New applications of Reflection Equation Algebras

Dimitry Gurevich
IITP

IHES
28-29 November 2022

What is the Schur-Weyl duality for the algebra $U(gl(N))$? Let V be the basic space, in which the algebra $U(gl(N))$ acts. This action can be extended onto $V^{\otimes k}$ for any integer $k > 0$ via the coproduct $\Delta : U(gl(N)) \to U(gl(N))^{\otimes 2}$, defined on the generators $X_i \in gl(N),\ i = 1...N$ in the usual way

$$\Delta(X_i) \to X_i \otimes 1 + 1 \otimes X_i.$$

Also, the symmetric group $\mathbb{S}_k$ acts onto $V^{\otimes k}$ via permuting the factors by means of the usual flip $P : V^{\otimes 2} \to V^{\otimes 2}$. Namely, we have the permutations P_{12}, P_{23} and so on.
The SW duality states that these actions commute with each other. Moreover, they are centralizers of each other.

Also, the following holds

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} V_\lambda \otimes M_\lambda, \qquad (1)$$

where $\lambda = (\lambda_1 \geq ... \geq \lambda_N)$ runs over all partitions of the integer k, V_λ is an irreducible $U(gl(N))$-module, labeled by the partition λ, and M_λ is an irreducible $\mathbb{S}_k$-module, also labeled by λ.

The question is: what is a quantum analog of this SW duality?
The group algebra $\mathbb{C}[\mathbb{S}_k]$ of the symmetric group can be deformed into the Hecke algebra $H_n(q)$. By what algebra that $U(gl(N))$ can be replaced?
In 1986 Jimbo suggested a form of the quantum SW duality, where the role of the algebra $U(gl(N))$ was attributed to the QG $U_q(gl(N))$. In 1991 Arum Ram exhibited a q-analog of the Frobenius formula, based on this duality.
Below, I exhibit another form of the q-duality, in which the role of the algebra $U(gl(N))$ plays the so-called Reflection Equation algebra. Also, a new q-Frobenius formula will be exhibited.
Now, recall the classical Frobenius formula.

Consider a set of commutative indeterminates $x_1 ... x_N$. To any partition $\lambda \vdash k$ there are associated some families of symmetric (i.e. invariant under an action of $\mathbb{S}_k$) polynomials in these indeterminates. We shall deal with two of them: the power sums $p_\lambda(x_1 ... x_N)$ and the Schur functions $s_\lambda(x_1 ... x_N)$. The famous Frobenius formula is

$$p_\nu(x_1 ... x_N) = \sum_{\lambda \vdash k} \chi_\nu^\lambda s_\lambda(x_1 ... x_N),$$

where χ_ν^λ is the character of the symmetric group $\mathbb{S}_N$ in the representation M_λ evaluated on the element whose cyclic type is $\nu = (\nu_1 ... \nu_N)$.

The power sums are defined as follows

$$p_k(x) = \sum_i x_i^k, \quad p_\nu(x) := p_{\nu_1} \dots p_{\nu_N}.$$

The Schur polynomials can be defined in different ways. For instance, they can be expressed by means of the Jacobi-Trudi formulae via the elementary symmetric polynomials or the full symmetric polynomials

$$e_k(x) = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k}, \quad h_k(x) = \sum_{i_1 \le \dots \le i_k} x_{i_1} \dots x_{i_k}.$$

Now, we say a couple of words on the Hecke algebras and their representation theory.

The Artin braid group B_N is the group generated by the unit e and $N-1$ invertible elements

$$\tau_1, \ldots, \tau_{N-1}$$

subject to the following relations

$$\tau_i\,\tau_j = \tau_j\,\tau_i \text{ if } |i-j| \geq 2 \text{ and } \tau_i\,\tau_{i+1}\,\tau_i = \tau_{i+1}\,\tau_i\,\tau_{i+1}, \quad i \leq N-2.$$

The last relation is called braid one.
Since there exists natural imbeddings $B_N \to B_{N+1}$ we can consider the direct limit of B_N.
If we impose the condition $\tau_i^2 = e$ for any i, we get the symmetric group $\mathbb{S}_N$.

Also, we consider the following quotients of its group algebra $\mathbb{C}[B_N]$. Let $q \in \mathbb{C}$. Impose the condition

$$(\tau_i - q\, e)(\tau_i + q^{-1}\, e) = 0, \ \forall i.$$

We get an algebra $H_N(q)$ called Iwahori-Hecke algebra. Observe that for $q = \pm 1$ we get the algebra $\mathbb{C}[\mathbb{S}_N]$. Also, there are of interest the so-called Birman-Murakami-Wenzl algebras (see articles by Ogievetsky-Pyatov in ariXiv).
We deal with q generic.

Now, consider some special reps of the braid groups.
Let V be a vector space over the field $\mathbb{C}$. We say that a linear invertible operator $R : V^{\otimes 2} \to V^{\otimes 2}$ is a braiding, if it is subject to the following relation in $V^{\otimes 3}$

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

where $I : V \to V$ is the identity operator.
The simplest example is the usual flip $R = P$ which acts as follows $P(x \otimes y) = y \otimes x$ for any $x, y \in V$, or a super-twist denoted $P_{m|n}$.
In order to define it we assume V to be a super-space $V = V_0 \oplus V_1$. The component V_0 is called even and that V_1 odd.
We say that $x \in V_0$ is of parity 0 and that $x \in V_1$ is of parity 1.
The parity is denoted $\overline{x}$. Then the super-flip is defined as follows

$$P_{m|n}(x \otimes y) = (-1)^{\overline{x}\,\overline{y}} y \otimes x.$$

This notation means that $dim V_0 = m$ and $dim V_1 = n$.

Note that $P_{m|n}^2 = I$. Also, note that the usual flip P is a particular case of a super-flip $n = 0$. The braidings R subject to the condition $R^2 = 1$ are called involutive symmetries. The braidings subject to the Hecke condition

$$(q\,I - R)(q^{-1}\,I + R) = 0,\;\; q \in \mathbb{C},\;\; q \neq 0,\;\; q \neq \pm 1$$

are called Hecke symmetries.
For any braiding R we denote $R_k : V^{\otimes p} \to V^{\otimes p},\;\; k = 1, ...p - 1$ the operator R acting on the components numbers k and $k + 1$. Thus,

$$R_k = I_{1\ldots k-1} \otimes R \otimes I_{k+2\ldots p}.$$

Observe that the map

$$\rho_R : \tau_k \to R_k,\;\;\; k = 1...p - 1$$

is a representation of the braid group, called R-matrix one.

Observe a very special property of this representation:

$$\rho_R(\tau_2) = P_1\, \rho_R(\tau_1)\, P_1,$$

$$\rho_R(\tau_3) = P_2\, \rho_R(\tau_2)\, P_2 = P_2\, P_1\, \rho_R(\tau_1)\, P_1\, P_2, \ldots$$

In the same way, if R is an involutive symmetry, it defines a representation of the algebra $\mathbb{C}[\mathbb{S}_N]$, and if R is a Hecke symmetry, it defines a representation of the Hecke algebra.
By fixing in the space V a basis $\{x_1 \ldots x_N\}$ and the corresponding basis $\{x_i \otimes x_j\}$ in the space $V^{\otimes 2}$ we can represent the operators R_i by matrices. Let us exhibit two examples of Hecke symmetries

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

The first Hecke symmetry comes from the Quantum Group $U_q(sl(2))$, the second one – from $U_q(sl(1|1))$. For $q = 1$ we get respectively a usual flip and a super-flip.
Note that if R is a braiding, then the operator $\mathcal{R} = R\,P$, where P is the usual flip, meets the so-called Quantum Yang-Baxter equation

$$\mathcal{R}_{12}\,\mathcal{R}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{23}\,\mathcal{R}_{13}\,\mathcal{R}_{12}.$$

Note that for $q = 1$ the matrix $\mathcal{R}$, corresponding to the first Hecke symmetry above, turns into the identity matrix. So by assuming that $q = 1 + \hbar$, we can expand the matrix $\mathcal{R}$ as follows

$$\mathcal{R} = I + \hbar\, r + \ldots$$

The matrix r meets the so-called classical YB equation $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$

However, there exist involutive and Hecke symmetries which are deformations neither of the usual flips nor of the super-flips. So, the following problem is of interest; what involutive and Hecke symmetries could be. Consider two related algebras: "R-symmetric" and "R-skew-symmetric" ones

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \ \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle,$$

where $T(V) = \bigoplus V^{\otimes k}$ is the free tensor algebra of V.
Also, consider the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \ P_-(t) = \sum_k dim \bigwedge\nolimits_R^{(k)}(V)t^k,$$

where the upper index (k) labels the homogenous components.
If R is involutive, we put $q = 1$ in these formulae.

Examples. If R is a deformation of the usual flip P and $dim V = N$, then

$$P_-(t) = (1+t)^N.$$

If R is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $N \geq 2$ there exit involutive and Hecke symmetries such that

$$P_-(t) = 1 + Nt + t^2.$$

Here $dim V = N$.
If $P_-(t)$ is a polynomial, R is called *even*.

Proposition. (G)

For a generic q the following holds $P_-(-t)P_+(t) = 1$.

Proposition. (Phung Ho Hai)

The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function:

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \dots + a_r\, t^r}{1 - b_1\, t + \dots + (-1)^s\, b_s\, t^s} = \frac{\prod_{i=1}^r (1 + x_i t)}{\prod_{j=1}^s (1 - y_j t)},$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

We call the couple $(r|s)$ bi-rank. In this sense all involutive and Hecke symmetries are similar to super-flips, for which the role of the bi-rank is played by the super-dimension $(m|n)$.
Note that all numerical characteristic of the related objects are expressed via the bi-rank $(r|s)$ of the initial involutive or Hecke symmetry R.
The bi-rank enters the quantum dimension of V and other spaces and other numerical characteristics.
Example.
The usual dimension of a super-space $V = V_0 \oplus V_1$ with super-dimension $(m|n)$ is $N = m + n$, whereas its quantum dimension is $m - n$.

Let us recall that for a generic q the algebra $H_k(q)$ is semisimple and isomorphic to the group algebra of the symmetric group $\mathbb{S}_k$. Then according to the Wedderburn-Artin theorem, the finite dimensional semisimple Hecke algebra $H_n(q)$ is isomorphic to a direct product of matrix algebras over $\mathbb{C}$:

$$H_n(q) \simeq M_{\lambda(1)}(\mathbb{C}) \times M_{\lambda(2)}(\mathbb{C}) \times \cdots \times M_{\lambda(s)}(\mathbb{C}), \qquad (2)$$

where $\{\lambda(1), \lambda(2), \ldots, \lambda(s)\}$ is a (unordered) set of *all* possible partitions of the integer n: $\lambda(k) \vdash n$. The dimension of the simple component M_λ is d_λ^2, where d_λ is the number of *standard Young tables* corresponding to the partition λ.

In each matrix algebra $M_\lambda,\ \lambda \in \{\lambda(1)...\lambda(s)\}$ there exists a basis $E^\lambda_{i\,j}$ consisting of the matrix units. Denote by $E^\lambda_{i\,j}$ the matrix with all trivial entries apart from that belonging to the row number i and the column number j, where 1 is placed.

By doing so in each component in the above product, we have a basis in the Hecke algebra $H_n(q)$, with the following multiplication table

$$E^\lambda_{k\,i}\,E^\mu_{r\,p} = \delta^{\lambda\,\mu}\,\delta_{i\,r}\,E^\lambda_{k\,p}.$$

Note that the diagonal elements $E^\lambda_{ii},\ i = 1...d_\lambda$ are primitive orthogonal idempotents defining a resolution of the unity. Namely, we have

$$e = \sum_{\lambda\vdash n}\sum_{i=1}^{d_\lambda} E^\lambda_{ii}, \qquad E^\lambda_{ii}E^\mu_{jj} = \delta^{\lambda\mu}\delta_{ij}E^\lambda_{ii}. \tag{3}$$

Now, consider the left regular representation π of the Hecke algebra $H_n(q)$ onto itself in the basis $E^{\lambda}_{k\,i}$. Let $z \in H_n(q)$ be an arbitrary element. Its image $\pi(z)$ in the basis $\{E^{\lambda}_{k\,i}\}$ for a fixed λ is represented by a matrix:

$$\pi(z)\, E^{\lambda}_{k\,i} = \sum_{r} Z^{\lambda}_{k\,r}\, E^{\lambda}_{r\,i} \qquad (4)$$

of dimension $d_\lambda \times d_\lambda$.

Observe that there exist different ways to construct the matrix unit $E_{i\,j}^{\lambda}$ in terms of the generators τ_i. In [OP] Lecture on Hecke algebra there is exhibited a method, using the so-called Jucys-Murphy (JM) elements, which are defined by recursion as follows

$$J_1 = I,\;\; J_2 = R_1^2,\;\; J_3 = R_2\, J_2\, R_2, ..., J_{k+1} = R_k\, J_k\, R_k.$$

Note that an analog of this construction in the algebra $\mathbb{C}[\mathbb{S}_N]$ is exhibited in the paper
Vershik, Okounkov A new approach to representation theory of symmetric groups.

Now, let R be an involutive or Hecke symmetry. Let L be a matrix subject to

$$RL_1RL_1 - L_1RL_1R = 0, \; L = (l_i^j), \; 1 \le i,j \le m.$$

This algebra is called RE one and denoted $\mathcal{L}(R)$.
If $\hat{L}$ is subject to

$$R\hat{L}_1R\hat{L}_1 - \hat{L}_1R\hat{L}_1R = R\hat{L}_1 - \hat{L}_1R, \; \hat{L} = (\hat{l}_i^j), \; 1 \le i,j \le m,$$

the algebra is called modified RE algebra and denoted $\hat{\mathcal{L}}(R)$.
Nevertheless, these algebras are isomorph to each other. Their isomorphism can be realised via the following relations between the generating matrices

$$L = I - (q - q^{-1})\,\hat{L}.$$

Observe that this isomorphism fails if $q = \pm 1$.

Nevertheless, if $R \to P$ as $q \to 1$, the RE algebras $\mathcal{L}(R)$ tends to the algebra $Sym(gl(N))$, whereas the modified RE algebra $\hat{\mathcal{L}}(R)$ tends to that $U(gl(N)))$.

Our next aim is to describe the center of the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ and to introduce analogs of the symmetric polynomials on the first of them.

I remind that the center of the algebra $U(gl(N))$ is generated by the power sums $Tr\hat{L}^k$, where $\hat{L}$ is the generating matrix of this algebra.

Now, we introduce the so-called R-trace of matrices

$$Tr_R M = Tr C\, M,$$

where the matrix $C = (C_i^j)$ is completely defined by a given braiding R (it can be defined for all braidings, which are skew-invertible). Now, we introduce power sums in the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ as follows

$$p_k(L) = Tr_R\, L^k = Tr\, C\, L^k, \;\; p_k(\hat{L}) = Tr_R\, \hat{L}^k = Tr\, C\, \hat{L}^k.$$

They are central in the corresponding algebras. Note that if $R = P$,then $C = I$ and we get the classical power sums.

Now, we want to describe other elements of the center of the algebra $\mathcal{L}(R)$.

Below, we use the following notations

$$L_{\overline{1}} = L_1,\ L_{\overline{2}} = R_{12}\, L_{\overline{1}}\, R_{12}^{-1},\ L_{\overline{3}} = R_{23}\, L_{\overline{2}}\, R_{23}^{-1} = R_{23}\, R_{12}\, L_{\overline{1}}\, R_{12}^{-1}\, R_{23}^{-1}, \ldots$$

Also, we use the notation

$$L_{\overline{1\to k}} = L_{\overline{1}}\, L_{\overline{2}} \ldots L_{\overline{k}}.$$

This string of the matrices $L_{\overline{i}}$ will play an important role in what follows.

The following claim can be found in the paper by Isaev-Pyatov.

Proposition.

Let $z \in H_k(q)$ be an arbitrary element. Then the element

$$ch(z) := Tr_{R(1\dots k)}\, \rho_R(z) L_{\overline{1\to k}}$$

is central in the algebra $\mathcal{L}(R)$.

The map

$$ch : H_k(q) \to \mathcal{L}(R), \;\; z \mapsto ch(z)$$

is called characteristic.

Observe that the power sums $p_k(L)$ in the algebra $\mathcal{L}(R)$ can be written in this form. Namely, we have

$$p_k(L) = Tr_{R(1\dots k)}\, \rho_R(z) L_{\overrightarrow{1\to k}}, \;\; z = \tau_{k-1}\, \tau_{k-2} \dots \tau_1.$$

The element $z = \tau_{k-1}\, \tau_{k-2} \dots \tau_1$ is called the Coxeter one.

However, in the RE algebras (and only in them) the power sums $p_k(L)$ can be reduced to the form $Tr_R L^k$, similar to the classical one.

Now, introduce the Schur polynomials (functions) in this algebra. They are defined in the same manner but with $z = E_{ii}^{\lambda}$, where E_{ii}^{λ} is the above idempotents. Namely, we put

$$s_\lambda(L) = Tr_{R(1...k)}\, \rho_R(z) L_{\overline{1\to k}}, \;\; z = E_{ii}^{\lambda}.$$

Note that this Schur polynomial does not depend on i, though the idempotents E_{ii}^{λ} for different i are not equivalent to each other.

Now, we exhibit our q-version of the Frobenius formula:

$$p_\nu(L) = \sum_{\lambda \vdash k} \chi_\nu^\lambda s_\lambda(L),$$

where χ_ν^λ is the character of the Hecke algebra $H_k(q)$ in the representation, labeled by λ, on the element whose cyclic type is $\nu = (\nu_1 ... \nu_N)$ and

$$p_\nu(L) := p_{\nu_1}(L) p_{\nu_2}(L) ... p_{\nu_N}(L).$$

As for the cyclic type, we define it only for some special elements. Let us remove some factors from a Coxeter element $\tau_{k-1}\,\tau_{k-2}...\tau_1$ by keeping instead of the removed elements blank spots. Thus, the element $z \in H_n(q)$ obtained in this way consists of a few strings $\tau_{k_1}\,\tau_{k_1-1}...\tau_{k_2},\; k_2 \le k_1$, separated by blank spots. The remaining elements z will be called the Coxeter elements with gaps.
Then to each black spot we assign the number 1 and to each string entering the element z we assign its length plus 1. By the cyclic type of z we means the family of these numbers ν_i, ordered downward.
We say the family $\nu = (\nu_1...\nu_k)$ is the cyclic type of the element z.

By comparing our construction and that by Arum Ram we want to observe that in the algebra $\mathcal{L}(R)$ the analogs of the symmetric polynomial are naturally defined whereas in the QG it is not so. Besides, it is not clear how it is possible to generalize his construction to the cases, related to Hecke symmetries, different from these coming from the QG.

Now, we want to discuss the question: whether it is possible to represent the quantum symmetric polynomials via "eigenvalues"?

Observe that in the algebra $\mathcal{L}(R)$ there are analogs of the Newton identities

$$p_k - q p_{k-1}\, e_1 + (-q)^2 p_{k-2}\, e_2 + \ldots + (-q)^{k-1} p_1\, e_k + (-1)^k k_q e_k = 0,$$

$k = 1.2\ldots$ and the Cayley-Hamilton identity

$$L^m - q\, L^{m-1}\, e_1 + (-q)^2 L^{m-2}\, e_2 + \ldots + (-q)^{m-1} L\, e_{m-1} + (-q)^m\, I\, e_m = 0,$$

provided R is even of bi-rank $(m|0)$.
Here, $e_k = e_k(L)$ are elementary symmetric polynomials (functions), i.e. particular cases of the Schur polynomials, respective to one-column diagrams.

Let $\mu_i, \ i = 1...m$ be indeterminates meeting the following system

$$\sum_i \mu_i = e_q(L), \ \sum_{i<j} \mu_i \, \mu_j = q^2 \, e_2(L), \ \dots, \mu_1 \, \mu_2 ... \mu_m = q^m \, e_m(L).$$

We call μ_i (quantum) "eigenvalues" of the matrix L. These indeterminates are assumed to be central in the algebra $\mathcal{L}(R)[\mu_1 ... \mu_m]$.

If R is not even and its bi-rank is $(m|n)$, then the eigenvalues can be defined in a similar way. In this case we have two families of them $\mu_1, ..., \mu_m$ (even eigenvalues) and $\nu_1, ..., \nu_n$ (odd eigenvalues) such that any symmetric polynomial can be expressed via these quantities.

For the power sums we have

$$p_k(L) = Tr_R\, L^k = \sum_{i}^{m} \mu_i^k\, d_i + \sum_{j}^{n} \nu_j^k\, \tilde{d}_j,$$

$$d_i = q^{-1} \prod_{p=1, p\neq i}^{m} \frac{\mu_i - q^{-2}\mu_p}{\mu_i - \mu_p} \prod_{j=1}^{n} \frac{\mu_i - q^2\nu_j}{\mu_i - \nu_j},$$

$$\tilde{d}_j = -q \prod_{i=1}^{m} \frac{\nu_j - q^{-2}\mu_i}{\nu_j - \mu_i} \prod_{p=1,\, p\neq j}^{n} \frac{\nu_j - q^2\nu_p}{\nu_j - \nu_p},$$

In the limit $q = 1$ we get the formula, corresponding to the involutive symmetry R

$$p_k(L) = \sum_{i}^{m} \mu_i^k - \sum_{j}^{n} \nu_j^k.$$

In the even case these polynomials coincide with the Hall-Littlewood polynomials up to a numerical factor and identification $t = q^{-2}$.

Observe that all these polynomials are super-symmetric in $q^{-1}\,\mu_i$ and $q\,\nu_j$.
Recall that by definition a polynomial in two sets in indeterminates μ_i and ν_j is called super-symmetric if it is symmetric in μ_i and ν_j separately and the polynomial in which one puts $\mu_1 = \nu_1 = s$ does not depend on s.

The second application of the RE algebras is a q-analog of the Capelli formula. In order to introduce it we need the partial derivative in the generators of the RE algebra. These derivatives are introduced via the following system

$$R\,L_1\,R\,L_1 = L_1\,R\,L_1\,R,$$
$$R^{-1}\,D_1\,R^{-1}\,D_1 = D_1\,R^{-1}\,D_1\,R^{-1},$$
$$D_1\,R\,M_1\,R = R\,M_1\,R^{-1}\,D_1 + R.$$

The first line defines a RE algebra $\mathcal{L}(R)$. The second line define a RE algebra $\mathcal{D}(R^{-1})$. The third line is the so-called permutation relations between two algebras.

The entries ∂_i^j of the matrix D, generating the algebra $\mathcal{D}(R^{-1})$, play the role of partial derivatives in l_i^j:

$$\partial_i^j(l_k^l) = \delta_i^l\delta_k^j.$$

The action of the partial derivatives on higher degree elements can be deduced from the permutation relations.

Note that in the classical limit (that is while $R = P$) the above system defines Weyl-Heisenberg algebra.

Theorem.

Let $L = \|l_i^j\|_{1\le i,j\le N}$ be the generating matrix of an algebra $\mathcal{L}(R)$ and $D = \|\partial_i^j\|_{1\le i,j\le N}$ be the matrix composed from the partial derivatives. Then the matrix

$$\hat{L} = L\,D$$

generates the modified RE algebra.

This theorem states that in our q-setting, the situation is similar to the classical one. Recall that in the classical setting the matrix $\hat{L} = L\,D$, where L is a matrix with commutative entries l_i^j and D is the matrix composed from the partial derivatives $\partial_i^j = \partial_{l_j^i}$, generates the algebra $U(gl(N))$.

Now, exhibit the classical Capelli identity. Let $\hat{L} = L\,D$. Then we have

$$rDet(\hat{L} + K) = detL\,detD,$$

where K is the diagonal matrix $diag(0, 1, ..., n-1)$ and $rDet$ is the so-called row-determinant.
Observe that the term $rDet(\hat{L} + K)$ in the l.h.s. can be written in the following form

$$cDet(\hat{L} + K) = detL\,detD,$$

where K is the diagonal matrix $diag(n-1, ...1, 0)$ and $cDet$ is the so-called column-determinant. Also, the matrix form

$$Tr_{1..N}A^{(N)}\hat{L}_1\,(\hat{L} + I)_2\,(\hat{L} + 2I)_3...(\hat{L} + (N-1)I)_N$$

is possible.

Proposition.

In the RE algebra the following holds

$$Tr_{R(1\dots m)}\, A^{(m)}\, \hat{L}_1\, (\hat{L}_{\overline{2}} + q\, I)(\hat{L}_{\overline{3}} + q^2\, 2_q\, I)\dots(\hat{L}_{\overline{m}} + q^{m-1}\, (m-1)_q\, I) =$$

$$q^{-m}\, det_R\, L\, det_{R^{-1}}\, D.$$

Here m is the rank of R. (Note that in the classical case $m = N$.)

Here, the determinants are the highest elementary polynomials, which can be defined for any even symmetry.

Observe that there are known numerous attempts to generalize the classical Capelli identity.
I want to only mention the paper by Noumi, Umeda, Wakayama (1994). Their construction is related to the RTT algebra. Their R is the standard Hecke symmetry, i.e., it comes from the QG $U_q(sl(N))$. Whereas ours is valid in general situation.

Also, observe that similarly to the Casimir operators coming from $U(gl(N))$ it is possible to define their q-analogs on the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$. Note that the eigenvalues of the operators $Tr\hat{L}^k$, where $\hat{L}$ is the generating matrix of $U(gl(N))$ in irreps of $U(gl(N))$ were computed by Perelomov-Popov. We have computed the eigenvalues of the two lowest Casimir operators $Tr_R\hat{L}^k$ defined in the algebras $\hat{\mathcal{L}}(R)$. Also, by using realization $\hat{L} = L\,D$ it is possible to define normal ordering

$$: Tr_R\hat{L}^k :=: Tr_R(LD)^k :$$

These operators are q-analogs of the so-called cut-and-join operators.