value of the chemical potential
the RRG gets clusterized into bipartite clusters

Kelly,Trugenberger,Biancolana 20'
Valba, A.G. 21'

Bipartite clusters are of the special type - hypercubes and correspond
to a thermofield double state

The number of bipartite clusters is fixed by the node degree in RRG ensemble

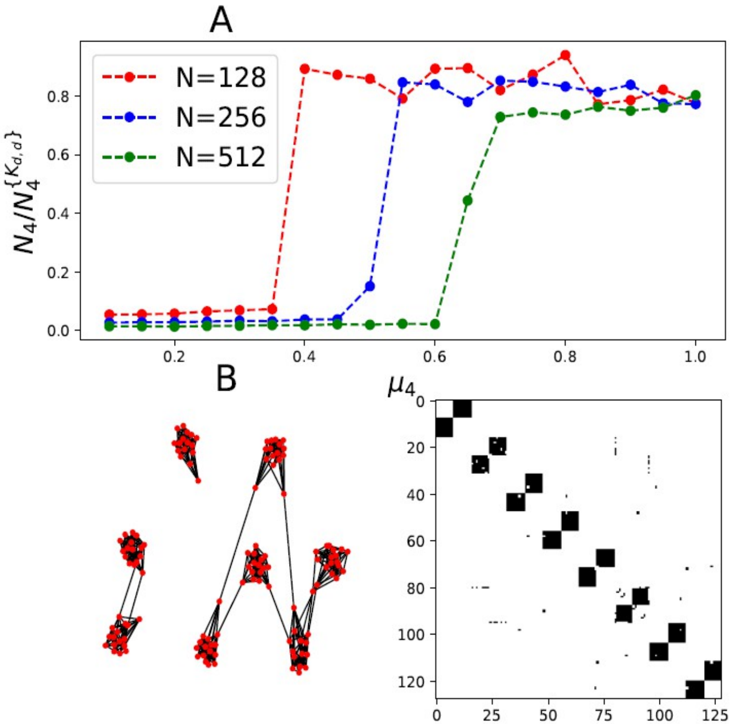


Figure 1. A. The dependence of number of 4-cycles on the chemical potential μ_4 for RRG of different sizes and $d = 8$; B. Ground state and its adjacency matrix at critical μ_4.

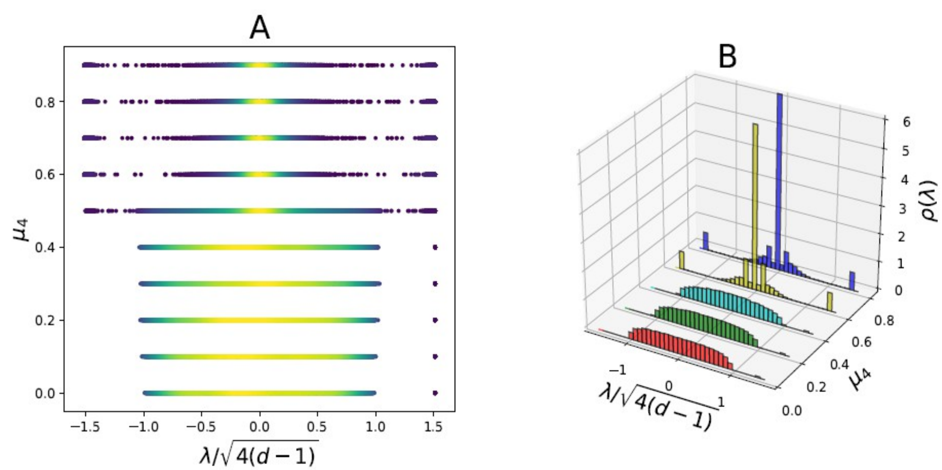
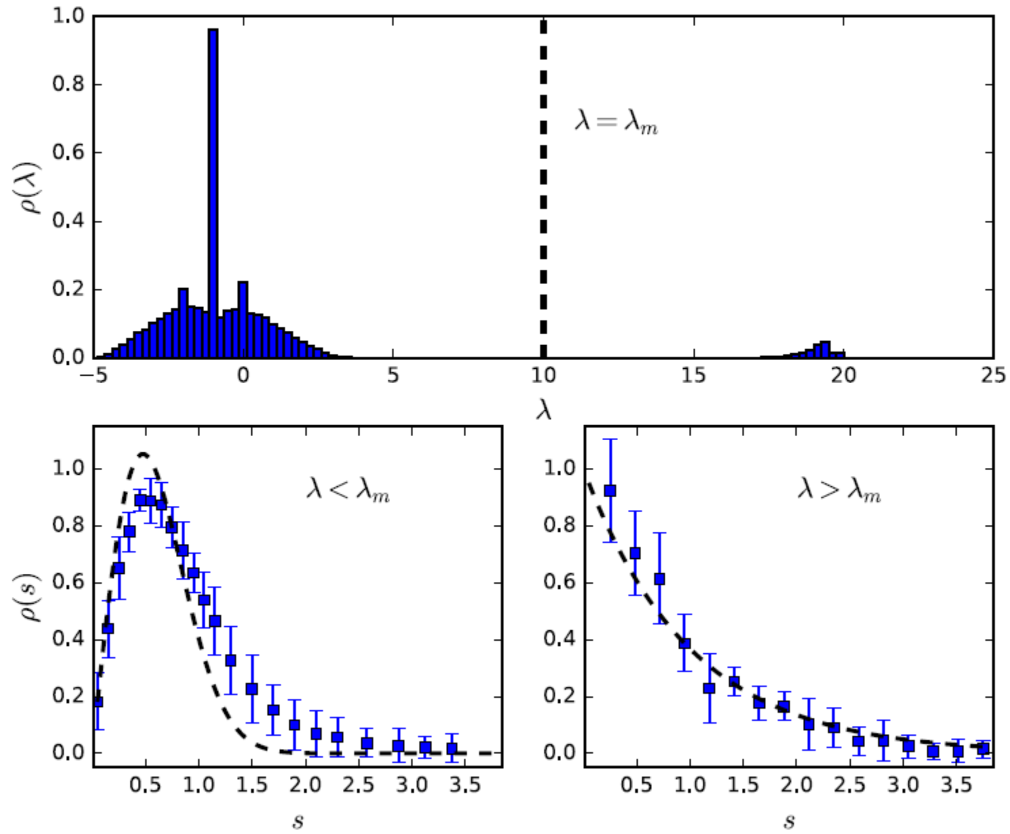


Figure 2. The dependence of the spectral density of adjacency matrix on the chemical potential μ_4 for RRG of $N = 256, d = 8$.

Clusters and localization in RRG perturbed by 3-cycles, no disorder.



Clusters due to the 3-cycles correspond to the localized soft modes
in the one- particle spectrum
on RRG describing a Hilbert space of many-body system

Avetisov, Nechaev, Valba, A.G. 16'

Conclusion 2

- If we consider the leading expansion of the fermionic determinant at large M we get this model with 3-cycles
- 2d gravity with fermions provides interpolation between the single tree at M=0 and large number of trees at large M
- Fragmentation of the Hilbert space — mechansm for the MBL phase

Thank you for attention

Stop the Russian agression in Ukraine!