A gentle introduction to template games: a homotopy model of linear logic

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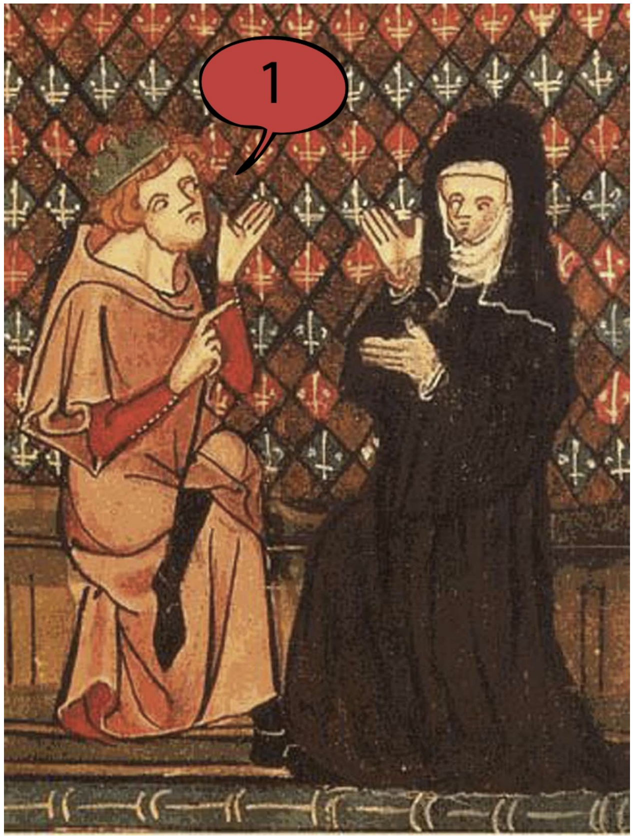
Understanding logic in space and time



What are the principles at work in a dialogue game?

Understanding logic in space and time

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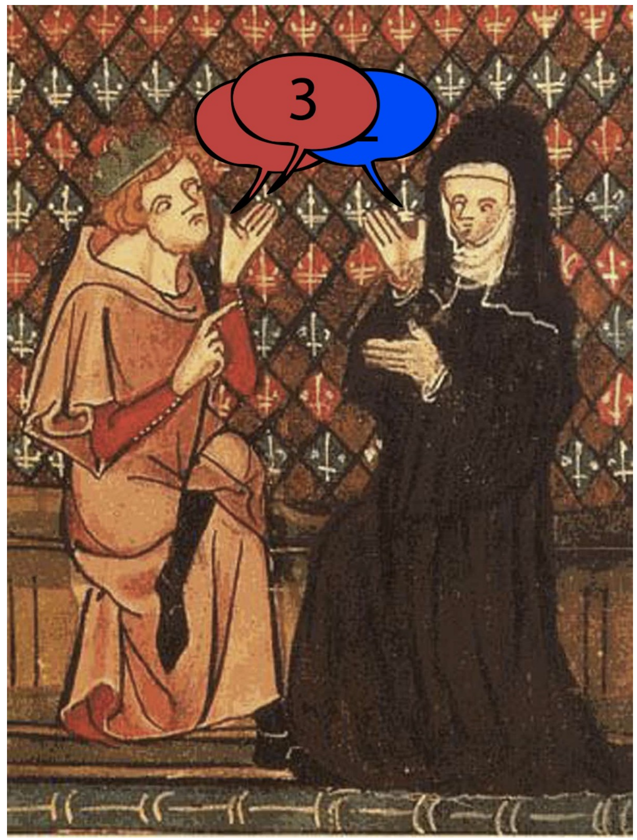

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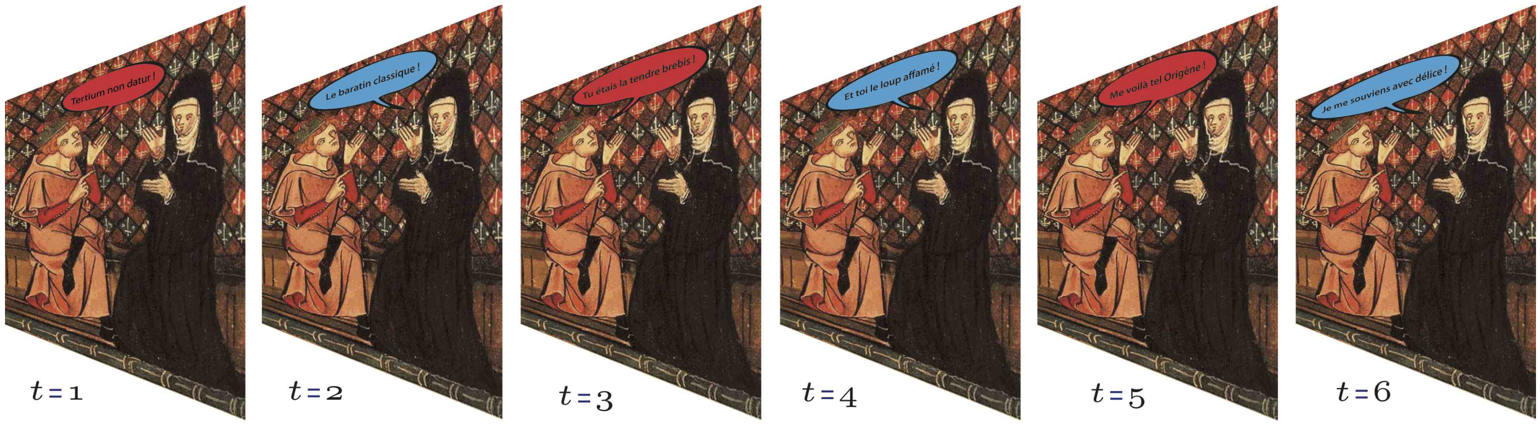
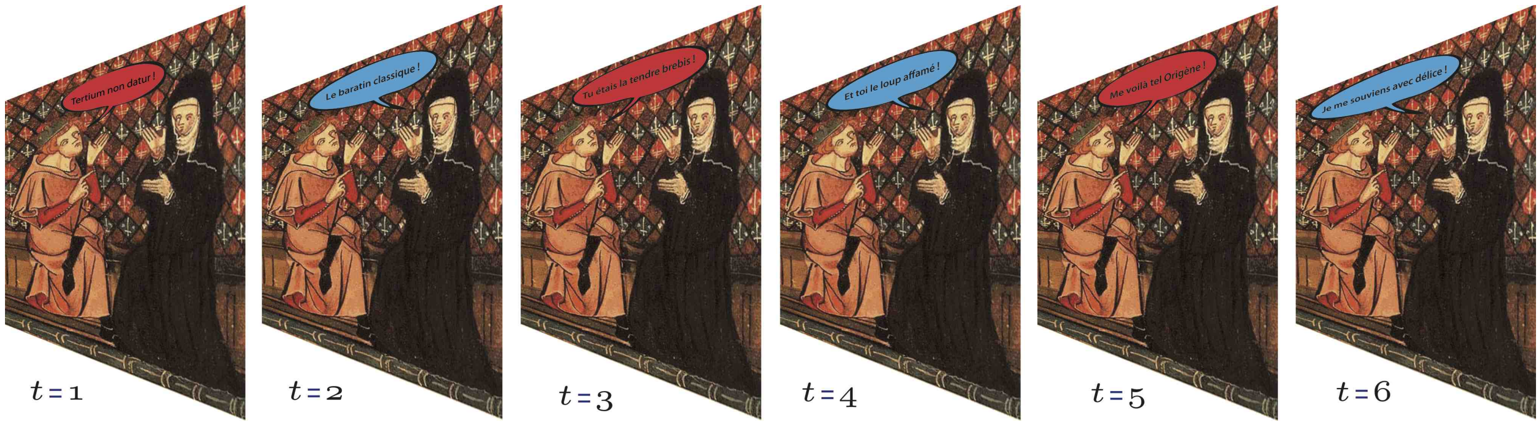


Purpose of this talk:

Understand how different proofs and programs may be

- combined together in space
- synchronized together in time

in the rich and modular ecosystem provided by **game semantics.**



Purpose of this talk:

Understand how different proofs and programs may be

- combined together in space
- synchronized together in time

in the rich and modular ecosystem provided by **linear logic.**

Linear logic

Seen through the lens of game semantics

Starting point: game semantics

Every proof of formula A initiates a dialogue where

Proponent tries to convince Opponent

Opponent tries to refute Proponent

An interactive approach to logic and programming languages

The formal proof of the drinker's formula

$$
\begin{array}{cl}
\hline
A(x_0) \vdash A(x_0) & \text{Axiom} \\
\hline
A(x_0) \vdash A(x_0), \forall x.A(x) & \text{Right Weakening} \\
\hline
\vdash A(x_0), A(x_0) \Rightarrow \forall x.A(x) & \text{Right } \Rightarrow \\
\hline
\vdash A(x_0), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Right } \exists \\
\hline
\vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Right } \forall \\
\hline
A(y_0) \vdash \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Left Weakening} \\
\hline
\vdash A(y_0) \Rightarrow \forall x.A(x), \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Right } \Rightarrow \\
\hline
\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\}, \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Right } \exists \\
\hline
\vdash \exists y.\{A(y) \Rightarrow \forall x.A(x)\} & \text{Contraction}
\end{array}
$$

The proof interpreted as a winning strategy

Step 1.

Prover picks randomly a customer y in the café,

Step 2.

Refutator contradicts Prover by exhibiting a customer x such that

x is not drinking while y is drinking !

Step 3.

Prover declares that his/her first choice of customer y was indeed wrong...

and **picks as new witness** $y' = x$ the customer exhibited by Refutator !

Step 4.

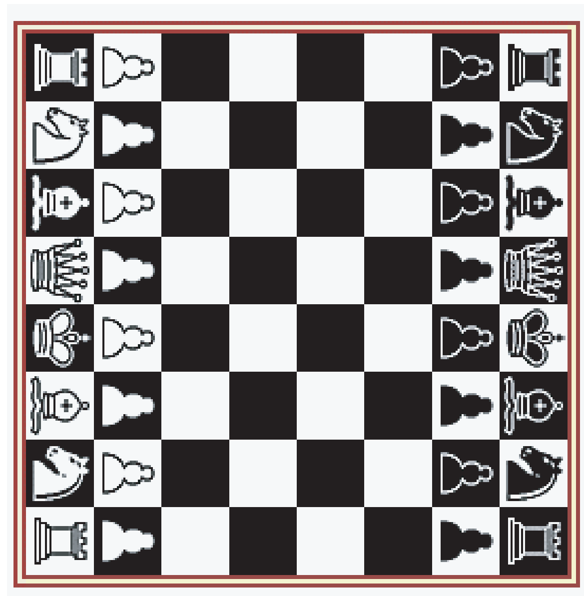
Refutator has to admit defeat and Prover wins the game...

Duality

Proponent
Program

plays the game

A



Opponent
Environment

plays the game

$\neg A$

Negation permutes the rôles of Proponent and Opponent

Duality

Opponent
Environment

plays the game

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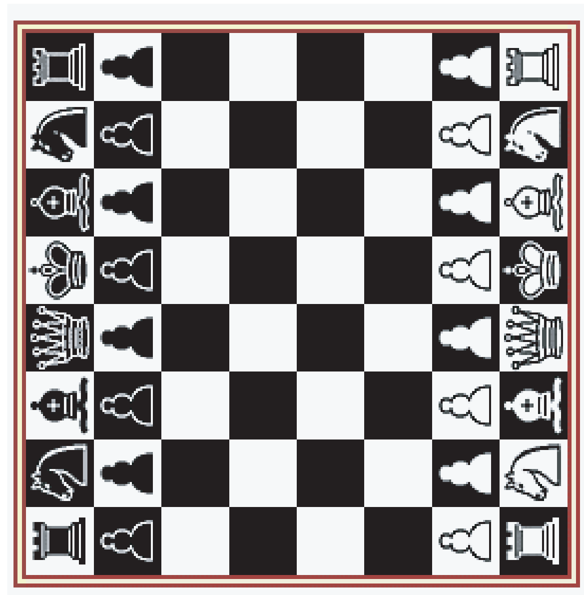


Proponent
Program

plays the game

A

Negation permutes the rôles of Opponent and Proponent

Sum

$\oplus$

Proponent selects the board which will be played

Sum

$\oplus$

A form of constructive disjunction

Product

&

Opponent selects the board which will be played

Product

&

A form of constructive conjunction

Tensor product

$\otimes$

The two games are played in parallel
Opponent is allowed to switch board but not Player

Tensor product

$$\otimes$$

A form of classical conjunction

Parallel product

⅋

The two games are played in parallel
Player is allowed to switch board but not Opponent

Parallel product

⅋

A form of classical disjunction

The law of excluded middle

Karpov **Korchnoi**

⅋

Player wins by playing Karpov against Korchnoi

The exponential modality

 $\otimes$  $\otimes$  $\otimes \cdots$

Opponent opens as many copies as necessary to beat Proponent
but is not allowed to open an infinite number of copies

Hence, the modality is {
coinductive from the point of view of Player,
inductive from the point of view of Opponent.

A beautiful isomorphism of linear logic

For every pair of formulas A and B of linear logic

$$!A \otimes !B \quad \cong \quad !(A \,\&\, B)$$

reminiscent of the isomorphism

$$\wp A \times \wp B \quad \cong \quad \wp(A + B)$$

This isomorphism is the origin for the name of **exponential** modality

Template games

Categorical combinatorics of synchronization

The category of polarities

We introduce the category

$$\pm_{\text{game}}$$

freely generated by the graph

$$\langle\ominus\rangle \;\underset{P}{\overset{O}{\leftrightarrows}}\; \langle\oplus\rangle$$

> the category $\pm_{\text{game}}$ will play a fundamental role in the talk

Template games

First idea:

Define a **game** as a category A equipped with a functor

$$\begin{array}{c} A \\ \Big\downarrow \lambda_A \\ \pm_{\text{game}} \end{array}$$

to the category $\pm_{\text{game}}$ freely generated by the graph

$$\langle \ominus \rangle \underset{P}{\overset{O}{\rightleftarrows}} \langle \oplus \rangle$$

Inspired by the notion of **coloring** in graph theory

Positions and trajectories

It is convenient to use the following terminology

objects	$\leftrightarrow$	positions
morphisms	$\leftrightarrow$	trajectories

and to see the category A as an **unlabelled** transition system.

The polarity functor

The polarity functor

$$\lambda_A \quad : \quad A \longrightarrow \pm_{\text{game}}$$

assigns a polarity $\oplus$ or $\ominus$ to every position of the game A.

Definition. A position $a \in A$ is called

Player	when its polarity	$\lambda_A(a) = \oplus$	is positive
Opponent	when its polarity	$\lambda_A(a) = \ominus$	is negative

Opponent moves

Definition. An **Opponent move**

$$m \quad : \quad a^{\oplus} \longrightarrow b^{\ominus}$$

is a trajectory of the game A transported to the edge

$$O \quad : \quad \langle \oplus \rangle \longrightarrow \langle \ominus \rangle$$

of the template category $\pm_{\text{game}}$.

Player moves

Definition. A **Player move**

$$m \quad : \quad a^{\ominus} \longrightarrow b^{\oplus}$$

is a trajectory of the game A transported to the edge

$$P \quad : \quad \langle \ominus \rangle \longrightarrow \langle \oplus \rangle$$

of the template category $\pm_{\text{game}}$.

Silent trajectories

Definition. A **silent move**

$$m \quad : \quad a \longrightarrow b$$

is a trajectory of the game A transported to an identity morphism

$$id_{\langle\oplus\rangle} \quad : \quad \langle\oplus\rangle \longrightarrow \langle\oplus\rangle$$

$$id_{\langle\ominus\rangle} \quad : \quad \langle\ominus\rangle \longrightarrow \langle\ominus\rangle$$

of the template category $\pm_{\text{game}}$.

The template of strategies

Categorical combinatorics of synchronization

The template of strategies

In order to describe the strategies between two games

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!\!\!+\quad\;\; B$$

we introduce the **template of strategies**

$$\pitchfork_{\mathrm{strat}}$$

defined as the category freely generated by the graph

$$\langle \ominus, \ominus \rangle \;\underset{O_s}{\overset{P_s}{\rightleftarrows}}\; \langle \oplus, \ominus \rangle \;\underset{P_t}{\overset{O_t}{\rightleftarrows}}\; \langle \oplus, \oplus \rangle$$

The template of strategies

Each of the four labels

$$O_s \qquad P_s \qquad O_t \qquad P_t$$

describes a specific kind of Opponent and Player move

O_s	:	Opponent move	played at	the source game
P_s	:	Player move	played at	the source game
O_t	:	Opponent move	played at	the target game
P_t	:	Player move	played at	the target game

which may appear on the interactive trajectory played by a strategy

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\quad\;\; B.$$

The template of strategies

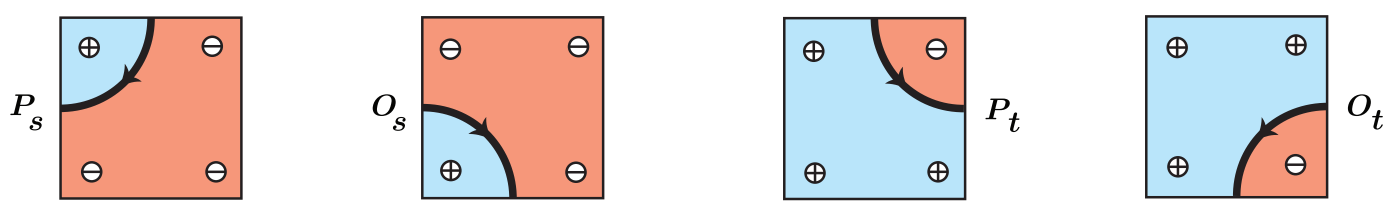
The four generators

$$\langle \ominus, \ominus \rangle \;\underset{O_s}{\overset{P_s}{\leftrightarrows}}\; \langle \oplus, \ominus \rangle \;\underset{P_t}{\overset{O_t}{\leftrightarrows}}\; \langle \oplus, \oplus \rangle$$

of the category

$$\pitchfork\,\mathrm{strat}$$

may be depicted as follows:

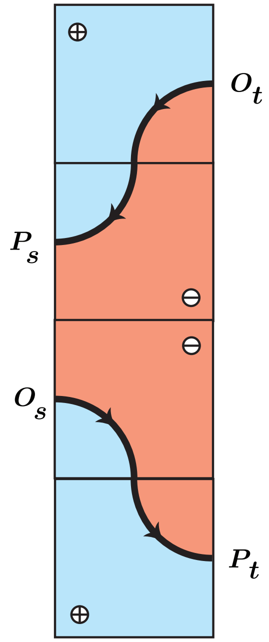


The template of strategies

In that graphical notation, the sequence

$$O_t \cdot P_s \cdot O_s \cdot P_t$$

is depicted as



The template of strategies

The category $\pitchfork_{\text{strat}}$ comes equipped with a span of functors

$$\pitchfork_{\text{game}} \xleftarrow{s=(1)} \pitchfork_{\text{strat}} \xrightarrow{t=(2)} \pitchfork_{\text{game}}$$

defined as the projection $s = (1)$ on the first component:

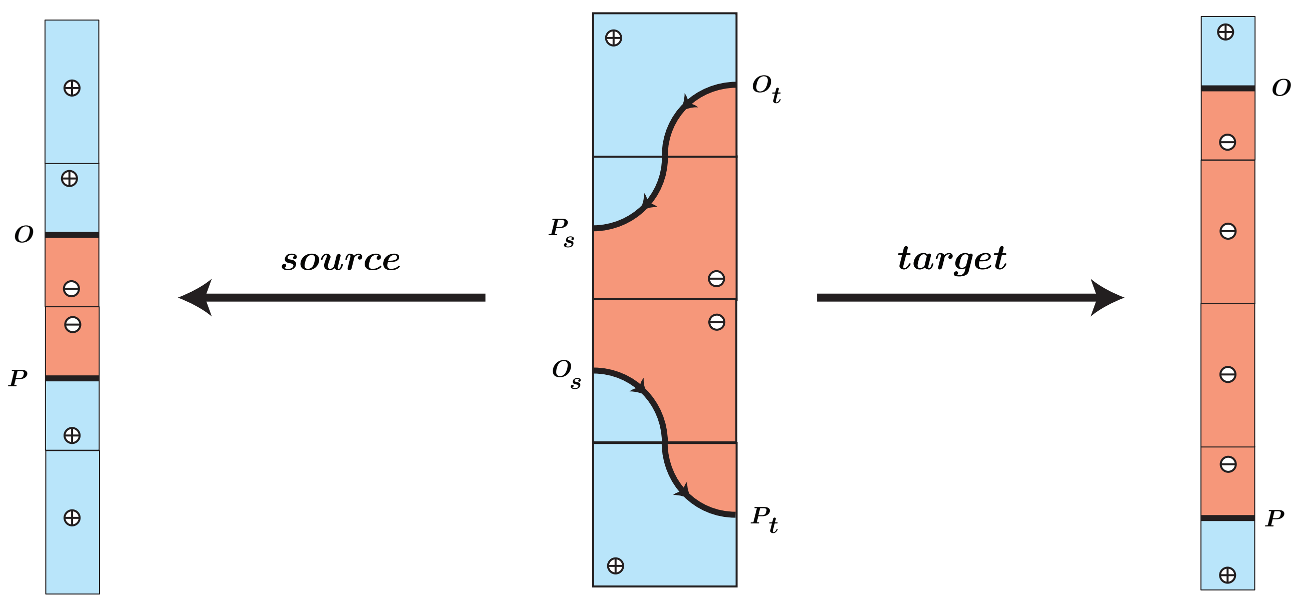
$$\begin{array}{rcl} \langle \ominus, \ominus \rangle & \mapsto & \langle \ominus \rangle \\ \langle \oplus, \ominus \rangle , \langle \oplus, \oplus \rangle & \mapsto & \langle \oplus \rangle \end{array} \qquad \begin{array}{c} O_s \mapsto P \qquad P_s \mapsto O \\ O_t \, , \, P_t \mapsto id_{\langle \oplus \rangle} \end{array}$$

and as the projection $t = (2)$ on the second component:

$$\begin{array}{rcl} \langle \oplus, \oplus \rangle & \mapsto & \langle \oplus \rangle \\ \langle \ominus, \ominus \rangle , \langle \oplus, \ominus \rangle & \mapsto & \langle \ominus \rangle \end{array} \qquad \begin{array}{c} O_t \mapsto O \qquad P_t \mapsto P \\ O_s \, , \, P_s \mapsto id_{\langle \ominus \rangle} \end{array}$$

The template of strategies

The two functors s and t are illustrated below:



Strategies between games

Second idea:

Define a **strategy** between two games

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!\!\!\!\!+\quad\; B$$

as a **span of functors**

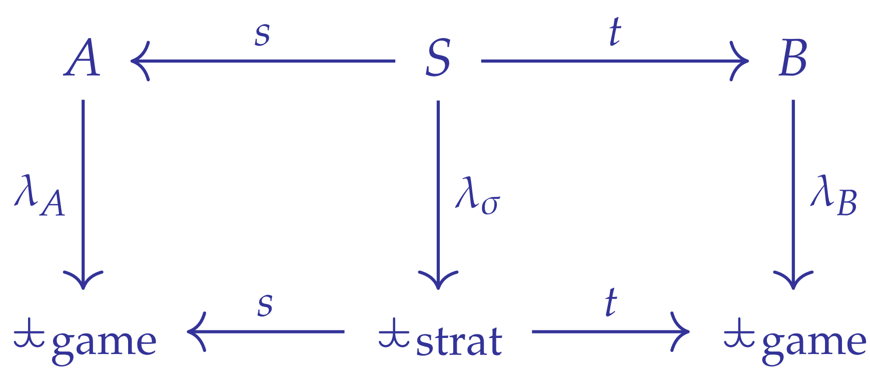
$$A \xleftarrow{\quad s \quad} S \xrightarrow{\quad t \quad} B$$

together with a **scheduling functor**

$$S \xrightarrow{\quad \lambda_\sigma \quad} \pm_{\text{strat}}$$

Strategies between games

making the diagram below commute



Key idea:

Every trajectory $s \in S$ induces a pair of trajectories $s_A \in A$ and $s_B \in B$.

The functor λ_σ describes how s_A and s_B are scheduled together by σ.

Support of a strategy

Terminology. The category S defining the span

$$A \xleftarrow{\ s\ } S \xrightarrow{\ t\ } B$$

is called the **support** of the strategy

$$\sigma \quad : \quad A \longrightarrow B$$

Basic intuition:

« the support S contains the trajectories played by σ »

A typical scheduling $B \cdot A \cdot A \cdot B$

A trajectory $s \in S$ of the strategy σ with schedule

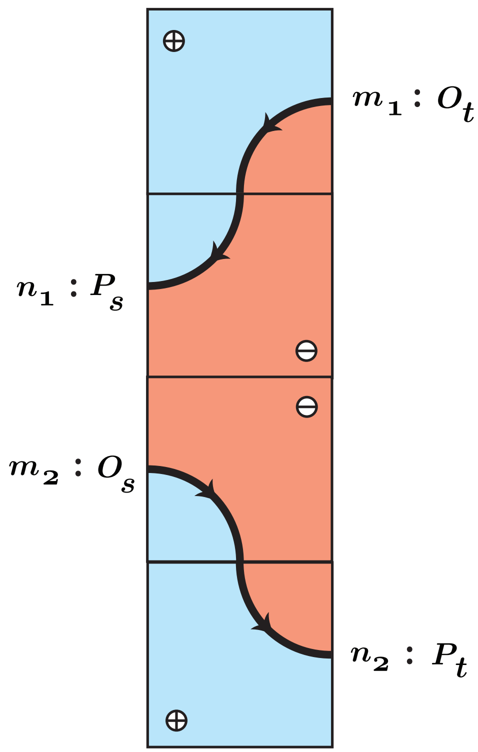
$$\langle \oplus, \oplus \rangle \xrightarrow{O_t} \langle \oplus, \ominus \rangle \xrightarrow{P_s} \langle \ominus, \ominus \rangle \xrightarrow{O_s} \langle \ominus, \oplus \rangle \xrightarrow{P_t} \langle \oplus, \oplus \rangle$$

is traditionally depicted as

	A	$\xrightarrow{\sigma}$	B
first move m_1 of polarity O_t			m_1
second move n_1 of polarity P_s	n_1		
third move m_2 of polarity O_s	m_2		
fourth move n_2 of polarity P_t			n_2

A typical scheduling $B{\cdot}A{\cdot}A{\cdot}B$

Thanks to the approach, one gets the more informative picture:



Simulations

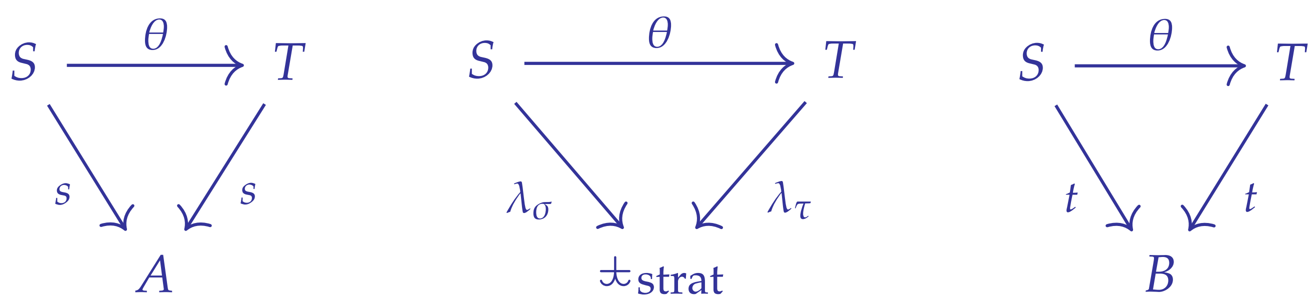
Definition: A **simulation** between strategies

$$\theta \; : \; \sigma \Longrightarrow \tau \; : \; A \longrightarrow\!\!\!\!\!\!+\;\; B$$

is a **functor** from the support of σ to the support of τ

$$\theta \; : \; S \longrightarrow T$$

making the three triangles commute



The category of strategies and simulations

Suppose given two games A and B.

The category **Games** (A, B) has **strategies** between A and B

$$\sigma, \tau \quad : \quad A \longrightarrow\!\!\!\!\!\!\!+\;\; B$$

as objects and **simulations** between strategies

$$\theta \ : \ \sigma \Longrightarrow \tau \ : \ A \longrightarrow\!\!\!\!\!\!\!+\;\; B$$

as morphisms.

The bicategory Games

A bicategory of games, strategies and simulations

The bicategory **Games** of games and strategies

At this stage, we want to turn the family of categories

$$\textbf{Games}\,(A, B)$$

into a **bicategory**

$$\textbf{Games}$$

of games and strategies.

The bicategory **Games** of games and strategies

To that purpose, we need to define a composition functor

$$\circ_{A,B,C} \; : \; \textbf{Games}(B,C) \times \textbf{Games}(A,B) \longrightarrow \textbf{Games}(A,C)$$

which composes a pair of strategies

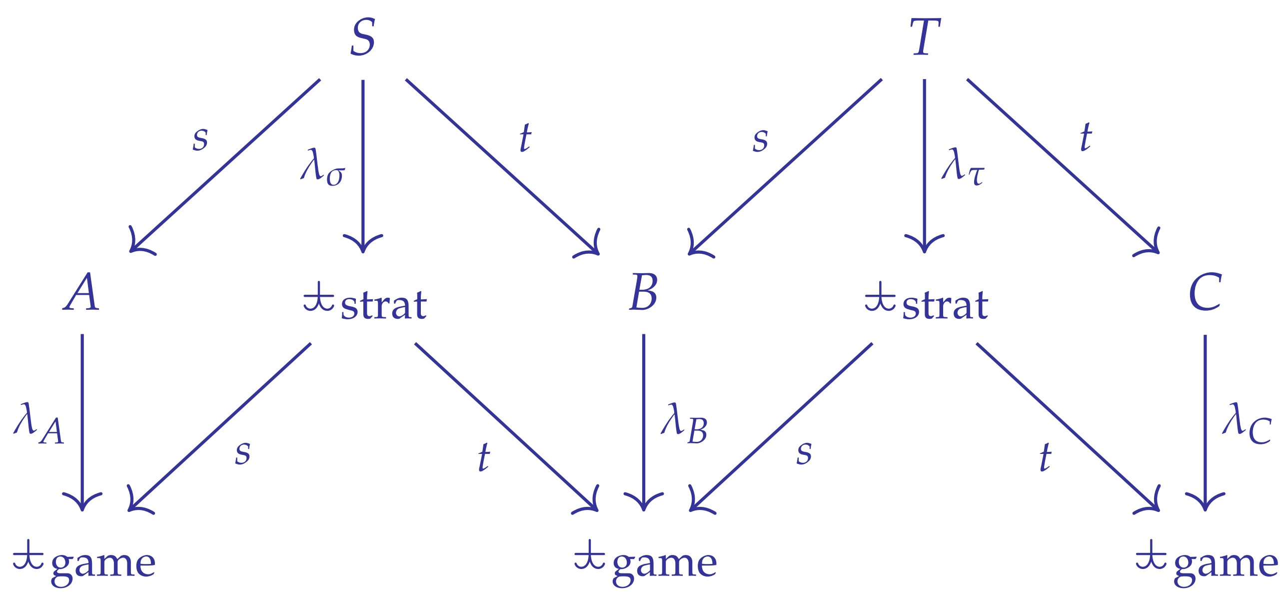
$$\sigma \; : \; A \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\; B \qquad\qquad \tau \; : \; B \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\; C$$

into a strategy

$$\sigma \circ_{A,B,C} \tau \; : \; A \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\; C$$

Composition of strategies

The construction starts by putting the pair of functorial spans side by side:



Fine, but how shall one carry on and perform the composition?

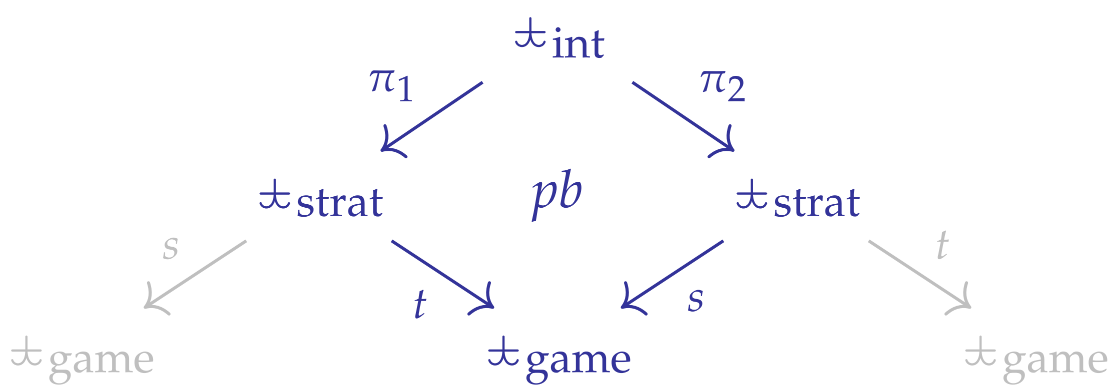
The template of interactions

Third idea:

We define the **template of interactions**

$$\maltese_{\mathrm{int}}$$

as the category obtained by the pullback diagram below



The template of interactions

Somewhat surprisingly, the category

$$\pitchfork_{\mathrm{int}}$$

is simple to describe, as the **free category** generated by the graph

$$\langle \ominus, \ominus, \ominus \rangle \underset{O_s}{\overset{P_s}{\rightleftarrows}} \langle \oplus, \ominus, \ominus \rangle \underset{P|O}{\overset{O|P}{\rightleftarrows}} \langle \oplus, \oplus, \ominus \rangle \underset{P_t}{\overset{O_t}{\rightleftarrows}} \langle \oplus, \oplus, \oplus \rangle$$

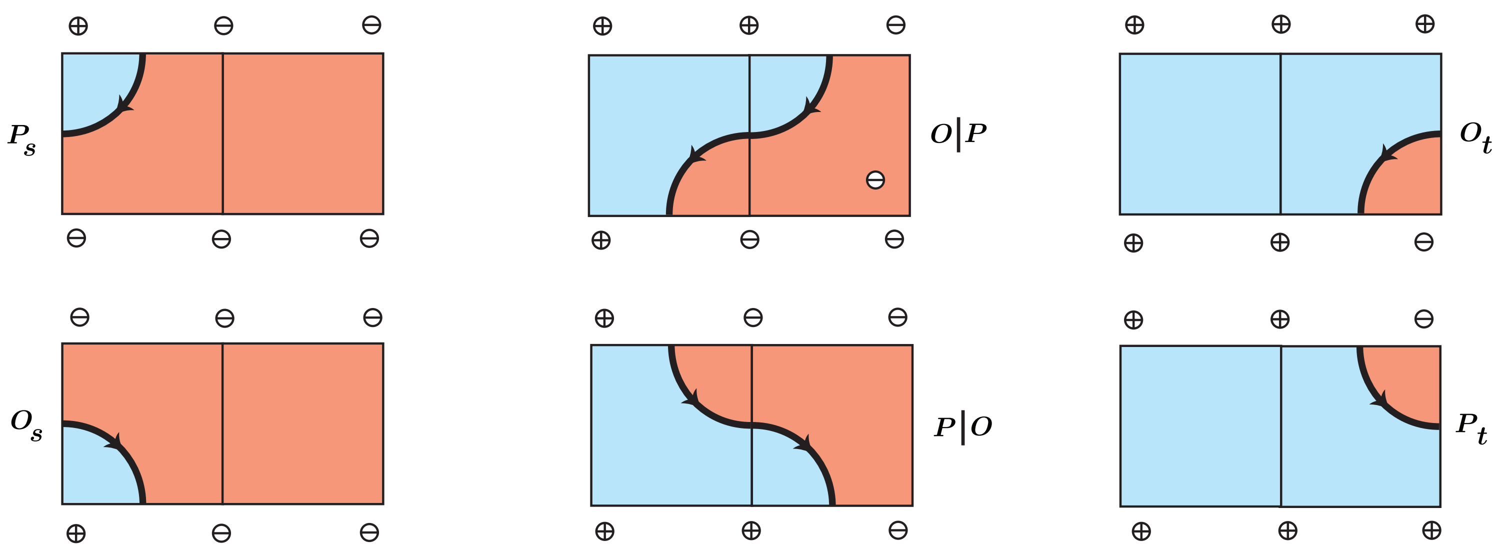
with four states or positions.

The template of interactions

The six generators

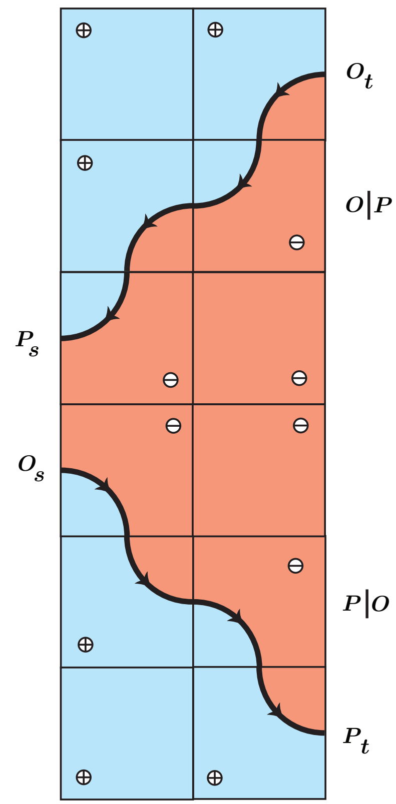
$$\langle\ominus,\ominus,\ominus\rangle \underset{O_s}{\overset{P_s}{\leftrightarrows}} \langle\oplus,\ominus,\ominus\rangle \underset{P|O}{\overset{O|P}{\leftrightarrows}} \langle\oplus,\oplus,\ominus\rangle \underset{P_t}{\overset{O_t}{\leftrightarrows}} \langle\oplus,\oplus,\oplus\rangle$$

may be depicted as follows:



A typical interaction $C \cdot B \cdot A \cdot A \cdot B \cdot C$

This typical sequence of interactions is depicted as follows:



The template of interactions

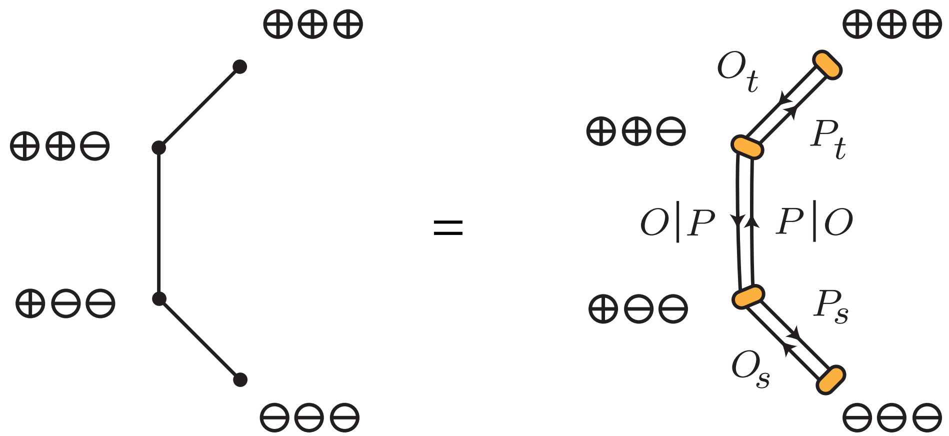
We find illuminating to depict the canonical functor

$$\pm_{\mathrm{int}} \xrightarrow{(1223)} \pm_{\mathrm{strat}} \times \pm_{\mathrm{strat}}$$

induced by the pullback diagram in the following way:

The template of interactions

In order to fully appreciate the diagram, one needs to “fatten” it



in such a way as to recover the template of interactions

$$\langle\ominus,\ominus,\ominus\rangle \;\underset{O_s}{\overset{P_s}{\leftrightarrows}}\; \langle\oplus,\ominus,\ominus\rangle \;\underset{P|O}{\overset{O|P}{\leftrightarrows}}\; \langle\oplus,\oplus,\ominus\rangle \;\underset{P_t}{\overset{O_t}{\leftrightarrows}}\; \langle\oplus,\oplus,\oplus\rangle$$

Key observation

The template $\pitchfork_{\mathrm{int}}$ of interactions comes equipped with a functor

$$hide \quad : \quad \pitchfork_{\mathrm{int}} \longrightarrow \pitchfork_{\mathrm{strat}}$$

which makes the diagram below commute:

$$\begin{array}{ccccc}
\pitchfork_{\mathrm{strat}} & \xleftarrow{(12)} & \pitchfork_{\mathrm{int}} & \xrightarrow{(23)} & \pitchfork_{\mathrm{strat}} \\
{\scriptstyle (1)}\downarrow & & \downarrow{\scriptstyle hide} & & \downarrow{\scriptstyle (2)} \\
\pitchfork_{\mathrm{game}} & \xleftarrow{s=(1)} & \pitchfork_{\mathrm{strat}} & \xrightarrow{t=(2)} & \pitchfork_{\mathrm{game}}
\end{array}$$

and thus defines a map of span.

Key observation

The functor

$$hide \quad : \quad \pitchfork_{\mathrm{int}} \longrightarrow \pitchfork_{\mathrm{strat}}$$

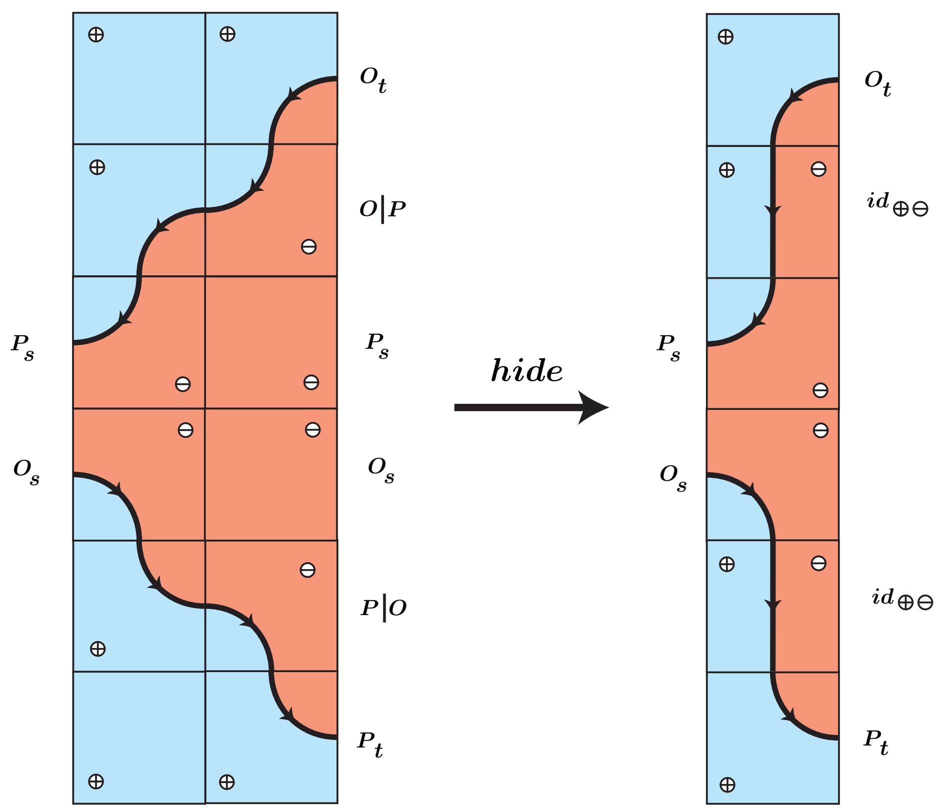
is defined by **projecting** the positions of the interaction category

$$\langle\, \varepsilon_1 \,,\, \varepsilon_2 \,,\, \varepsilon_3 \,\rangle$$

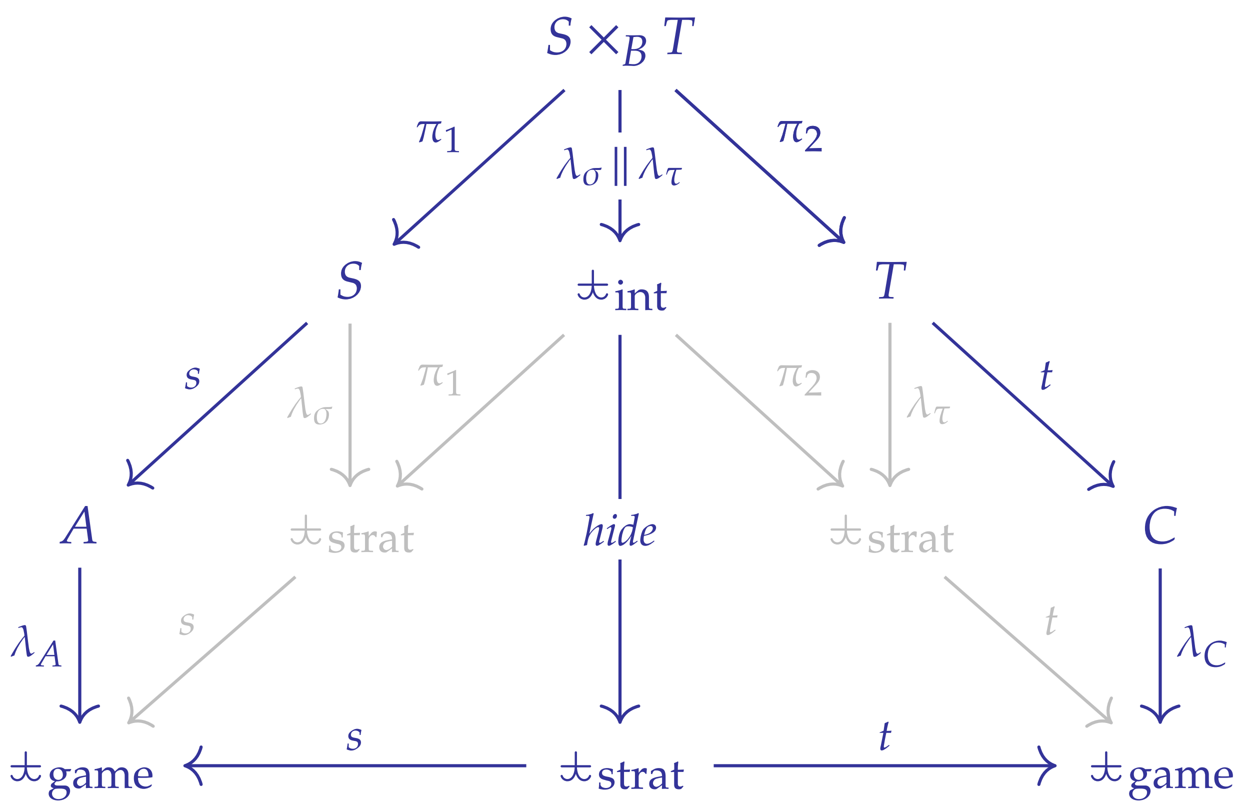
on their first and third components:

$$\begin{array}{rclcrclcrcl}
\langle \ominus, \ominus, \ominus \rangle & \mapsto & \langle \ominus, \ominus \rangle & & O_s & \mapsto & O_s & & P_s & \mapsto & P_s \\
\langle \oplus, \ominus, \ominus \rangle \,,\, \langle \oplus, \oplus, \ominus \rangle & \mapsto & \langle \oplus, \ominus \rangle & & O|P \,,\, P|O & \mapsto & id_{\langle \oplus, \ominus \rangle} & & & & \\
\langle \oplus, \oplus, \oplus \rangle & \mapsto & \langle \oplus, \oplus \rangle & & O_s & \mapsto & O_s & & P_s & \mapsto & P_s
\end{array}$$

Illustration



Composition of strategies



Composition of strategies

This definition of composition implements the slogan that

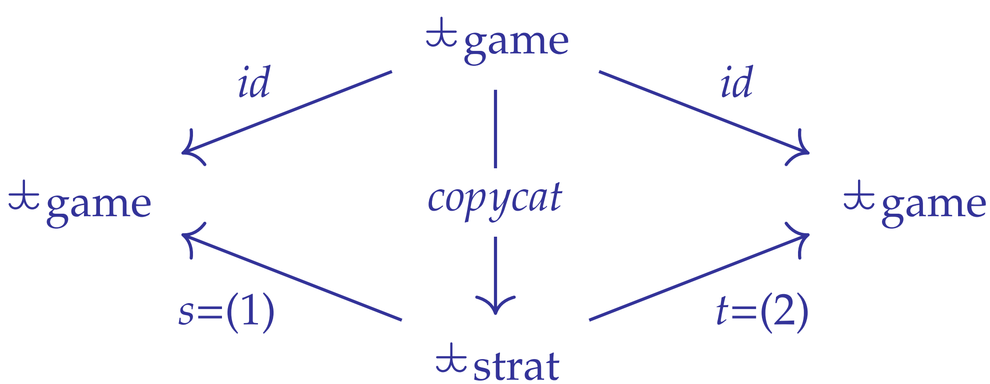
composition = synchronization + hiding

What about identities?

There exists a functor

$$\mathit{copycat} \quad : \quad \text{⨦}_{\text{game}} \longrightarrow \text{⨦}_{\text{strat}}$$

which makes the diagram commute:



and thus defines a morphism of spans.

What about identities?

The functor

$$copycat \quad : \quad \pm_{\text{game}} \longrightarrow \pm_{\text{strat}}$$

is defined by **duplicating** the positions of the polarity category

$$\langle \varepsilon \rangle$$

in the following way:

$$\begin{array}{rcl} \langle \ominus \rangle & \mapsto & \langle \ominus, \ominus \rangle \\ \langle \oplus \rangle & \mapsto & \langle \oplus, \oplus \rangle \end{array} \qquad \begin{array}{rcl} O & \mapsto & O_t \cdot P_s \\ P & \mapsto & O_s \cdot P_t \end{array}$$

A synchronous copycat strategy

The functor

$$copycat \quad : \quad ⚻_{\mathrm{game}} \longrightarrow ⚻_{\mathrm{strat}}$$

transports the edge

$$\langle \ominus \rangle \xleftarrow{\;O\;} \langle \oplus \rangle$$

to the trajectory consisting of two moves

$$\langle \ominus, \ominus \rangle \xleftarrow{\;P_s\;} \langle \oplus, \ominus \rangle \xleftarrow{\;O_t\;} \langle \oplus, \oplus \rangle$$

A synchronous copycat strategy

The functor

$$copycat \quad : \quad \pm_{\mathrm{game}} \longrightarrow \pm_{\mathrm{strat}}$$

transports the edge

$$\langle \ominus \rangle \xrightarrow{\;P\;} \langle \oplus \rangle$$

to the trajectory consisting of two moves

$$\langle \ominus, \ominus \rangle \xrightarrow{\;O_s\;} \langle \oplus, \ominus \rangle \xrightarrow{\;P_t\;} \langle \oplus, \oplus \rangle$$

The identity strategy

Given a game A, the copycat strategy

$$\mathbf{cc}_A \quad : \quad A \longrightarrow A$$

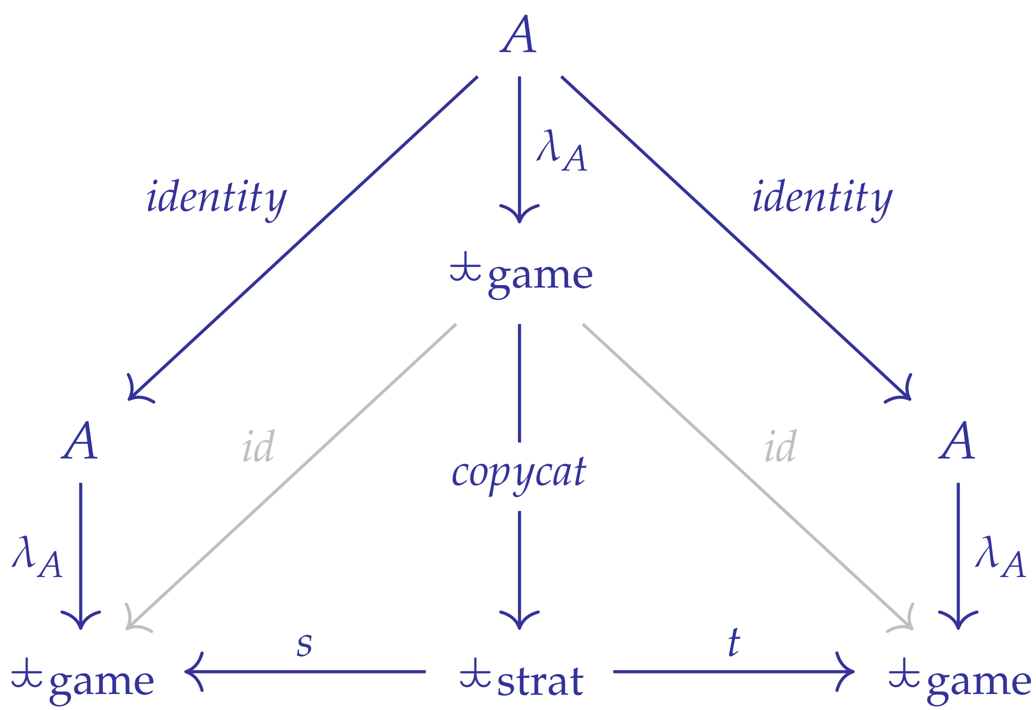
is defined as the functional span

$$A \xleftarrow{\;\textit{identity}\;} A \xrightarrow{\;\textit{identity}\;} A$$

together with the scheduling functor

$$\lambda_{\mathbf{cc}_A} \;=\; A \xrightarrow{\;\lambda_A\;} \pitchfork_{\mathrm{game}} \xrightarrow{\;\textit{copycat}\;} \pitchfork_{\mathrm{strat}}$$

Identity strategy



Discovery of an unexpected structure

Key observation: the categories

$$⫚[0] = ⫚_{\mathrm{game}} \qquad ⫚[1] = ⫚_{\mathrm{strat}} \qquad ⫚[2] = ⫚_{\mathrm{int}}$$

and the span of functors

$$⫚[0] \xleftarrow{\ s\ } ⫚[1] \xrightarrow{\ t\ } ⫚[0]$$

define an **internal category** in $\mathbf{Cat}$ with composition and identity

$$⫚[2] \xrightarrow{\ \mathit{hide}\ } ⫚[1] \qquad\qquad ⫚[0] \xrightarrow{\ \mathit{copycat}\ } ⫚[1]$$

As an immediate consequence...

Theorem A. The construction just given defines a **bicategory**

Games

of games, strategies and simulations.

Main technical result of the paper

Theorem B. The bicategory

Games

of games, strategies and simulations is **symmetric monoidal.**

Main technical result of the paper

Theorem C. The bicategory

Games

of games, strategies and simulations is **star-autonomous.**

All these results are based on the same recipe!

One constructs an **internal category** of tensorial schedules

$$⫛^{\otimes}$$

together with a pair of **internal functors**

$$⫛ \times ⫛ \xleftarrow{\;pick\;} ⫛^{\otimes} \xrightarrow{\;pince\;} ⫛$$

where *pick* and *pince* are moreover required to be **acute**.

All these results are based on the same recipe!

One constructs an **internal category** of cotensorial schedules

$$\pitchfork^{\wp}$$

together with a pair of **internal functors**

$$\pitchfork \times \pitchfork \xleftarrow{\;\;\textit{pick}\;\;} \pitchfork^{\wp} \xrightarrow{\;\;\textit{pince}\;\;} \pitchfork$$

where *pick* and *pince* are moreover required to be **acute.**

All these results are based on the same recipe!

One constructs an **internal functor**

$$reverse \quad : \quad \pitchfork^{op} \longrightarrow \pitchfork$$

which reverses the polarity of every position and move

$$\begin{array}{ccc} \oplus & \mapsto & \ominus \\ \ominus & \mapsto & \oplus \end{array} \qquad\qquad \begin{array}{ccc} O & \mapsto & P \\ P & \mapsto & O \end{array}$$

Acute internal functors

Definition An internal functor

$$F \quad : \quad \pitchfork_1 \longrightarrow \pitchfork_2$$

is **acute** when the two diagrams

$$\begin{array}{ccc} \pitchfork_1[2] & \xrightarrow{F[2]} & \pitchfork_2[2] \\ {\scriptstyle hide_1}\big\downarrow & & \big\downarrow{\scriptstyle hide_2} \\ \pitchfork_1[1] & \xrightarrow[F[1]]{} & \pitchfork_2[1] \end{array} \qquad \begin{array}{ccc} \pitchfork_1[0] & \xrightarrow{F[0]} & \pitchfork_2[0] \\ {\scriptstyle copycat_1}\big\downarrow & & \big\downarrow{\scriptstyle copycat_2} \\ \pitchfork_1[1] & \xrightarrow[F[1]]{} & \pitchfork_2[1] \end{array}$$

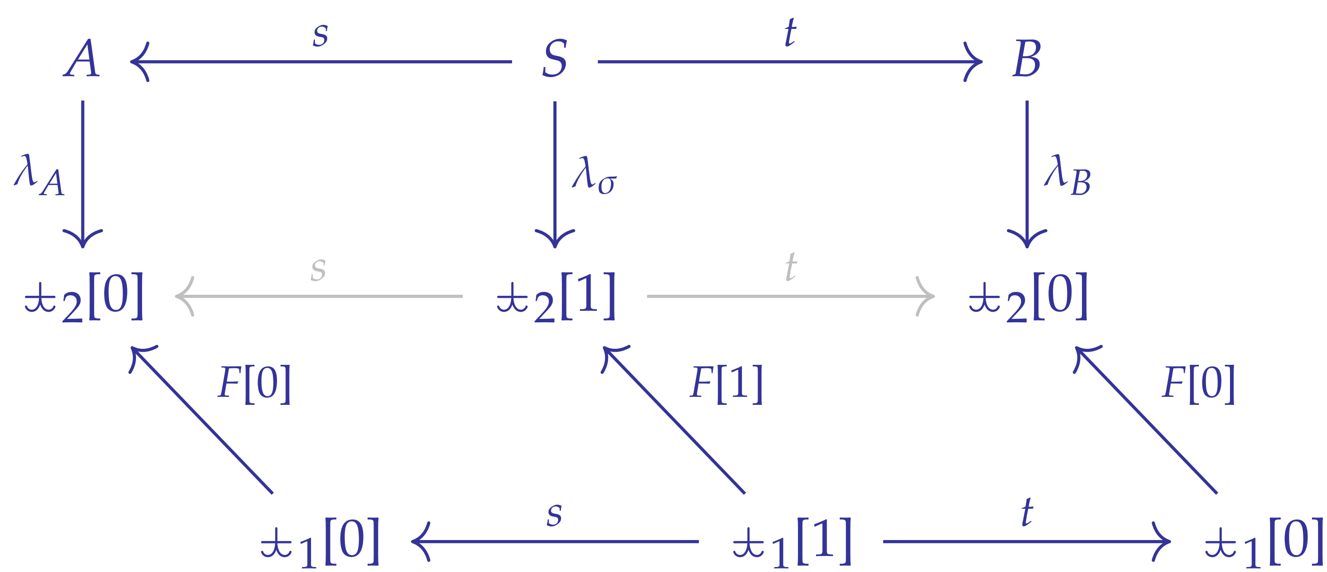
are pullback diagrams.

The backward action

Every acute internal functor $F : \pm_1 \to \pm_2$ induces a homomorphism

$$F^{\triangleleft} \quad : \quad \textbf{Games}(\pm_2) \longrightarrow \textbf{Games}(\pm_1)$$

defined by **pullback** on games and strategies:



The forward action

Every acute internal functor $F : \pm_1 \to \pm_2$ induces a homomorphism

$$F^{\triangleright} \quad : \quad \mathbf{Games}(\pm_1) \longrightarrow \mathbf{Games}(\pm_2)$$

defined by **postcomposition** on games and strategies:

$$\begin{array}{ccccc}
A & \xleftarrow{\;s\;} & S & \xrightarrow{\;t\;} & B \\
\downarrow{\scriptstyle \lambda_A} & & \downarrow{\scriptstyle \lambda_\sigma} & & \downarrow{\scriptstyle \lambda_B} \\
\pm_1[0] & \xleftarrow{\;s\;} & \pm_1[1] & \xrightarrow{\;t\;} & \pm_1[0] \\
\downarrow{\scriptstyle F[0]} & & \downarrow{\scriptstyle F[1]} & & \downarrow{\scriptstyle F[0]} \\
\pm_2[0] & \xleftarrow{\;s\;} & \pm_2[1] & \xrightarrow{\;t\;} & \pm_2[0]
\end{array}$$

The template of tensorial schedules

The recipe for the tensor product

The template of tensorial schedules

We consider the category

$$\pm^{\otimes}_{\text{game}}$$

freely generated by the graph

$$\langle \ominus, \oplus \rangle \underset{P_l}{\overset{O_l}{\leftrightarrows}} \langle \oplus, \oplus \rangle \underset{P_r}{\overset{O_r}{\rightleftarrows}} \langle \oplus, \ominus \rangle$$

The template of tensorial schedules

Idea: The three positions

$$\langle \ominus, \oplus \rangle \qquad \langle \oplus, \oplus \rangle \qquad \langle \oplus, \ominus \rangle$$

represent the three polarities

$$\langle \varepsilon_1, \varepsilon_2 \rangle$$

possibly reached by a position $a_1 \otimes a_2$ in the game

$$A_1 \otimes A_2$$

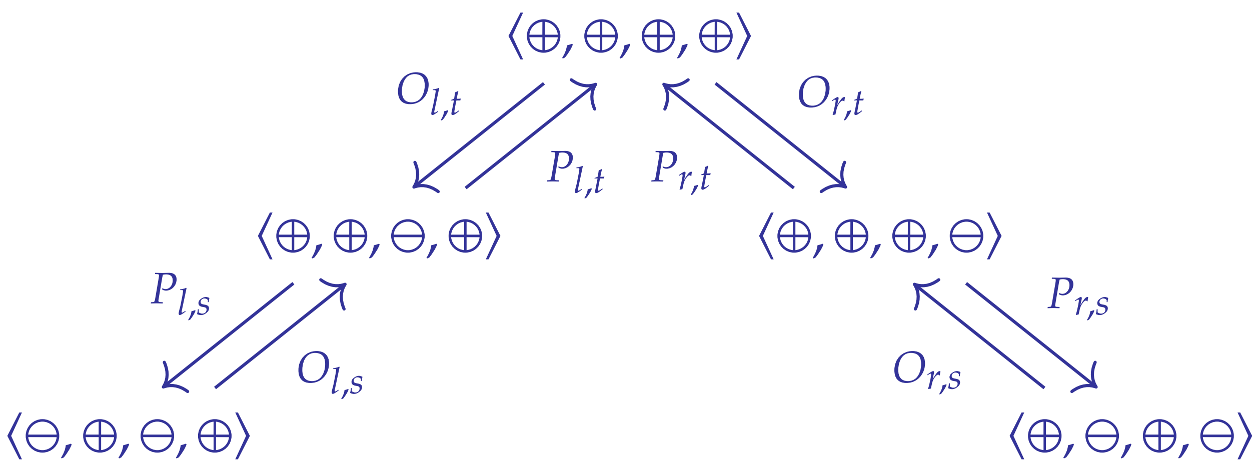
obtained by tensoring the games A_1 and A_2.

The template of tensorial schedules

The category

$$\pitchfork^{\otimes}_{\mathrm{strat}}$$

is freely generated by the graph



The template of tensorial schedules

The five positions of the category

$$\langle \ominus, \oplus, \ominus, \oplus \rangle \quad \langle \oplus, \oplus, \ominus, \oplus \rangle \quad \langle \oplus, \oplus, \oplus, \oplus \rangle \quad \langle \oplus, \oplus, \oplus, \ominus \rangle \quad \langle \oplus, \ominus, \oplus, \ominus \rangle$$

describe the five possible sequences of polarities

$$\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle$$

reached by a position of the games A_1, A_2, A_3, A_4 in a trajectory of

$$\sigma \quad : \quad A_1 \otimes A_2 \longrightarrow A_3 \otimes A_4$$

Key observation

Theorem. The categories

$$\pitchfork^{\otimes}[0] = \pitchfork^{\otimes}_{\mathrm{game}} \qquad\qquad \pitchfork^{\otimes}[1] = \pitchfork^{\otimes}_{\mathrm{sched}}$$

and the span of functors

$$\pitchfork^{\otimes}_{\mathrm{game}} \xleftarrow{\;s\;} \pitchfork^{\otimes}_{\mathrm{strat}} \xrightarrow{\;t\;} \pitchfork^{\otimes}_{\mathrm{game}}$$

define an **internal category** $\pitchfork^{\otimes}$ in the category $\mathbf{Cat}$.

A pair of internal functors

The internal category

$$⫝^{\otimes}$$

comes equipped with a pair of **internal functors**

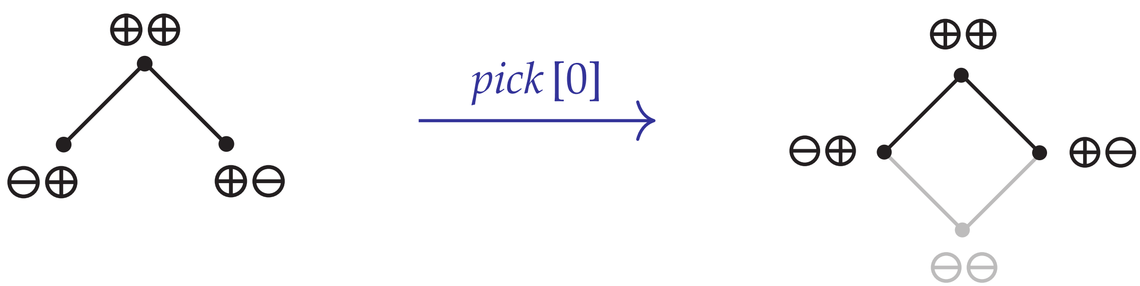
$$⫝ \times ⫝ \xleftarrow{\;pick\;} ⫝^{\otimes} \xrightarrow{\;pince\;} ⫝$$

The pick functor

The internal functor

$$pick \quad : \quad \pitchfork^{\otimes} \longrightarrow \pitchfork \times \pitchfork$$

is defined at dimension 0 by the functor:

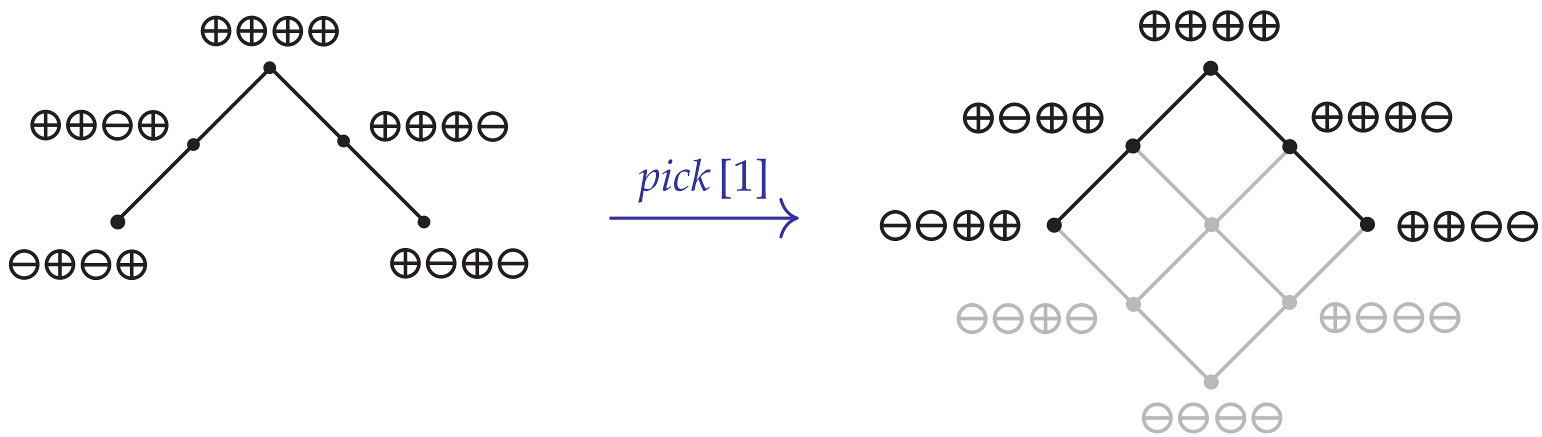


The pick functor

The internal functor

$$pick \quad : \quad \pm^{\otimes} \longrightarrow \pm \times \pm$$

is defined at dimension 1 by the functor:



The pince functor

The internal functor

$$pince \quad : \quad \pitchfork^{\otimes} \longrightarrow \pitchfork$$

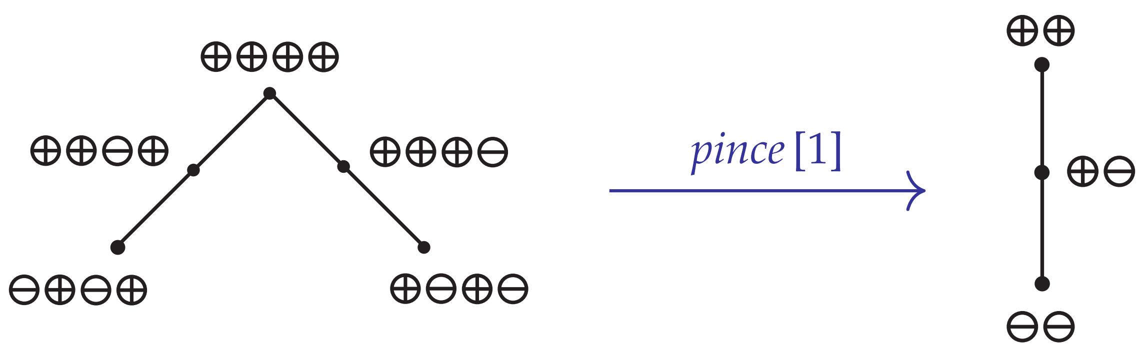
is defined at dimension 0 by the functor:

The pince functor

The internal functor

$$pince \quad : \quad ⫛^{\otimes} \longrightarrow ⫛$$

is defined at dimension 1 by the functor:

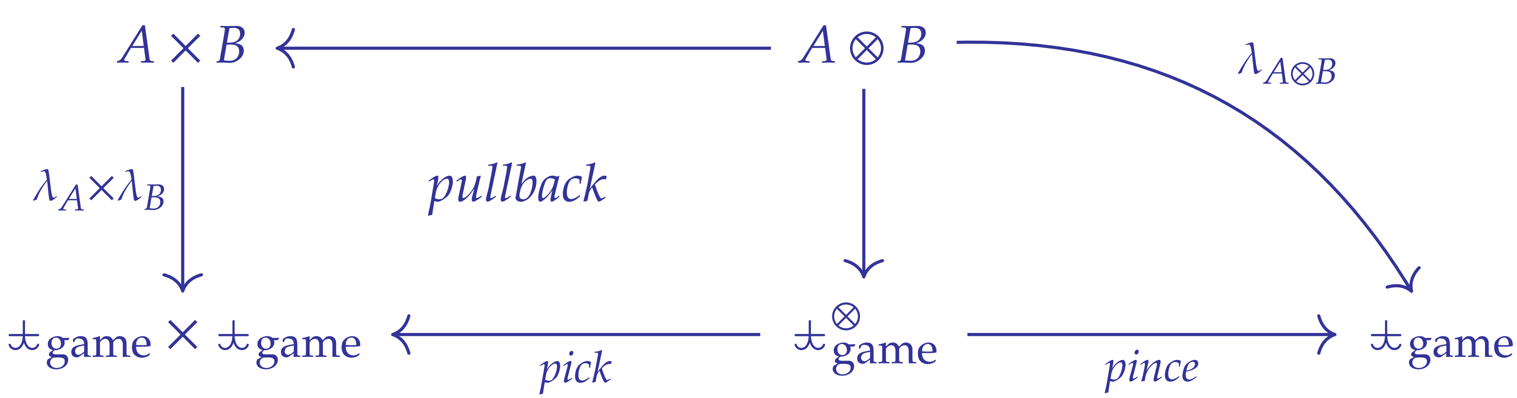


The tensor product of template games

The tensor product $A \otimes B$ of two template games

$$A \xrightarrow{\quad \lambda_A \quad} \pm_{\mathrm{game}} \qquad\qquad B \xrightarrow{\quad \lambda_B \quad} \pm_{\mathrm{game}}$$

is computed by pullback along *pick* followed by composition with *pince*:



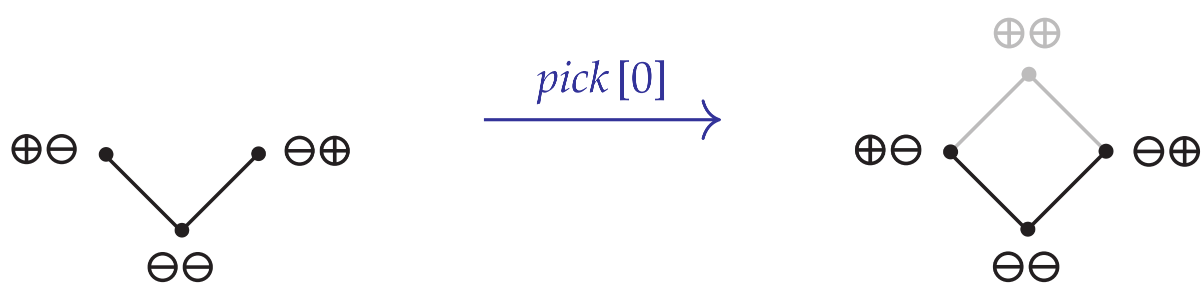
A categorical version of Milner's idea of **synchronization algebra.**

The pick functor

The internal functor

$$pick \quad : \quad \pitchfork^{\parr} \longrightarrow \pitchfork \times \pitchfork$$

is defined at dimension 0 by the functor:

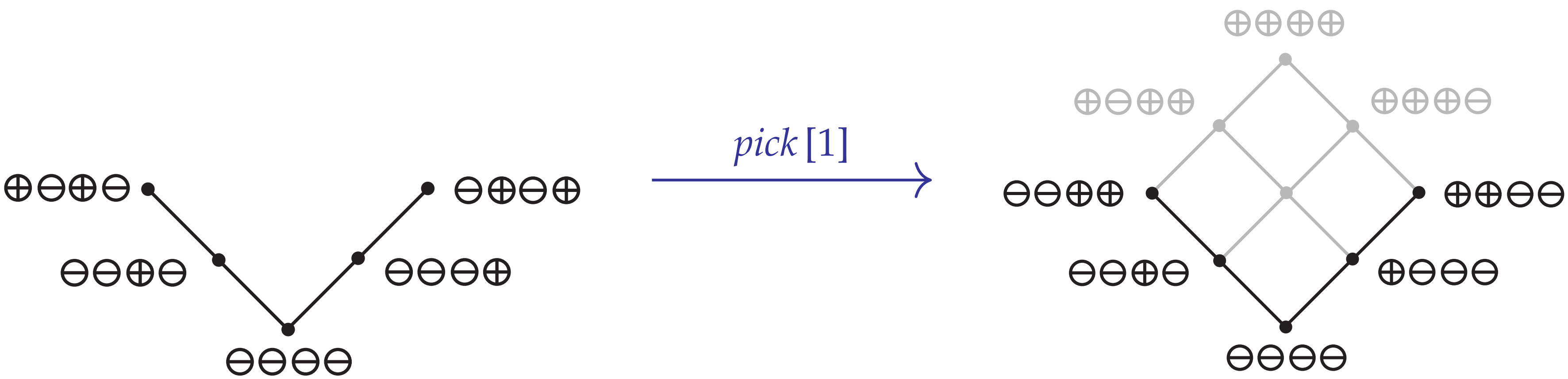


The pick functor

The internal functor

$$pick \quad : \quad \pm^{⅋} \longrightarrow \pm \times \pm$$

is defined at dimension 1 by the functor:

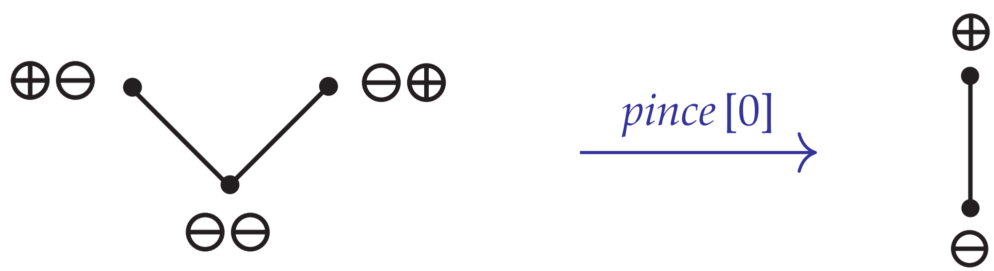


The pince functor

The internal functor

$$\mathit{pince} \quad : \quad \rotatebox[origin=c]{180}{$\pitchfork$}^{⅋} \longrightarrow \rotatebox[origin=c]{180}{$\pitchfork$}$$

is defined at dimension 0 by the functor:

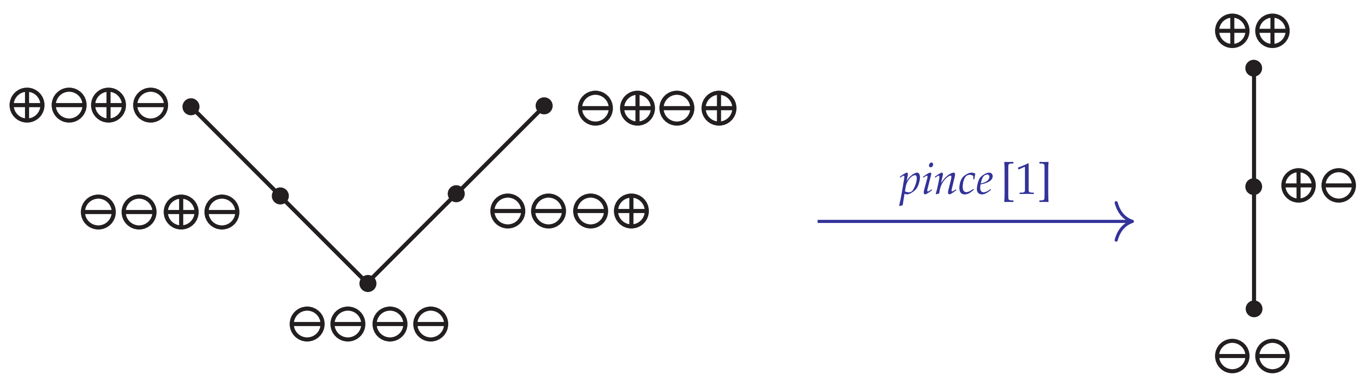


The pince functor

The internal functor

$$\mathit{pince} \quad : \quad ⫛^{⅋} \longrightarrow ⫛$$

is defined at dimension 1 by the functor:

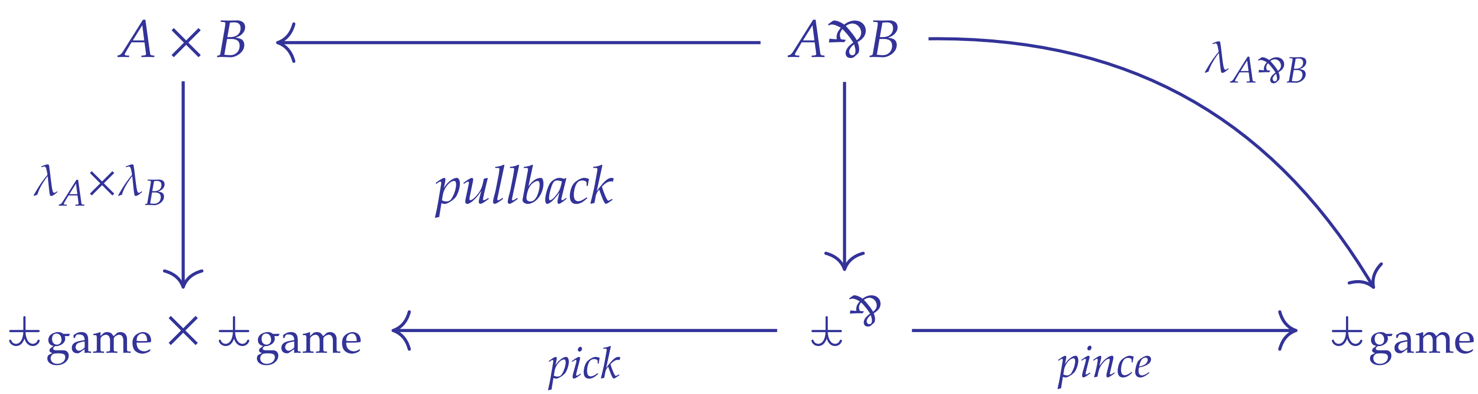


The cotensor product of template games

The cotensor product $A \parr B$ of two template games

$$A \xrightarrow{\lambda_A} \pm_{\mathrm{game}} \qquad\qquad B \xrightarrow{\lambda_B} \pm_{\mathrm{game}}$$

is computed by pullback along *pick* followed by composition with *pince*:



The distributivity law of linear logic

A game semantics of linear logic

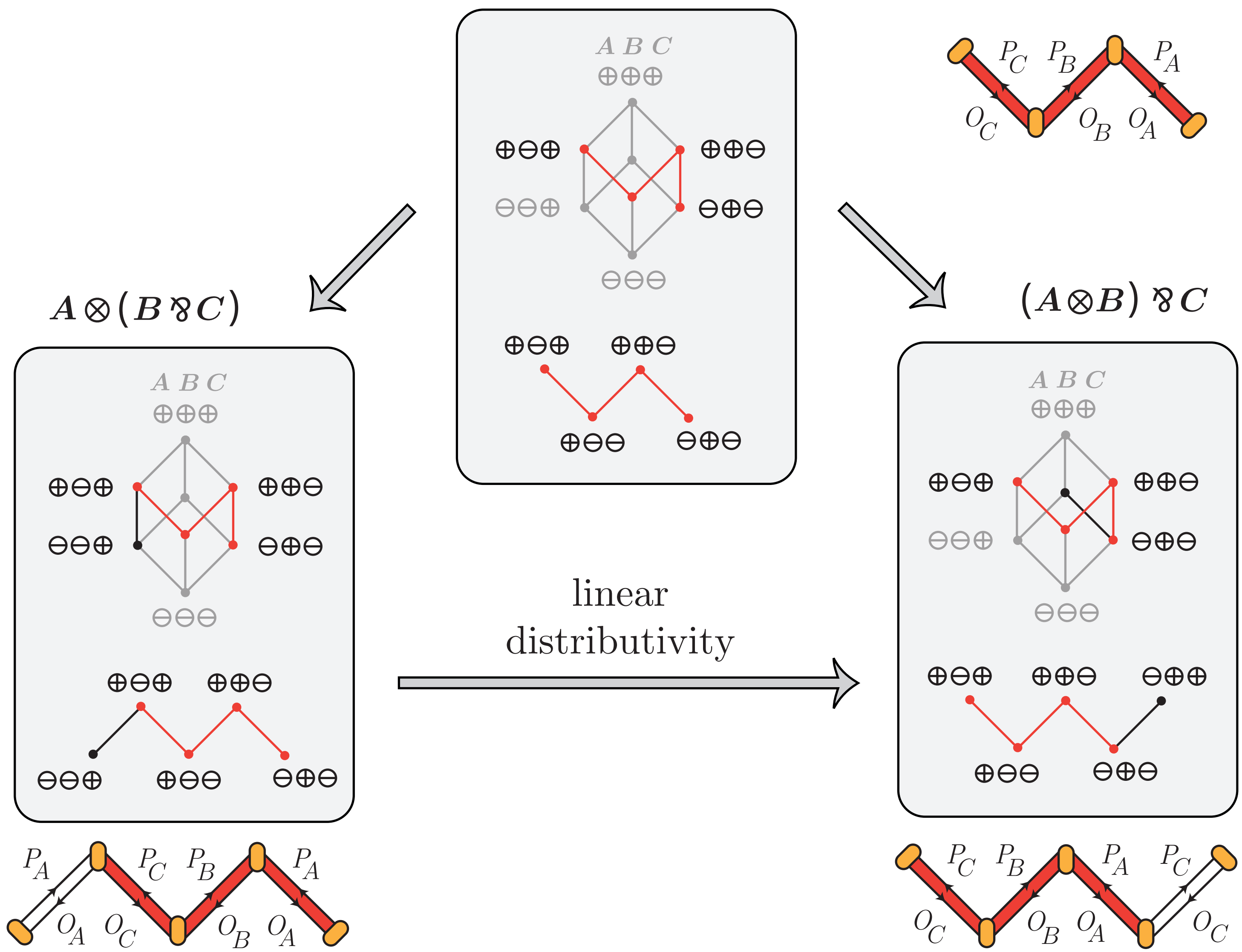
The distributivity law of linear logic

The main ingredient of linear logic

$$\kappa_{A,B,C} \quad : \quad A \otimes (B \parr C) \longrightarrow (A \otimes B) \parr C$$

cannot be interpreted in traditional game semantics.

When one interprets it in template games, here is what one gets...


A ⊗ (B ⅋ C)
(A ⊗ B) ⅋ C
linear distributivity
A B C
⊕⊕⊕
⊕⊖⊕
⊕⊕⊖
⊖⊖⊕
⊖⊕⊖
⊕⊖⊖
⊖⊖⊖
P_A
P_B
P_C
O_A
O_B
O_C

The exponential modality

The exponential modality

The construction of the exponential modality relies on the fact that

Property. The monad

$$\mathbf{Sym} \quad : \quad \mathbf{Cat} \longrightarrow \mathbf{Cat}$$

which associates to every category

$$\mathscr{C} \quad \in \quad \mathbf{Cat}$$

the freely generated symmetric monoidal category

$$\mathbf{Sym}(\mathscr{C}) \quad \in \quad \mathbf{Cat}$$

is a **cartesian monad**.

The exponential modality

From this follows that

Corollary. The monad

$$\mathbf{Sym} \quad : \quad \mathbf{Cat} \longrightarrow \mathbf{Cat}$$

transports the internal category of polarities

$$\pm$$

into an internal category

$$\mathbf{Sym}(\pm)$$

The exponential modality

The objects of

$$\mathbf{Sym}(\pm_{\mathrm{game}})$$

are the finite words

$$\varepsilon_1 \cdots \varepsilon_n$$

on the alphabet with two letters

$$\oplus \qquad \ominus$$

The template of exponential polarities

The category

$$\pm^{!}_{\mathrm{game}}$$

is defined as a the full subcategory of

$$\mathbf{Sym}(\pm_{\mathrm{game}})$$

with objects of the form

$$\oplus \cdots \oplus \cdots \oplus$$

containing **only positive polarities**, and objects of the form

$$\oplus \cdots \ominus \cdots \oplus$$

containing **exactly one negative polarity**.

The template of exponential schedules

The internal category

$$\text{よ}^{!}$$

is defined by restricting the internal category

$$\textbf{Sym}(\text{よ})$$

to the category of objects $\text{よ}^{!}_{\mathrm{game}}$ using the pullback

$$\begin{array}{ccc}
\text{よ}^{!}_{\mathrm{strat}} & \longrightarrow & \textbf{Sym}(\text{よ}_{\mathrm{strat}}) \\
\downarrow & & \downarrow \\
\text{よ}^{!}_{\mathrm{game}} \times \text{よ}^{!}_{\mathrm{game}} & \longrightarrow & \textbf{Sym}(\text{よ}_{\mathrm{game}}) \times \textbf{Sym}(\text{よ}_{\mathrm{game}})
\end{array}$$

A pair of internal functors

The internal category

$$\pitchfork^{!}$$

comes equipped with a pair of internal functors

$$\mathbf{Sym}(\pitchfork) \xleftarrow{\;pick\;} \pitchfork^{!} \xrightarrow{\;pince\;} \pitchfork$$

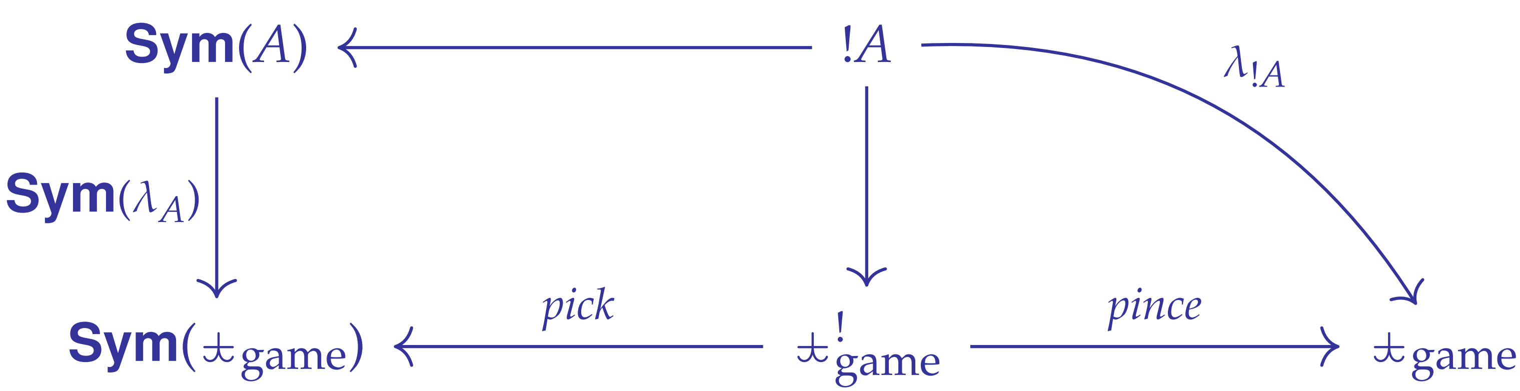
which defines an **exponential modality** of linear logic.

The exponential modality

The exponential of a template game

$$A \xrightarrow{\lambda_A} \pm_{\text{game}}$$

is simply computed by pullback followed by composition:



Main result

Theorem D. The symmetric monoidal category

Games

equipped with the exponential modality

$$!$$

defines a bicategorical (homotopy) model of differential linear logic.

Conclusion and perspectives

▷ games played on **categories** with **synchronous copycats**

▷ games played on **2-categories** with **asynchronous copycats**

▷ a number of **different templates** considered already:

$\pitchfork_{\mathrm{alt}}$	alternating games and strategies
$\pitchfork_{\mathrm{asynch}}$	asynchronous games and strategies
$\pitchfork_{\mathrm{span}}$	functorial spans with no scheduling

▷ a model of **differential linear logic** based on **homotopy theory**

▷ a model of **concurrent separation logic** based on **cobordisms** and **synchronization on machine states** with Léo Stefanesco.

Short selection of related papers

[1] PAM.
Categorical Combinatorics of Scheduling and Synchronization in Game Semantics.
POPL 2019

[2] PAM.
Template Games and Differential Linear Logic.
LICS 2019

[3] PAM.
Asynchronous Template Games and the Gray Tensor Product of 2-categories
LICS 2021

[4] Clovis Eberhart, Tom Hirschowitz and Alexis Laouar.
Template Games, Simple Games, and Day Convolution.
FSCD 2019

[5] Simon Castellan, Pierre Clairambault and Glynn Winskel.
Thin games with symmetry and concurrent Hyland-Ong games
LMCS 2020

Short selection of related papers

[1] Russ Harmer, Martin Hyland and PAM.
Categorical Combinatorics for Innocent Strategies.
LICS 2007

[2] PAM and Samuel Mimram.
Asynchronous Games: Innocence Without Alternation.
CONCUR 2007

[3] Sylvain Rideau and Glynn Winskel.
Concurrent Strategies.
LICS 2011

[4] PAM and Léo Stefanesco.
An Asynchronous Soundness Theorem for Concurrent Separation Logic.
LICS 2018

[5] PAM and Léo Stefanesco.
Concurrent Separation Logic Meets Template Games.
LICS 2020

Thank you !

The category of asynchronous graphs

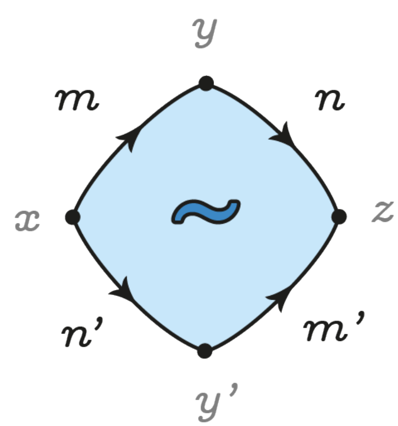
A primitive framework for concurrency theory

Asynchronous graphs

Definition. An **asynchronous graph** is defined as a graph

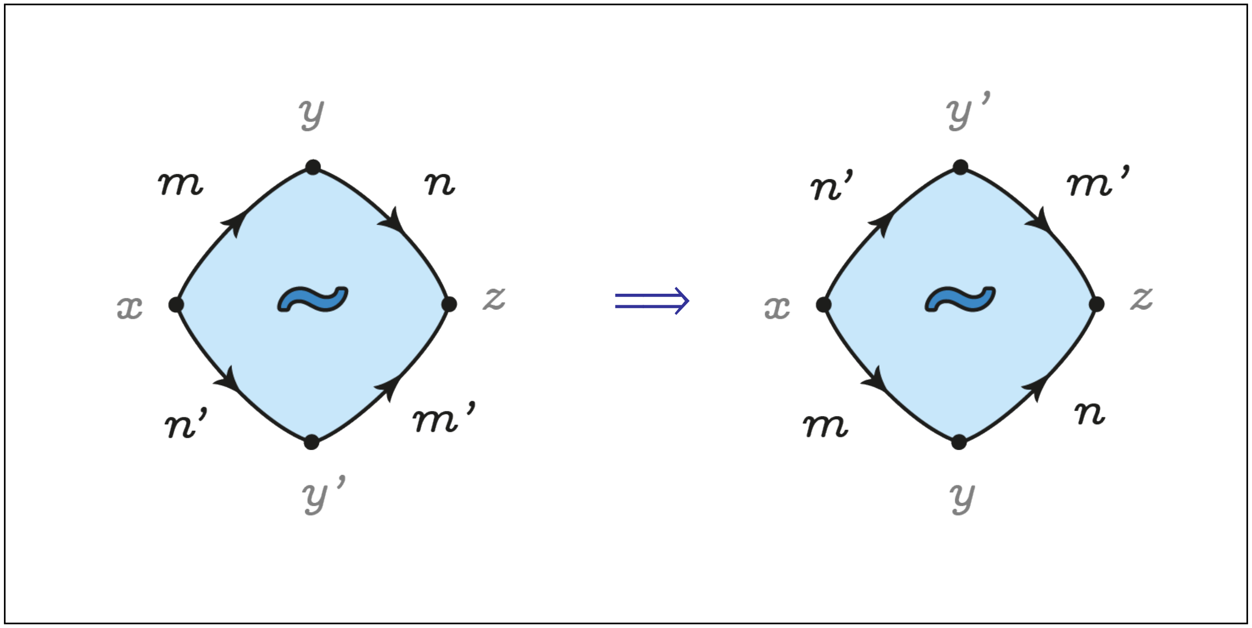
$$G = (V, E)$$

equipped with a set of **permutation tiles** of the form

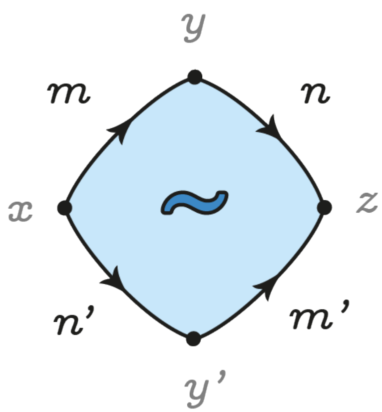


between **coinitial** and **cofinal** paths of length 2.

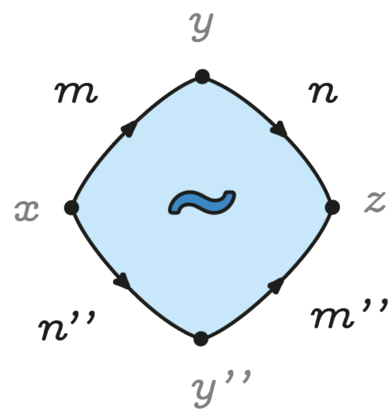
— Axiom 1 —
All permutations are symmetric



— Axiom 2 —
All permutations are deterministic

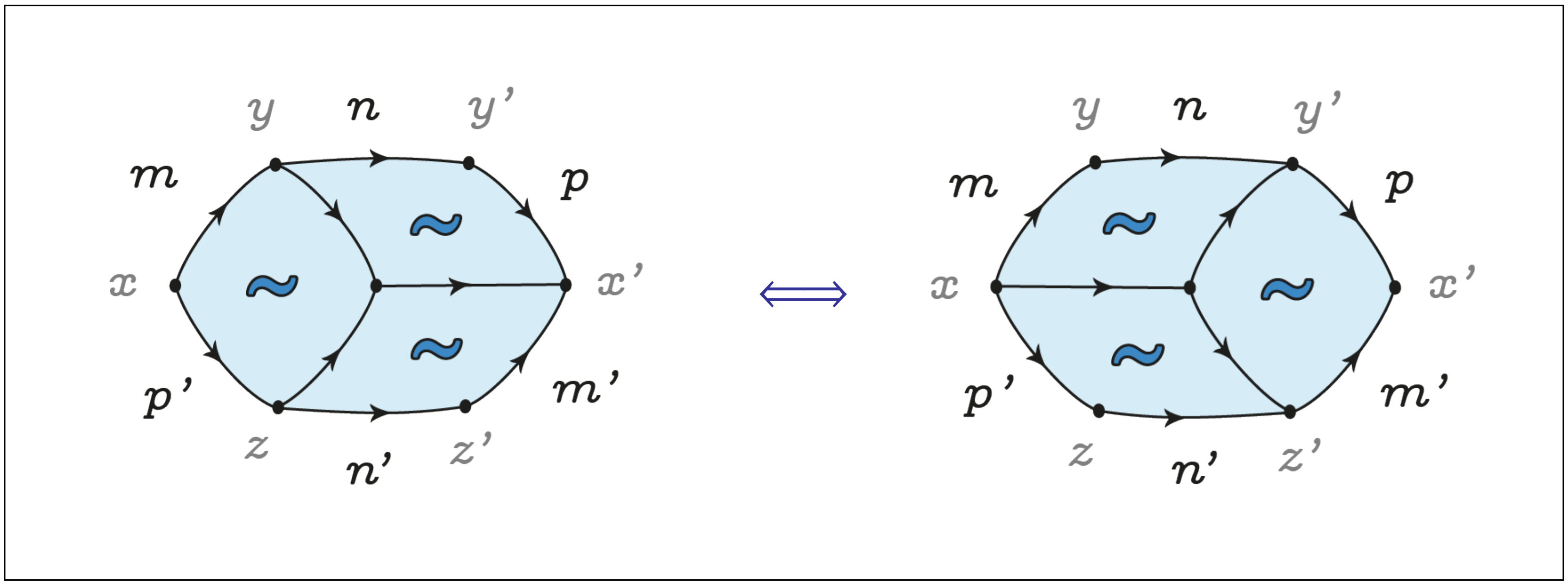


and



$$\Longrightarrow \quad n' \cdot m' = n'' \cdot m''$$

— Axiom 3 —
The cube axiom



The shuffle tensor product

The **shuffle tensor product**

$$G \sqcup\!\sqcup H \quad = \quad (G \sqcup\!\sqcup H, \diamond_{G \sqcup\!\sqcup H})$$

of two asynchronous graphs

$$G = (G, \diamond_G) \qquad\qquad H = (H, \diamond_H)$$

is the asynchronous graph

- whose vertices (x, y) are the pairs of vertices $x \in G$ and $y \in H$,

The shuffle tensor product

- whose edges are of two kinds: the pairs

$$(x, y) \xrightarrow{(u,y)} (x', y)$$

consisting of an edge in the graph G

$$x \xrightarrow{u} x'$$

and of a vertex $y \in H$; and pairs

$$(x, y) \xrightarrow{(x,v)} (x, y')$$

consisting of an edge in the graph H

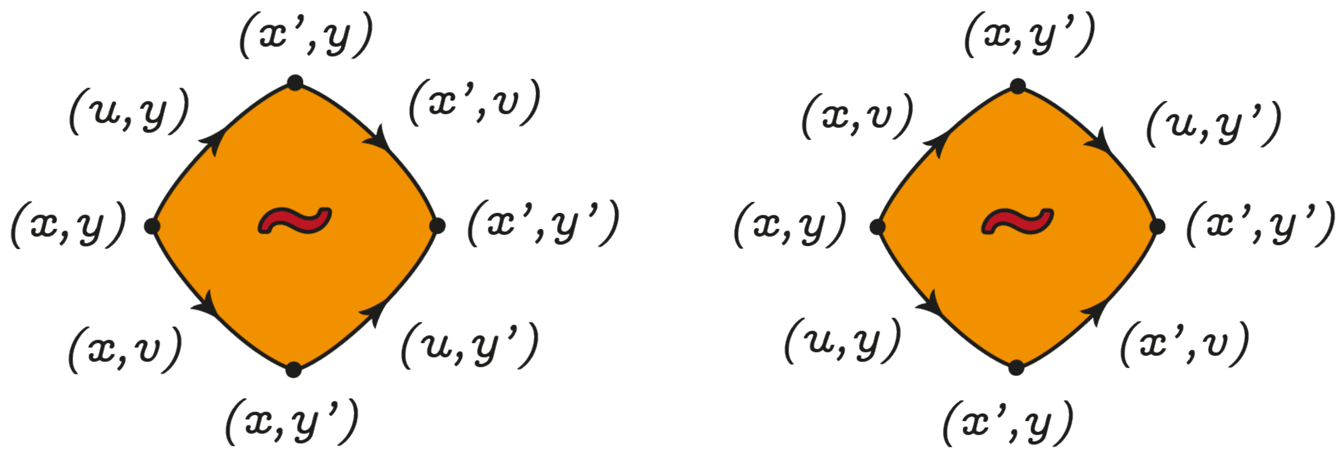
$$y \xrightarrow{v} y'$$

and of a vertex $x \in G$.

The shuffle tensor product

- whose permutation tiles are of three kinds:

1. two permutation tiles



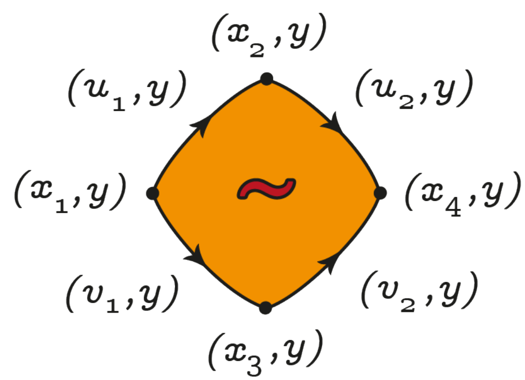
for every pair of edges

$$x \xrightarrow{u} x' \qquad\qquad y \xrightarrow{v} y'$$

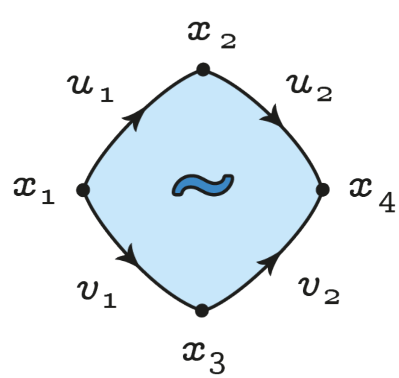
in the graphs G and H respectively ;

The shuffle tensor product

2. a permutation tile



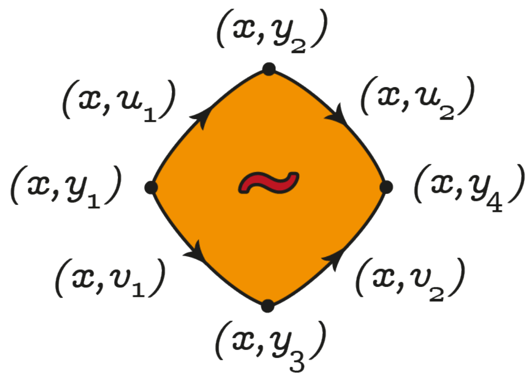
for every permutation tile



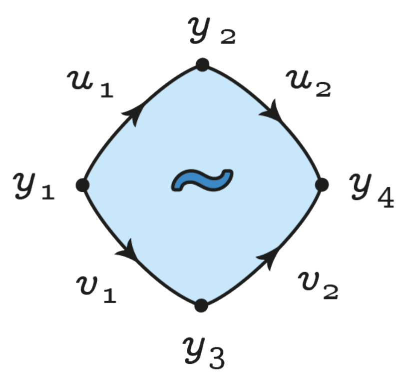
in the asynchronous graph G and every vertex $y \in H$;

The shuffle tensor product

3. a permutation tile



for every permutation tile



in the asynchronous graph H and every vertex $x \in G$.

The category of asynchronous graphs

The category **Asynch** of asynchronous graphs has its morphisms

$$f \quad : \quad (G, \diamond_G) \longrightarrow (H, \diamond_H)$$

graph homomorphisms

$$f \quad : \quad G \longrightarrow H$$

transporting every permutation tile of G to a permutation tile of H.

Theorem. The shuffle tensor product

$$G, H \quad \mapsto \quad G \sqcup\!\sqcup H \quad : \quad \textbf{Asynch} \times \textbf{Asynch} \longrightarrow \textbf{Asynch}$$

turns the category **Asynch** into a **symmetric monoidal category.**

Basic illustration

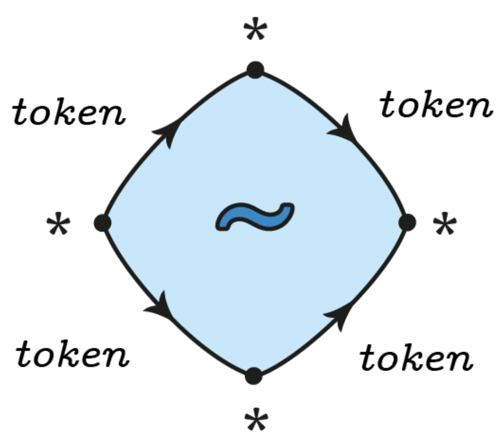
For every label token, the asynchronous graph

$$\pitchfork[\mathit{token}]$$

has a unique vertex $*$ and a unique edge

$$\mathit{token} \quad : \quad * \longrightarrow *$$

together with a unique permutation tile



Asynchronous graphs as 2-categories

A necessary step towards asynchronous template games

Asynchronous graphs seen as 2-categories

We make the basic observation that

every asynchronous graph $(G, \diamond_G)$ **generates a 2-category** $\langle G, \diamond_G \rangle$

The 2-category $\langle G, \diamond_G \rangle$ is defined in the following way:

- its objects = the vertices of the graph,
- its morphisms = the paths of the graph,
- its 2-cells = the **reshufflings** induced by the permutation tiles.

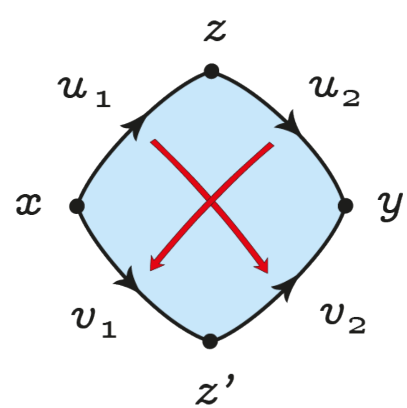
Reshufflings between paths

Definition: a **reshuffling** is a **bijective function**

$$\varphi \quad : \quad \{1, ..., n\} \quad \longrightarrow \quad \{1, ..., n\}$$

which "keeps track" of a sequence of tiles on a path of length n.

Typically, the reshuffling $\begin{pmatrix} 1 \mapsto 2 \\ 2 \mapsto 1 \end{pmatrix}$ is associated to any permutation tile:

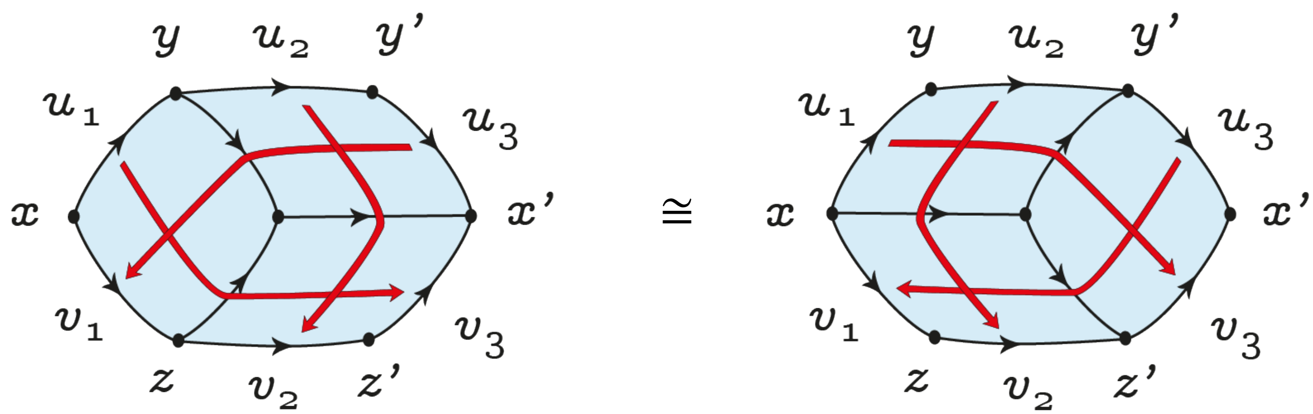


Reshufflings between paths

Similarly, the reshuffling on three indices

$$\begin{pmatrix} 1 \mapsto 3 \\ 2 \mapsto 2 \\ 3 \mapsto 1 \end{pmatrix} \quad : \quad \{1,2,3\} \quad \longrightarrow \quad \{1,2,3\}$$

keeps track and identifies the two sequences of tiles:



Related to the braid equation and the Yang-Baxter equation

From asynchronous graphs to 2-categories, functorially...

The translation induces a functor

$$\langle - \rangle \quad : \quad \mathbf{Asynch} \longrightarrow \mathbf{TwoCat}$$

where **TwoCat** is the category of 2-categories and 2-functors.

Key observation:

The functor $\langle - \rangle$ defines in fact a **symmetric monoidal functor**

$$\langle - \rangle \quad : \quad (\mathbf{Asynch}, \sqcup, \mathbf{I}) \longrightarrow (\mathbf{TwoCat}, \boxtimes, \mathbf{1})$$

equipped with a family of isomorphisms

$$\langle G \sqcup H \rangle \;\cong\; \langle G \rangle \boxtimes \langle H \rangle \qquad\qquad \langle \mathbf{I} \rangle \;\cong\; \mathbf{1}$$

where we write $\boxtimes$ for the **Gray tensor product** of 2-categories.

A homotopy structure on functorial spans

A homotopy model of differential linear logic

The natural model structure on $\mathbf{Cat}$

We distinguish three classes of functors

$$F \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

between small categories:

- ▷ the class $\mathscr{C}$ of **monos on objects**
- ▷ the class $\mathscr{F}$ of **isofibrations**
- ▷ the class $\mathscr{W}$ of **categorical equivalences**

Theorem [Joyal]

The category $\mathbf{Cat}$ of small categories and functors equipped with

$\mathscr{C}$: cofibrations $\quad$ $\mathscr{F}$: fibrations $\quad$ $\mathscr{W}$: weak equivalences

defines a Quillen model structure.

The Seely equivalence

The usual Seely isomorphism of linear logic

$$!(A \,\&\, B) \quad \cong \quad !A \otimes !B$$

is replaced in the 2-category $\mathbf{Cat}$ by a **categorical equivalence**

$$\textbf{Sym}\,(A + B) \xrightarrow{\quad deshuffle \quad} \textbf{Sym}\,A \times \textbf{Sym}\,B$$

which happens to be an **isofibration** and thus in $\mathscr{F} \cap \mathscr{W}$.

The categorical equivalence in the converse direction

$$\textbf{Sym}\,A \times \textbf{Sym}\,B \xrightarrow{\quad concat \quad} \textbf{Sym}\,(A + B)$$

happens to be a **mono on object** and thus in $\mathscr{C} \cap \mathscr{W}$.

In the case of distributors

Every functor between small categories

$$F \quad : \quad A \longrightarrow B$$

induces an adjoint pair $L_F \dashv R_F$ of distributors

$$L_F \ : \ A \longrightarrow\!\!\!\!\!\!\mapsto B \qquad\qquad R_F \ : \ B \longrightarrow\!\!\!\!\!\!\mapsto A$$

in the bicategory $\mathbf{Dist}$, where the distributors are defined as

$$L_F\,(b,a) \ = \ B\,(Fb,a) \quad : \quad B^{op} \times A \longrightarrow \mathbf{Set}$$

$$R_F\,(a,b) \ = \ B\,(a,Fb) \quad : \quad A^{op} \times B \longrightarrow \mathbf{Set}$$

In the case of functorial spans

Similarly, every functor between small categories

$$F \quad : \quad A \longrightarrow B$$

induces an adjoint pair $L_F \dashv R_F$ of categorical spans

$$L_F \;:\; A \longrightarrow\!\!\!\!\!\!\!\!\mapstochar\quad B \qquad\qquad R_F \;:\; B \longrightarrow\!\!\!\!\!\!\!\!\mapstochar\quad A$$

in the bicategory $\mathbf{Span}$, where the spans L_F and R_F are defined as

$$L_F \;=\; A \xleftarrow{\;id\;} A \xrightarrow{\;F\;} B$$

$$R_F \;=\; B \xleftarrow{\;F\;} B \xrightarrow{\;id\;} A$$

Same recipe for contractions and co-contractions

This enables one to deduce from the monoid structure in $\mathbf{Cat}$

$$\otimes_A \;:\; \mathbf{Sym}\,A \times \mathbf{Sym}\,A \longrightarrow \mathbf{Sym}\,A$$
$$I_A \;:\; \mathbf{1} \longrightarrow \mathbf{Sym}\,A$$

the comonoid structure in $\mathbf{Dist}$ of the exponential modality

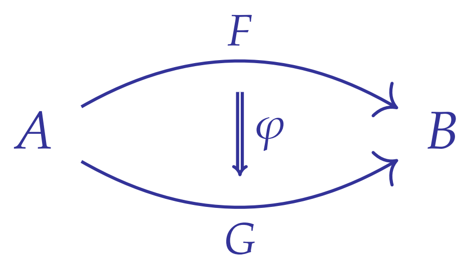
$$d_A = R_{\otimes_A} \;:\; \mathbf{Sym}\,A \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{Sym}\,A \otimes \mathbf{Sym}\,A$$
$$e_A = R_{I_A} \;:\; \mathbf{Sym}\,A \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{1}$$

as well as its monoid structure coming from the differential structure:

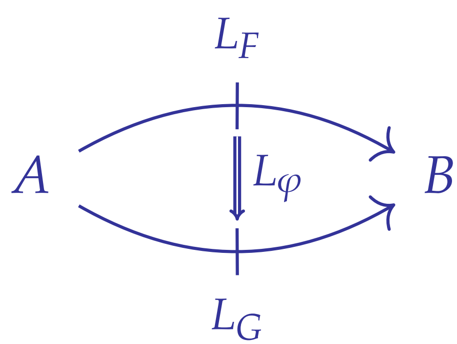
$$m_A = L_{\otimes_A} \;:\; \mathbf{Sym}\,A \otimes \mathbf{Sym}\,A \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{Sym}\,A$$
$$u_A = L_{I_A} \;:\; \mathbf{1} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbf{Sym}\,A$$

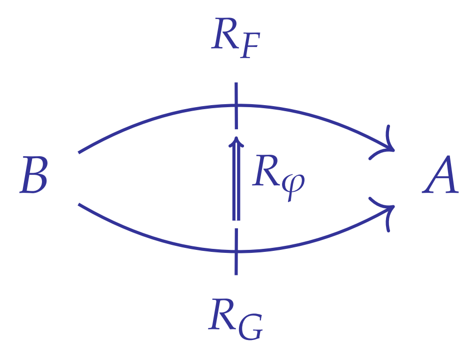
In the case of distributors

Every natural transformation in **Cat**



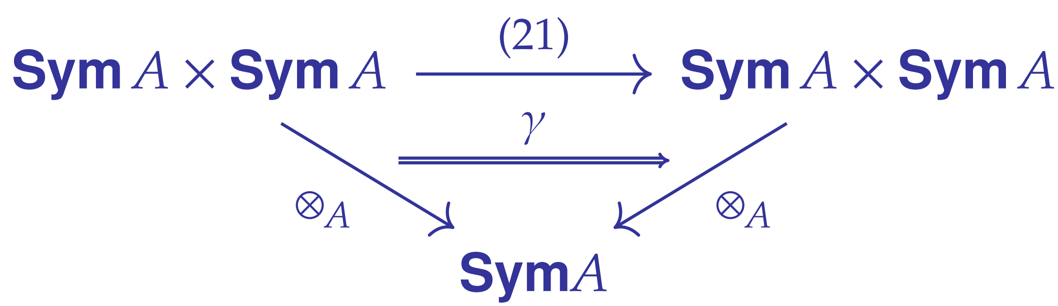
is transported to a pair of 2-cells in **Dist**



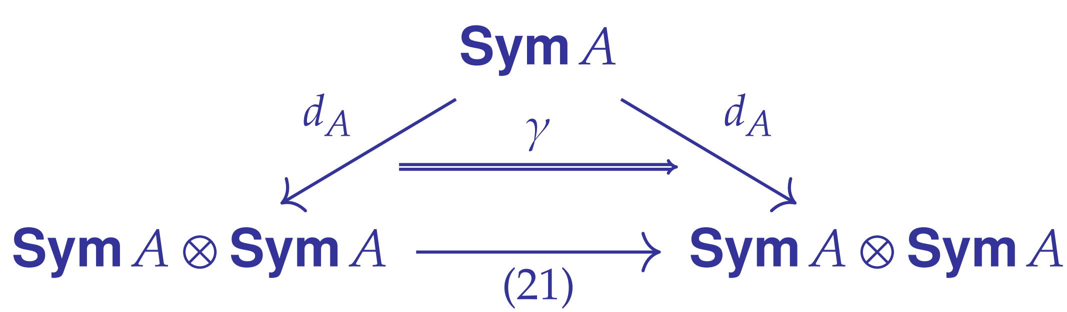


Commutativity up to an invertible 2-cell

The multiplication in **Cat** is commutative up to an isomorphism

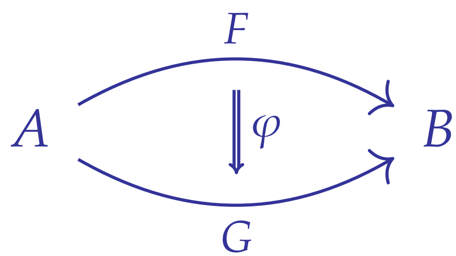


Hence, the comultiplication in **Dist** is commutative up to an isomorphism

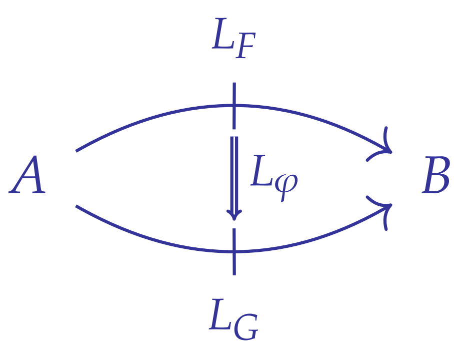


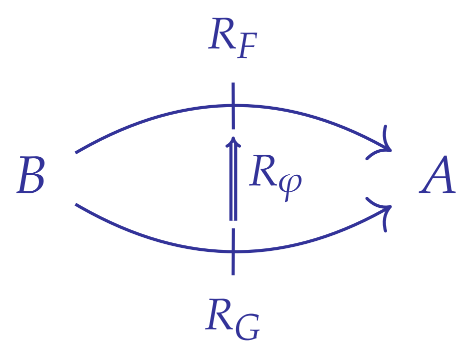
An apparent obstruction

In contrast to what happens with **Dist**, a natural transformation in **Cat**



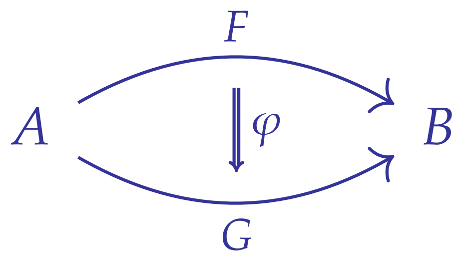
is **not transported** to a pair of 2-cells in the bicategory **Span(Cat)**





Resolving the obstruction up to homotopy

However, every natural isomorphism in **Cat**



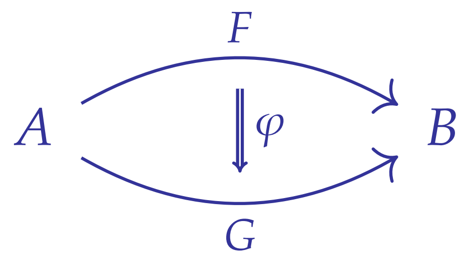
is transported to a pair of **cospans of simulations**

$$L_F \xrightarrow{inl} \widetilde{L}_\varphi \xleftarrow{inr} L_G \qquad R_F \xrightarrow{inl} \widetilde{R}_\varphi \xleftarrow{inr} R_G$$

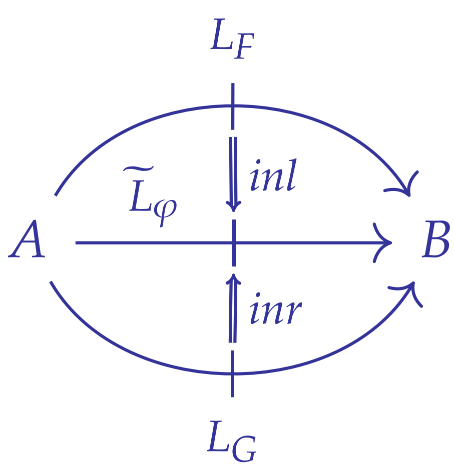
each of them defining a **cospan of 2-cells** in the bicategory **SpanCat**.

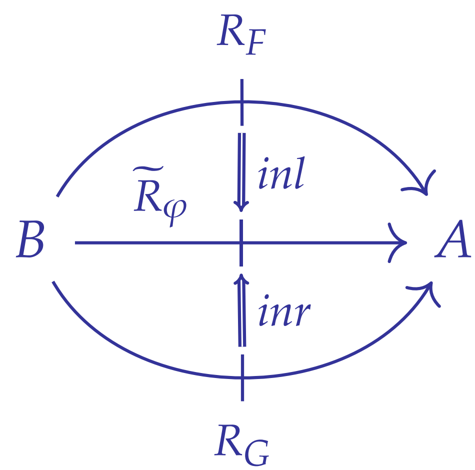
Resolving the obstruction up to homotopy

However, every natural isomorphism in **Cat**



is transported to a pair of **cospans of simulations**





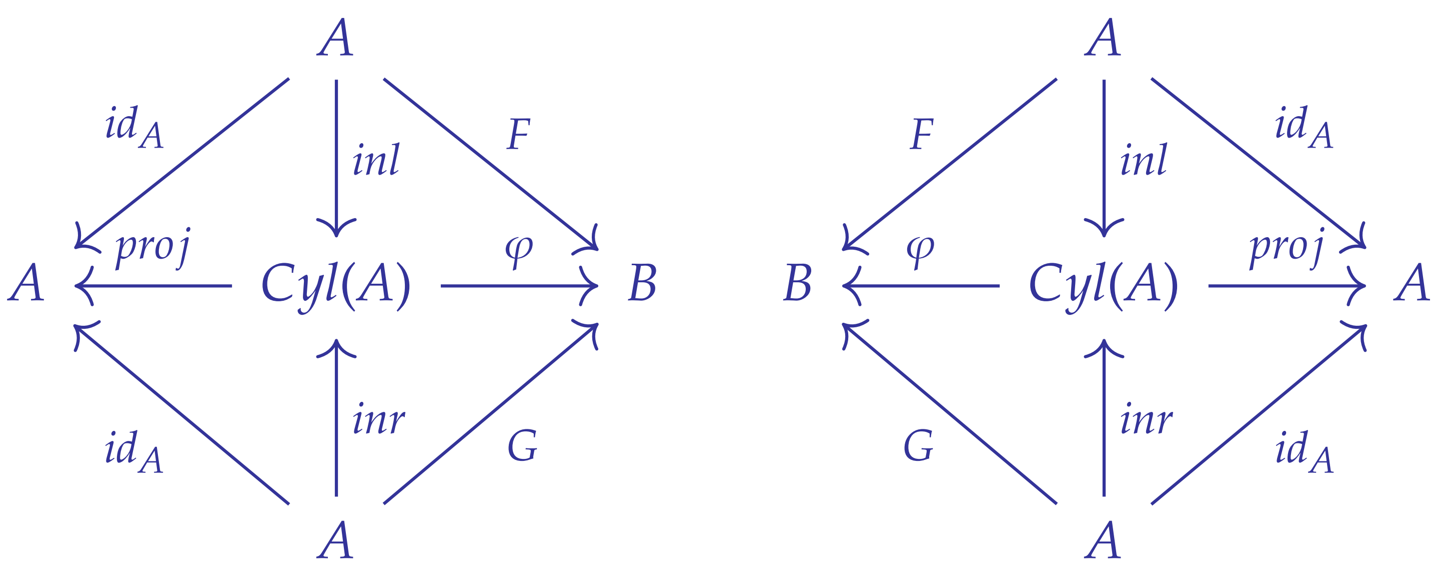
each of them defining a **cospan of 2-cells** in the bicategory **SpanCat**.

Resolving the obstruction up to homotopy

These **cospans** of 2-cells in **SpanCat**

$$L_F \xrightarrow{inl} \widetilde{L}_\varphi \xleftarrow{inr} L_G \qquad\qquad R_F \xrightarrow{inl} \widetilde{R}_\varphi \xleftarrow{inr} R_G$$

are defined as the following simulations



Resolving the obstruction up to homotopy

Here, $Cyl(A)$ denotes the **cylinder category** defined as

$$Cyl(A) \quad = \quad \mathbb{J} \times A$$

where the **interval category** $\mathbb{J}$ is the category

$$0 \xrightarrow{\ j\ } 1$$

with two objects 0 and 1 and an isomorphism $j : 0 \to 1$ between them.

The category $\mathbb{J}$ comes equipped with three functors

$$\mathbf{1} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{J} \xrightarrow{\ p\ } \mathbf{1}$$

Resolving the obstruction up to homotopy

The three functors

$$A \underset{inr}{\overset{inl}{\leftleftarrows}} Cyl(A) = A \times \mathbb{J} \xleftarrow{proj} A$$

are deduced from the three functors

$$\mathbf{1} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{J} \xrightarrow{p} \mathbf{1}$$

in the expected way:

$$inl = 0 \times A \qquad inr = 1 \times A \qquad proj = p \times A.$$

Resolving the obstruction up to homotopy

The two functorial spans

$$\widetilde{L}_\varphi \ : \ A \longrightarrow\!\!\!\!\!\!\!\!\!\!+\quad B \qquad\qquad \widetilde{R}_\varphi \ : \ B \longrightarrow\!\!\!\!\!\!\!\!\!\!+\quad A$$

are defined as

$$A \xleftarrow{\ proj\ } Cyl(A) \xrightarrow{\ \varphi\ } B \qquad\qquad B \xleftarrow{\ \varphi\ } Cyl(A) \xrightarrow{\ proj\ } A$$

where the functor

$$\varphi \ : \ Cyl(A) \longrightarrow B$$

internalizes the natural isomorphism $\varphi : F \Rightarrow G : A \rightarrow B$ and thus satisfies:

$$F = \varphi \circ inl \qquad\qquad G = \varphi \circ inr$$

required for the functors inl and inr to define simulations.