KONTSEVICH's $\star_{\text{aff}}$ mod $\bar{o}(\hbar^7)$, OR: $\star^{\text{red}}_{\text{aff}}$ mod $\bar{o}(\hbar^7)$ OVER $\mathbb{Q}$.

Arthemy KISELEV (IHÉS; GRONINGEN), joint with R. BURING (MAINZ).

- TOPIC: DEFORMATION QUANTIZATION. [H. WEYL (1946): $p\star q \neq q\star p$]
- GOAL: $\star_{\text{aff}}$ is in photonics.
- REF: arXiv:2209.14438 [q-alg].
- TOOLS: R.B. (2016–2022), B.–P.–P. (2017–18)

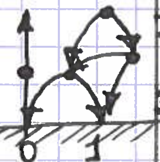
1 §1 TH. IF ASSOCIATIVE $\star = \times + \sum_{n=1}^{+\infty} \hbar^n \cdot B_n(\cdot,\cdot)$ ⟵ BI-DIFF. OPERATORS 1

ON $C^\infty(M^d)\times\mathbb{R}[[\hbar]]$ $\Rightarrow$ (skew bi-der) B_1^- is POISSON $\{\cdot,\cdot\}_{\mathcal{P}}$. 1

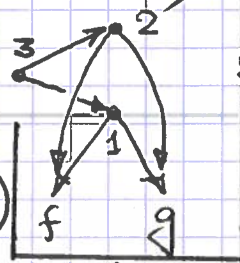
2 TH. [M.K.'97] $(\forall)\ (M^{d<\infty}_{\text{affine}}, \mathcal{P}) \Rightarrow \exists\ \star = \times + \hbar\{\cdot,\cdot\}_{\mathcal{P}} + \sum_{n\geqslant 1} \frac{\hbar^n}{n!} B_n(\cdot,\cdot)$; 2

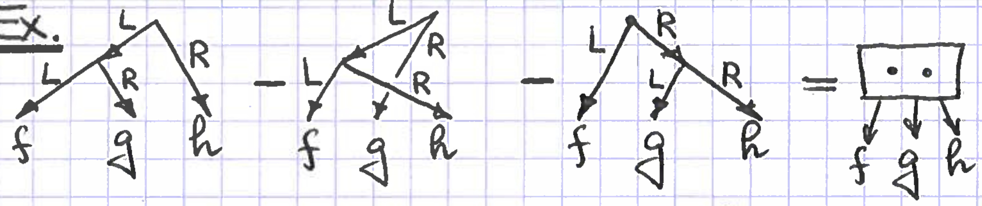
↳ ASSOCIATIVE UNITAL.

3 • $B_n = \sum_{\Gamma\in\text{Graphs}_{2,n}} w(\Gamma)\times B_\Gamma(\cdot,\cdot)$. 3

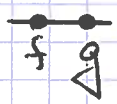
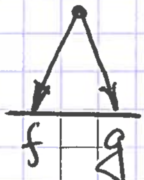
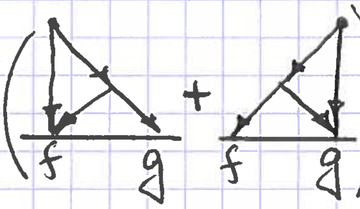
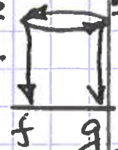
• WEIGHTS $w(\Gamma) = \langle\text{formula}\rangle_{\mathbb{H}^2}$.  3

4 §2 $\partial/\partial x^i \rightsquigarrow \xrightarrow{i}$; $\frac{1}{2}\mathcal{P}^{ij}(\underline{x})\partial_i\otimes\partial_j \rightsquigarrow$  (LEFT ≺ RIGHT) 4

5 DEF. K. GRAPHS } ⟵ BUILT of WEDGES. Ex. $\Gamma = \Theta\Gamma$ ("ZERO"):  5

6 Ex.  $= \mathrm{JAC}(\mathcal{P})(f,g,h) = \sum_{\circlearrowleft} \{\{f,g\}_{\mathcal{P}}, h\}_{\mathcal{P}}$. 6

7 Ex. 7

$\star =$  $+ \frac{\hbar^1}{1!}$  $+ \frac{\hbar^2}{2!}$  $+ \frac{\hbar^2}{3}\Big($  $\Big) + \frac{\hbar^2}{6}$ 

8 $+\bar{o}(\hbar^2)$. } ⟵ ?: higher orders? 8

9 • $\hbar^3 \approx$ (P.–V. [1998]); • $\hbar^4$: R.B. (2017); • $\hbar^{5\div 6}$: OXFORD (2018); • $\hbar^7$ 9

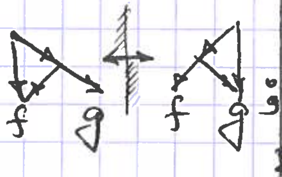
10 §3 TABLE. 10

	n:	1	2	3	4	5	6	7	
11	# K. GRAPHS (RELEVANT):	1	4	30	330	4893	91 489	2 049 704	11
12	# of them, basic prime & in-degree(•) ≤ 1:	1	3	8	23	59	171	477	12

13 REM. $\zeta(3)$ shows up in w("linear" graph $\mid n=7$). ⟵ {WILLWACHER, FELDER (2008). 13

14 REM. $\zeta(3)$ shows up at $n\geqslant 5$ in some $w(\Gamma)\neq 0$ in $\star$. 14

§4 PROPERTIES OF $w(\Gamma)$. $\Rightarrow$ (linear constraints)

(a) Basic: LEFT $\rightleftarrows$ RIGHT in $\bullet$; MIRROR REFLECTION: 

$w(\text{if } \Gamma \overset{\checkmark}{=} \ominus\Gamma) \equiv 0$, etc. $\Rightarrow$ BASIC GRAPHS.

(b) OVER SINKS: $w(\Gamma_1 \bar{\times} \Gamma_2) \overset{\checkmark}{=} w(\Gamma_1) \overset{\mathbb{R}}{\times} w(\Gamma_2)$. AVK.

Ex. $w(\ldots) = w(\ldots) \times w(\ldots)$. $\Rightarrow$ PRIME GRAPHS.

(c) CYCLIC WEIGHT RELATIONS. $\} \leftarrow$ [T.W.+G.F.(2008)] $\leftarrow$ (2000).

$w(\Gamma) \overset{\checkmark}{=} \Sigma\, (\pm)\, w($In Γ, send [$\varnothing$... ALL] LEFT EDGES TO SINK 0,

& CYCLICALLY PERMUTE SINKS $0, 1, \ldots, m-1$). $\leftarrow$ {If $m=2$ (here), $\Rightarrow$ "SWAP" $0 \rightleftarrows 1$.

NB: ◢ – LINEAR SYSTEM W.R.T. in-degree($\bullet$) $\Rightarrow$ WELL DEF. (if) $P^{ij} = A\underline{x} + b$.

(d) $\mathrm{Assoc}(\star(P))(f,g,h) \doteq 0$ by $\mathrm{JAC}(P) = 0$.

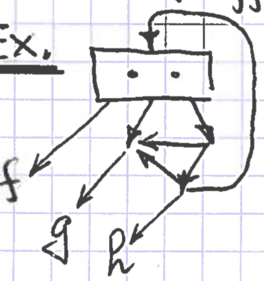
1] RESTRICT TO KNOWN $\{\cdot,\cdot\}_P \leftarrow$ E.g. $P(\underline{c}(\underline{x}), [\underline{a}(\underline{x})])$ · __ 2] "$\forall$" P.

§5 Ex. $\hbar^4$: 160,000 K. GRAPHS $\to$ 330 RELEVANT (connected, $\neq 0$, diff. order >0)

$\Rightarrow$ 10 master-param. $w(\Gamma)$. $\to$ {249 coeff $\neq 0$ at $\hbar^4$ in $\star$}. [2017]

• Affine: $\star_{\text{aff}}$ mod $\bar{o}(\hbar^7)$. $\} \leftarrow$ (1423) $w(\Gamma) \neq 0\ \} \leftarrow \mathbb{Q}$-linear$\left(1, \frac{\zeta(3)^2}{\pi^6}\right)$.

$\mathrm{Assoc}(\star_{\text{aff}}) = \Diamond(\ldots, \ldots = \mathrm{JAC}(P)) \longleftarrow$ LEIBNIZ GRAPHS.

Ex.  (LEIBNIZ GRAPH) $\Big|_{\forall \text{ Poisson } P} = 0\ \} \leftarrow$ POLY-DIFF. OPERATOR w.r.t. sinks $f,g,h,\ldots$ DUE TO $\mathrm{JAC}(P) = 0$.

§6 ASSOCIATIVITY MECHANISM: $\Diamond$ in R.H.S.

Prop. • IN $\star_{\text{full}}$ ($\Rightarrow$ in $\star_{\text{aff}}$) mod $\bar{o}(\hbar^{\leqslant 6})$: [Enough {all L. graphs from $\mathrm{Assoc}(\star(P))$}]

(BY CONTRACTING ONE INTERNAL EDGE) $\longrightarrow$

• FROM $\hbar^{\geqslant 7}$, FOR $\star_{\text{aff}}$ ($\Rightarrow \star_{\text{full}}$) NEED MORE LEIBNIZ GRAPHS.

§7 "REM." "$\forall$"$P \mid P^{ij} = A\underline{x} + \underline{b} \Rightarrow \star_{\text{aff}}(P_{\text{aff}})(\cdot,\cdot) \not\ni \zeta(3)^2/\pi^6$. } $\leftarrow$ (?)

Try "solving" $\star_{\text{aff}}$ mod $\bar{o}(\hbar^7)$ "=" lin. combinations of LEIBNIZ GRAPHS!

$\Rightarrow$: • ALL K. GRAPHS NEAR $\zeta(3)^2/\pi^6$ MERGE TO LEIBNIZ GRAPHS.

• MANY K. GRAPHS NEAR 1 in $w(\Gamma$ in $\star_{\text{aff}}$ mod $\bar{o}(\hbar^7))$ —//—//—//—.

Σ: $\star^{\text{red}}_{\text{aff}}$ mod $\bar{o}(\hbar^7)$. $\} \leftarrow$ # 326 $\mathbb{Q}$-coeffs $\neq 0$ (of K. GRAPHS).

Tiny f-la can be used in Physics. | • $\forall P_{\text{affine}}$, $\star_{\text{aff}}(P) \underset{\text{mod } \bar{o}(\hbar^7)}{\overset{\text{f-la}}{=}} \star^{\text{red}}_{\text{aff}}(P)$.