Hausdorff moment problems for combinatorial numbers: heuristics via Meijer G-functions

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1) Combinatorial numbers expressible by factorials (**)
2) Hausdorff moment problem for them
3) Integral representations
4) Properties of solutions:
   - solubility, • positivity, • algebraicity

   Relations between them (if any)
5) Outlook

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*) Collaborators (at various stages):

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(**) (The Bell numbers etc. do not enter this category)

Hausdorff Moment Problem

given $\varrho(n)$, $n = 0, 1, \ldots$,

find $W(x)$ satisfying

$(0, R)$ = support of $W(x)$

$$\varrho(n) = \int_0^R x^n W(x)\, dx$$

$$R = \lim_{n\to\infty} \left([\varrho(n)]^{1/n} \right) < \infty$$

$$\mathrm{ogf}(z) = \sum_{n=0}^{\infty} \varrho(n) z^n$$

Cauchy test for convergence of ogf

Solutions $W(x)$ always unique.

Tools:

Mellin transform and its inverse

Meijer G-functions
(defined as Mellin inverse)

Online Encyclopedia of Integer Sequences
(OEIS, N.J.A. SLOANE)

Notation:

Moments $\rho(n)$, $n=0,1\dots$

Ordinary Generating Functions

- $\mathrm{ogf}(z) = \sum_{n=0}^{\infty} \rho(n) z^n = g(z)$
- $W(x)$ = weight functions in moment problems

In order to avoid the proliferation of indices:
the same for all examples

Mellin Transform and its inverse:

$f(x)$ defined for $x \geqslant 0$.

$$\mathcal{M}[f(x);s] = \int_0^\infty x^{s-1} f(x)\,dx \equiv f^*(s)$$

$$\mathcal{M}^{-1}[f^*(s);x] = \underbrace{f(x)}_{x}$$

Ex.: $\mathcal{M}[e^{-x};s] = \int_0^\infty x^{s-1}e^{-x}\,dx \equiv \Gamma(s) \Longrightarrow \mathcal{M}^{-1}[\Gamma(s);x] = e^{-x}$

- $\boxed{\text{Meijer } G} = G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} \alpha_1 \ldots \alpha_n, & \alpha_{n+1} \ldots \alpha_p \\ \beta_1 \ldots \beta_m, & \beta_{m+1} \ldots \beta_q \end{matrix}\right) =$

MeijerG$([[\alpha_1 \ldots \alpha_n],[\alpha_{n+1} \ldots \alpha_p]],[[\beta_1 \ldots \beta_m],[\beta_{m+1} \ldots \beta_q]], z) =$

$$\mathcal{M}^{-1}\left[\frac{\prod_{j=1}^{m}\Gamma(\beta_j+s)\prod_{j=1}^{n}\Gamma(1-\alpha_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j-s)\prod_{j=n+1}^{p}\Gamma(\alpha_j+s)}; z\right]$$

Meijer G-functions are inverse Mellin transforms

of ratios of shifted Γ's

of variables s and $-s$

In our applications:

NO shifted $\Gamma(\ldots -s)$!

In our applications

- $\boxed{\text{Meijer G}} = G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} \alpha_1 \ldots \alpha_n, & \alpha_{n+1} \ldots \alpha_p \\ \beta_1 \ldots \beta_m, & \beta_{m+1} \ldots \beta_q \end{matrix}\right) =$

$$\text{MeijerG}\left([[\alpha_1 \ldots \alpha_n],[\alpha_{n+1} \ldots \alpha_p]],[[\beta_1 \ldots \beta_m],[\beta_{m+1} \ldots \beta_q]], z\right) =$$

$$\mathcal{M}^{-1}\left[\frac{\prod_{j=1}^{m} \Gamma(\beta_j + s) \cancel{\prod_{j=1}^{n} \Gamma(1-\alpha_j - s)}}{\cancel{\prod_{j=m+1}^{q} \Gamma(1-\beta_j - s)} \prod_{j=n+1}^{p} \Gamma(\alpha_j + s)} ; z\right]$$

(... ouff !)

Elementary Example : A000984

$$\frac{(2n)!}{n!\cdot n!} = \binom{2n}{n} = 1, 2, 6, 20, 70, 252, 924 \ldots$$

Central Binomial Coefficients

$$\text{ogf}(z) = {}_1F_0\left(\left[\tfrac{1}{2}\right], [\ ], 4z\right) \equiv \frac{1}{\sqrt{1-4z}}$$

$$\binom{2n}{n} = \int_0^4 x^n\, W(x)\, dx$$

$$= \int_0 x^{s-1} W(x)\, dx \leftarrow \text{Mellin Transform}$$

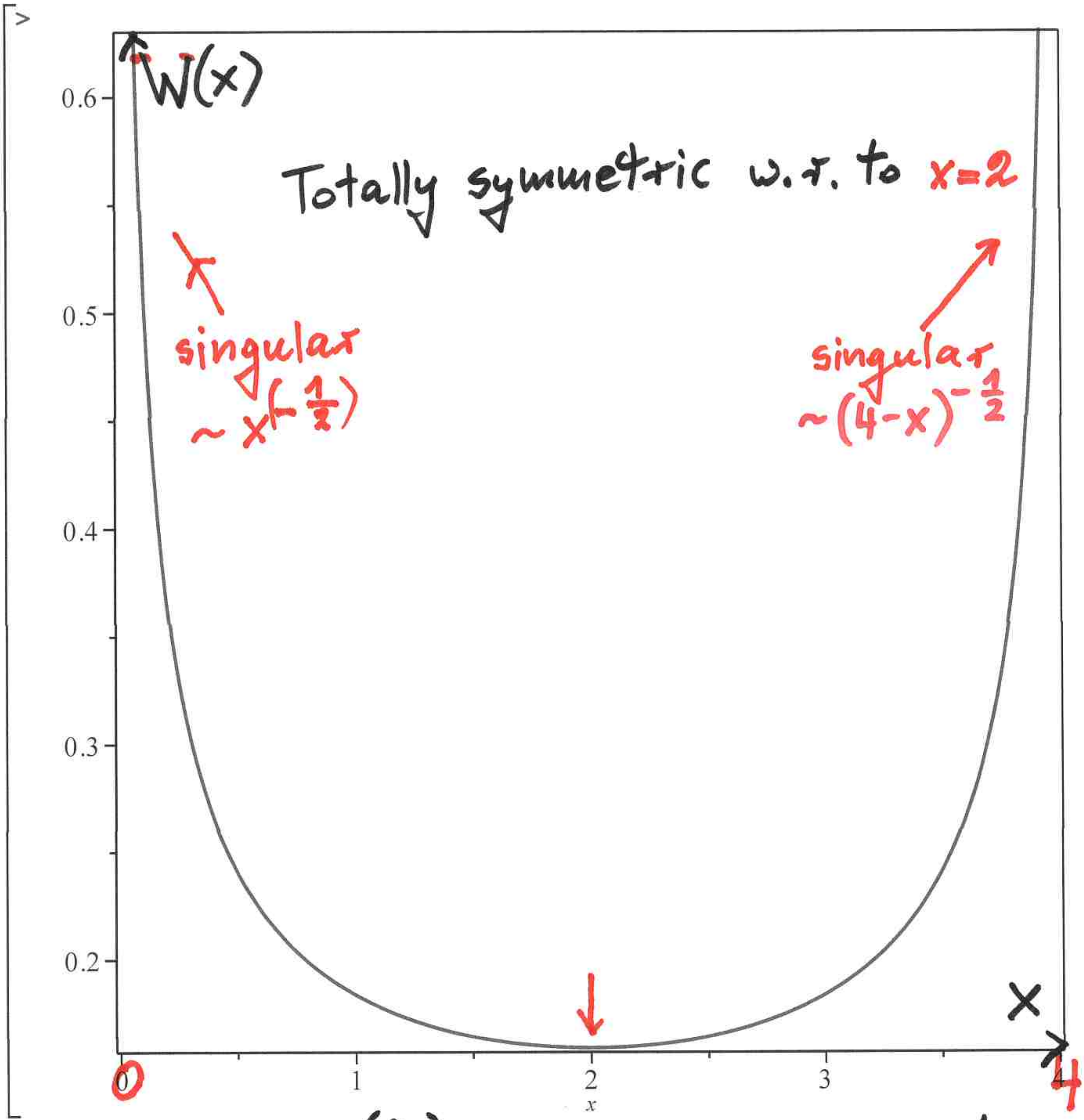
$$W(x) \cong \mathcal{M}^{-1}\left[\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}; x\right]$$

One ratio of Γ's

$$W(x) = \frac{1}{\pi\sqrt{x(4-x)}}$$

$W(x)$ is an algebraic function...

"arcsin" distribution



Moments $\binom{2n}{n}$, weight positive $= W(x) = \frac{1}{\pi\sqrt{x(4-x)}}$

—— * ——

$$\int_0^Y W(x)\,dx = \text{cts} + \frac{1}{\pi}\arcsin\left(\frac{Y}{2}-1\right) \;;\quad 0 < Y < 4$$

(• P. Lévy, 1939; • Paul Erdős (1969))

Extend simple moments $\binom{2n}{n}$ to parametrized moments:

$$\binom{2n+M}{n}, \quad M = 0, 1, 2, 3\ldots$$

$$\mathrm{Ogf}_M(z) = \sum_{n=0}^{\infty} \binom{2n+M}{n} z^n$$

$$= \underbrace{\frac{1}{\sqrt{1-4z}}}_{\text{ogf of } \binom{2n}{n}} \underbrace{\left(\frac{1-\sqrt{1-4z}}{2z}\right)}_{\text{ogf of Catalans}}^{M} \Longleftarrow \left(\begin{array}{c}\text{a bit} \\ \text{strange...}\end{array}\right)$$

———— * ————

Moment problem:

$$(*) \qquad \int_0^4 x^n W(M,x)\,dx = \underbrace{\binom{2n+M}{n}}_{\leftarrow 2\,\Gamma\text{'s}}, \quad n = 0, 1, \ldots$$

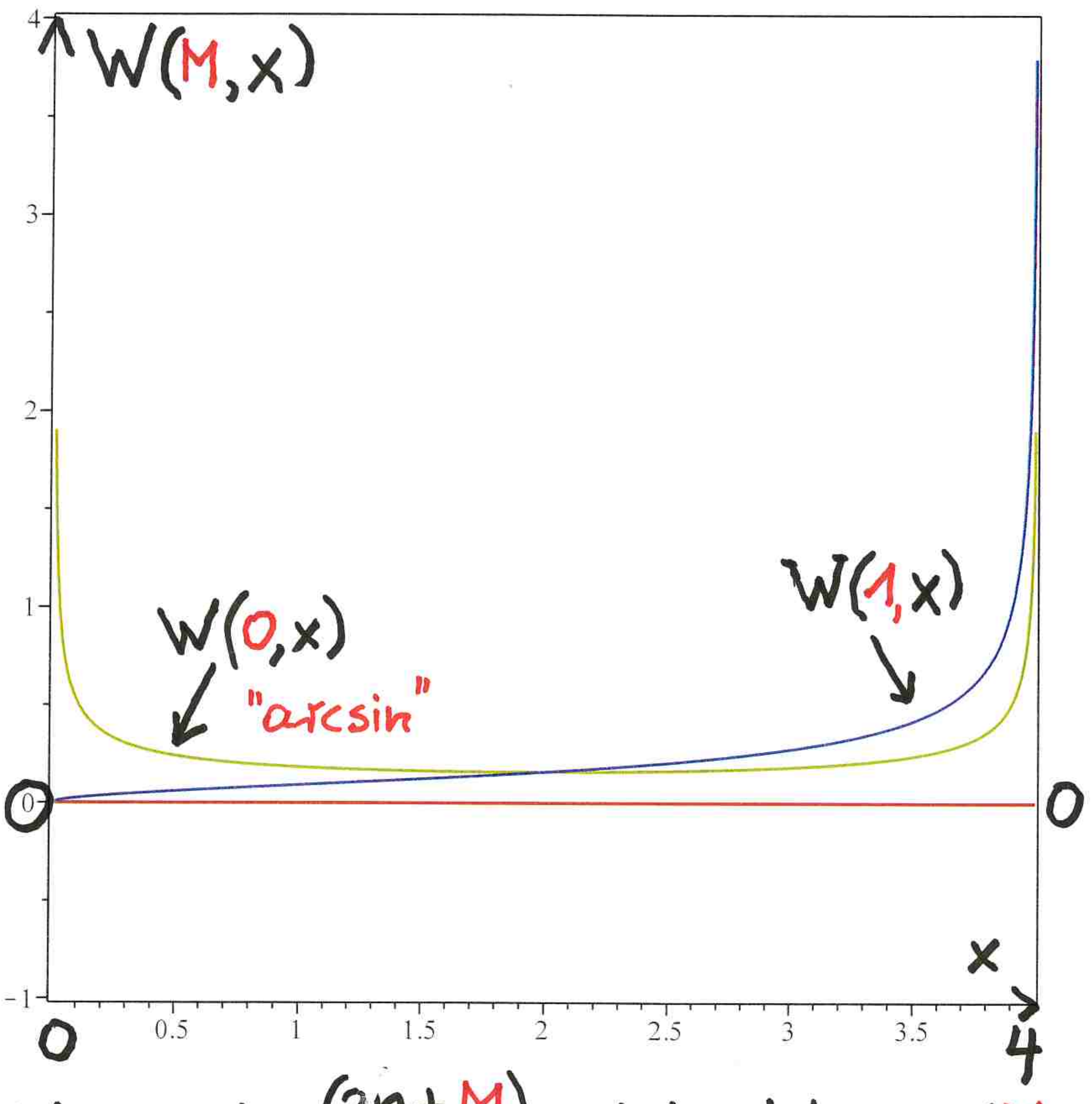
$$W(M,x) \sim \mathcal{M}^{-1}\left[\frac{\Gamma\left(s+\frac{M}{2}-\frac{1}{2}\right)\Gamma\left(s+\frac{M}{2}\right)}{\underbrace{\Gamma(s)\cdot\Gamma(s+M)}_{2\,\Gamma\text{'s}}}; \frac{x}{4}\right]$$

Two Gamma ratios

Exact solution:

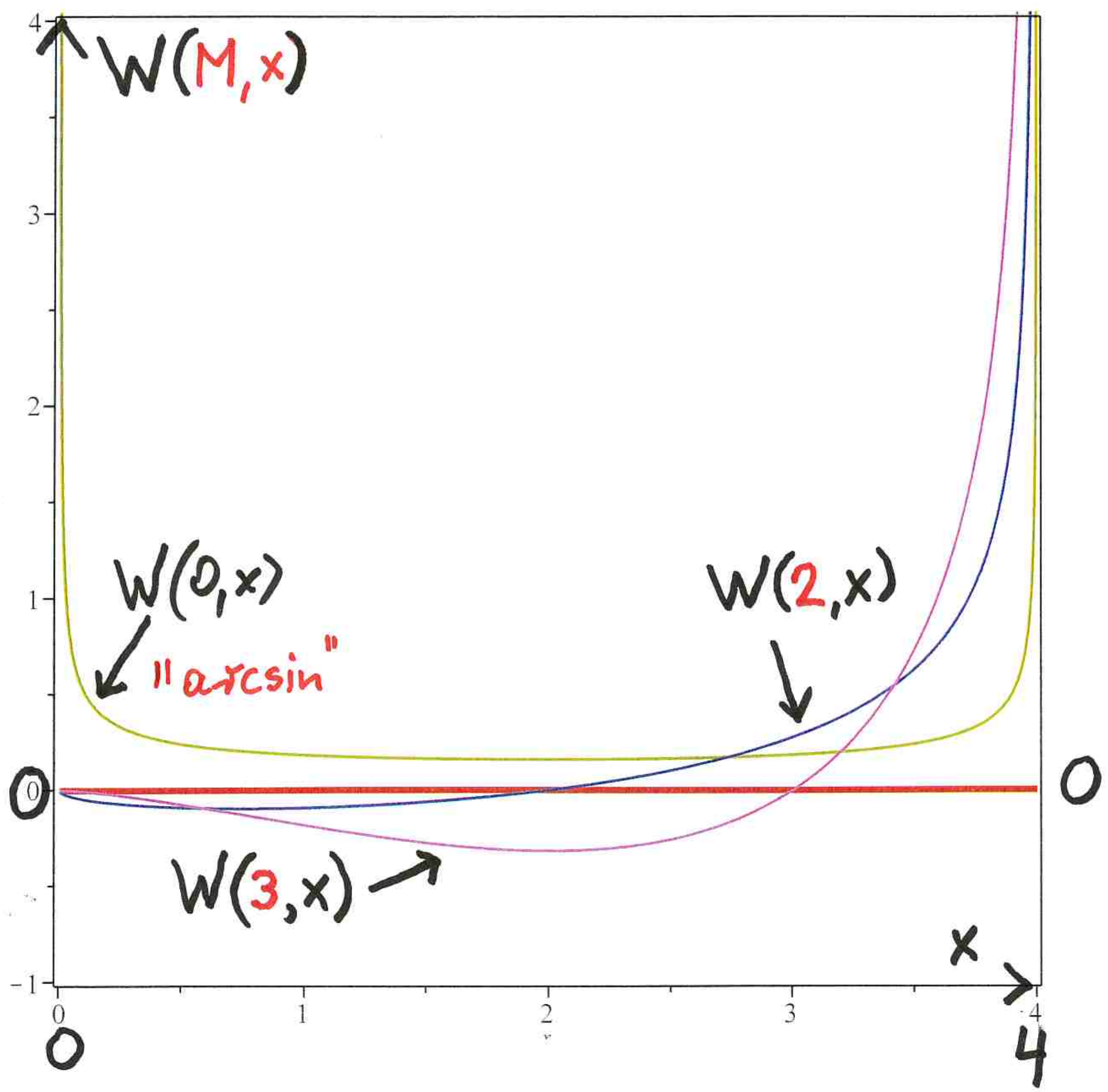
$$\boxed{W(M,x) = \frac{\cos\left(-\frac{M\pi}{2} + M\arcsin\left(\frac{\sqrt{x}}{2}\right)\right)}{\pi\sqrt{4-x}}\, x^{\frac{M-1}{2}} \quad \text{for } M = 0, 1, \ldots}$$

$\uparrow$ "arcsin"



Moments $\binom{2n+M}{n}$; Weights $W(M,x)$

For $M=0,1$ positive weights



Moments$\binom{2n+M}{n}$; Weights $W(M,x)$

For $M = 2,3,\ldots$ signed weights

⊛ Moments $\binom{2n+M}{n}$ ⊛

For all M both Ogf(M, z) and W(M, x) satisfy <u>quadratic</u> equations:

$W(5, x) :\equiv W(x)$:

$$4(x-4)W^2 + x^5(x^2-5x+5)^2 = 0 \quad (*)$$

$\mathrm{Ogf}(5, z) :\equiv g(z)$

$$1 + z^5(4z-1)g^2 + (4z-1)(z^2-3z+1)g^1 = 0 \quad (*)$$

How to find the Gamma ratios for: Kontsevich seq. A061162

$$\frac{(6n)!\,n!}{(3n)!\,(2n)!\,(2n)!} = \frac{\Gamma(6n+1)\,\Gamma(n+1)}{\Gamma(3n+1)\left[\Gamma(2n+1)\right]^2}$$

$$\underset{n=s-1}{=} \frac{\Gamma(6s-5)\,\Gamma(s)}{\Gamma(3s-2)\left[\Gamma(2s-1)\right]^2}$$

Now, Gauss-Legendre + simplify

$$\Longrightarrow \quad \sim \frac{\Gamma(s-\frac{5}{6})\cdot\Gamma(s-\frac{1}{6})}{\Gamma(s)\cdot\Gamma(s-\frac{1}{2})}$$

Two Gamma Ratios

Gamma shifts are:

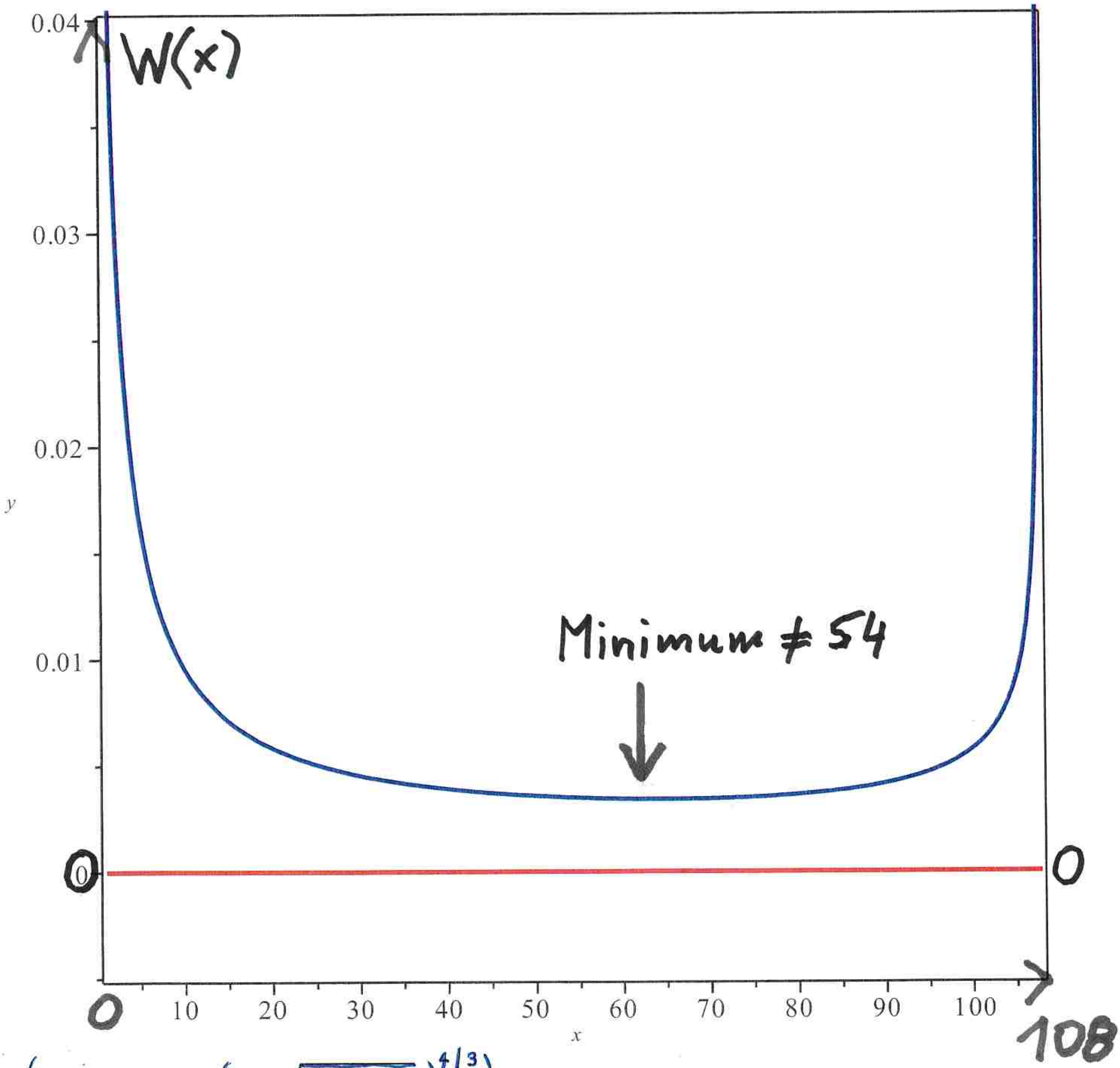
upper: $-\frac{5}{6}, -\frac{1}{6}$

Lower: $0, -\frac{1}{2}$

of Γ ratios not intuitive...

$$W(x) \sim \mathcal{M}^{-1}\left[\frac{\Gamma(s-\frac{5}{6})\,\Gamma(s-\frac{1}{6})}{\Gamma(s)\,\Gamma(s-\frac{1}{2})}; x\right]$$

Kontsevich: $\dfrac{(6n)!\,n!}{(3n)!(2n)!(2n)!}$ = A061162



$$\frac{\left(x^{2/3}+18\,2^{1/3}\left(1+\frac{\sqrt{324-3x}}{18}\right)^{4/3}\right)2^{1/3}\sqrt{3}}{24\,\pi\,x^{5/6}\sqrt{324-3x}\left(1+\frac{\sqrt{324-3x}}{18}\right)^{2/3}}=W(x)$$

Alg. eq. for $\pi\cdot W(x)\equiv V(x)$:

$$1024x^5(x-108)^3V^6+384x^4(x-108)^2V^4+36x^3(x-108)V^2+(x-216)^2=0$$

order 6

Alg. eq. for ogf$(z)\equiv g(z)$:

$$16(108z-1)^3g^6+24(108z-1)^2g^4+(972z-9)g^2+(216z-1)^2=0$$

Order 6

A005809

$\boxed{\text{Moments: } \binom{3n}{n}}$

Bober I'st series
$a=2,\ b=1$

$$\frac{2^{1/3}\left(6^{\frac{1}{3}}\left(9+\sqrt{3}\sqrt{27-4x}\right)^{2/3}+2\cdot 3^{2/3}x^{1/3}\right)}{4\pi\cdot\left(9+\sqrt{3}\sqrt{27-4x}\right)^{1/3}\sqrt{27-4x}\;x^{2/3}} = W(x)$$

$$\operatorname{ogf}(z) = {}_2F_1\left(\left[\tfrac{2}{3},\tfrac{1}{3}\right],\left[\tfrac{1}{2}\right],\tfrac{27z}{4}\right) \equiv g(z)$$

Alg. Eq. for $\pi\cdot W(x)$:

$$64x^4(4x-27)^3W^6 + 288x^3(4x-27)^2W^4 + 324x^2(4x-27)W^2 + 729 = 0$$

Cubic Eq. for $Y=(4x-27)\left(\frac{2}{3}W\right)^2$

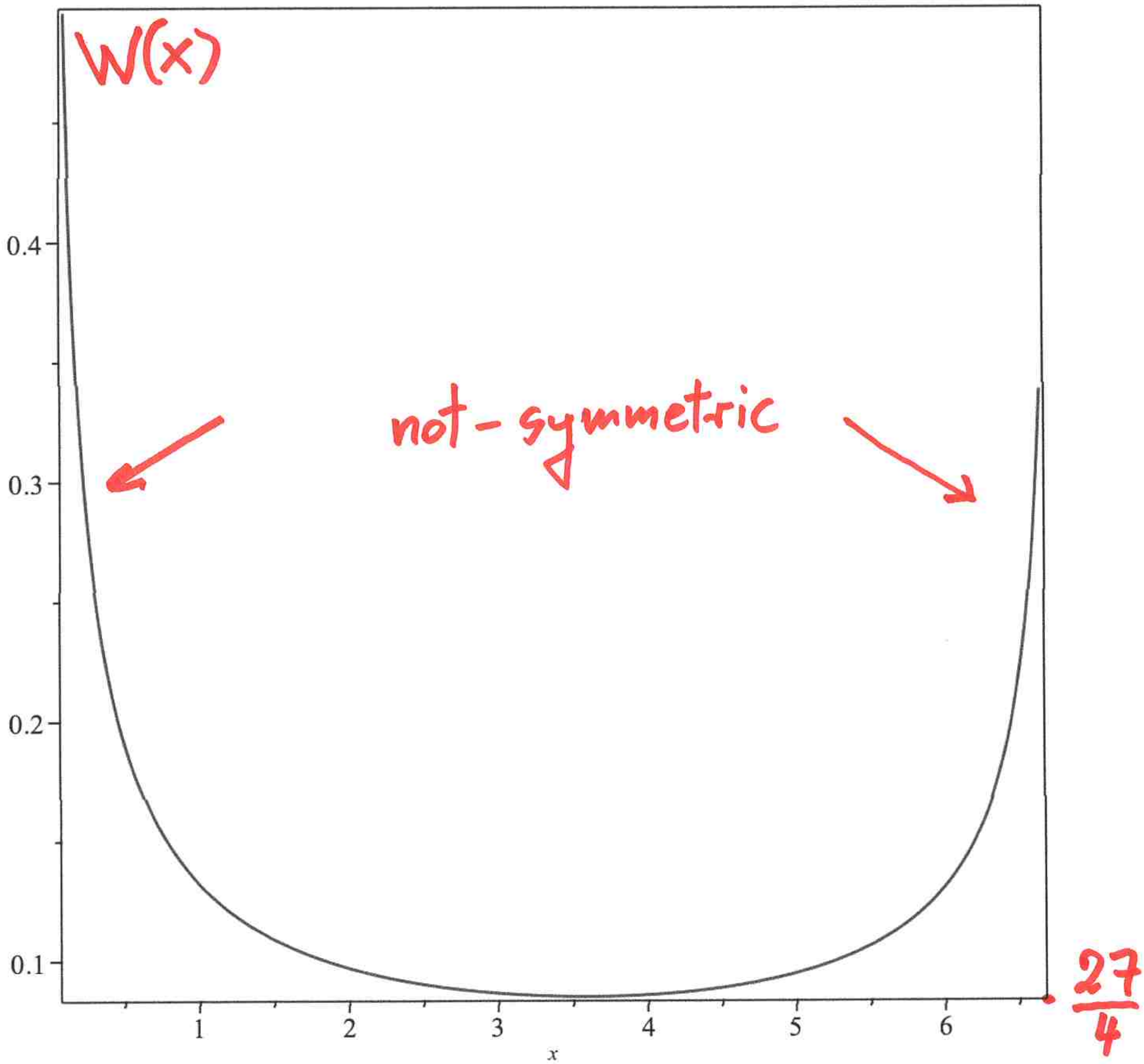
Alg. Eq. for $g(z)$:

$$\boxed{(27z-4)g^3 + 3g^1 + 1 = 0} \qquad (*)$$

(*) $\Rightarrow$ for any $p\geq 2$ (integer) the $g(z)$ of $\binom{pn}{n}$ satisfies the Al. Eq. of order p:

$$p^p z[g(z)]^p + [(p-1)g(z)+1]^p - p[(p-1)g(z)+1]^{p-1}g(z) = 0$$

Exact result!



Moments : $\binom{3n}{n}$ = A005809

= 1, 15, 84, 495, 3003..

Moments

$$\binom{3n-1}{n} = 1, 2, 10, 56, 330, 2002, 12376, \ldots$$
$$= A165817$$

- $\text{ogf}(z) = \underbrace{2z\,{}_3F_2\left([1, \tfrac{4}{3}, \tfrac{5}{3}], [\tfrac{3}{2}, 2], \tfrac{27z}{4}\right) + 1}_{\text{Relations via MeijerG functions do not work}} \equiv g(z)$

Alg. Eq. for $g(z)$:

$$\circledast \Rightarrow \boxed{-z + 9zg + (4-27z)g^2 + (4-27z)g^3 = 0} \Leftarrow \circledast$$
(cubic)

This is the particular case of exact relation for $p \geq 2$, of $\text{ogf}(z) = \sum_{n=0}^{\infty} \binom{pn-1}{n} z^n \equiv G(p,z)$

For $G(p,z) = \mathbf{g}$ we have

$$(p\mathbf{g}-1)[(p-1)\mathbf{g}]^{p-1} - [(p-1)\mathbf{g}]^p - z(p\mathbf{g}-1)^p = 0,$$

i.e. algebraic equation of order p. (Theorem?)

$\circledast$ is for $p=3$

For $p=4$: $\binom{4n-1}{n}$

$$\Rightarrow (4g-1)(3g)^3 - (3g)^4 - z(4g-1)^4 = 0$$

A025174

Moments: $\boxed{\binom{3n+2}{n}}$, not in the Bober series

but: for the weight from Mellin inverse $W(x)$

Define: $\boxed{Y(x) \equiv \dfrac{(4x-27)^3[W(x)]^6}{6^4 x^2}}$

Algebraic equation for $W(x)$ can be rewritten as a cubic equation for Y:

$$\rightarrow \quad Y^3 + (2x+3)Y^2 + (x^2-6x+3)Y + 1 = 0$$

To be compared with the algebraic equation for the ogf$(z) \equiv g$:

$$\rightarrow \quad z^2(27z-4)g^3 + z(27z-4)g^2 + (9z-1)g^1 + 1 = 0$$

Remark: Very close relationship with:

$\dfrac{(6n)!\,n!}{(3n)!(2n)!(2n)!}$ for which the new variable $Y(x) \sim \underline{x(108-x)W^2(x)}$

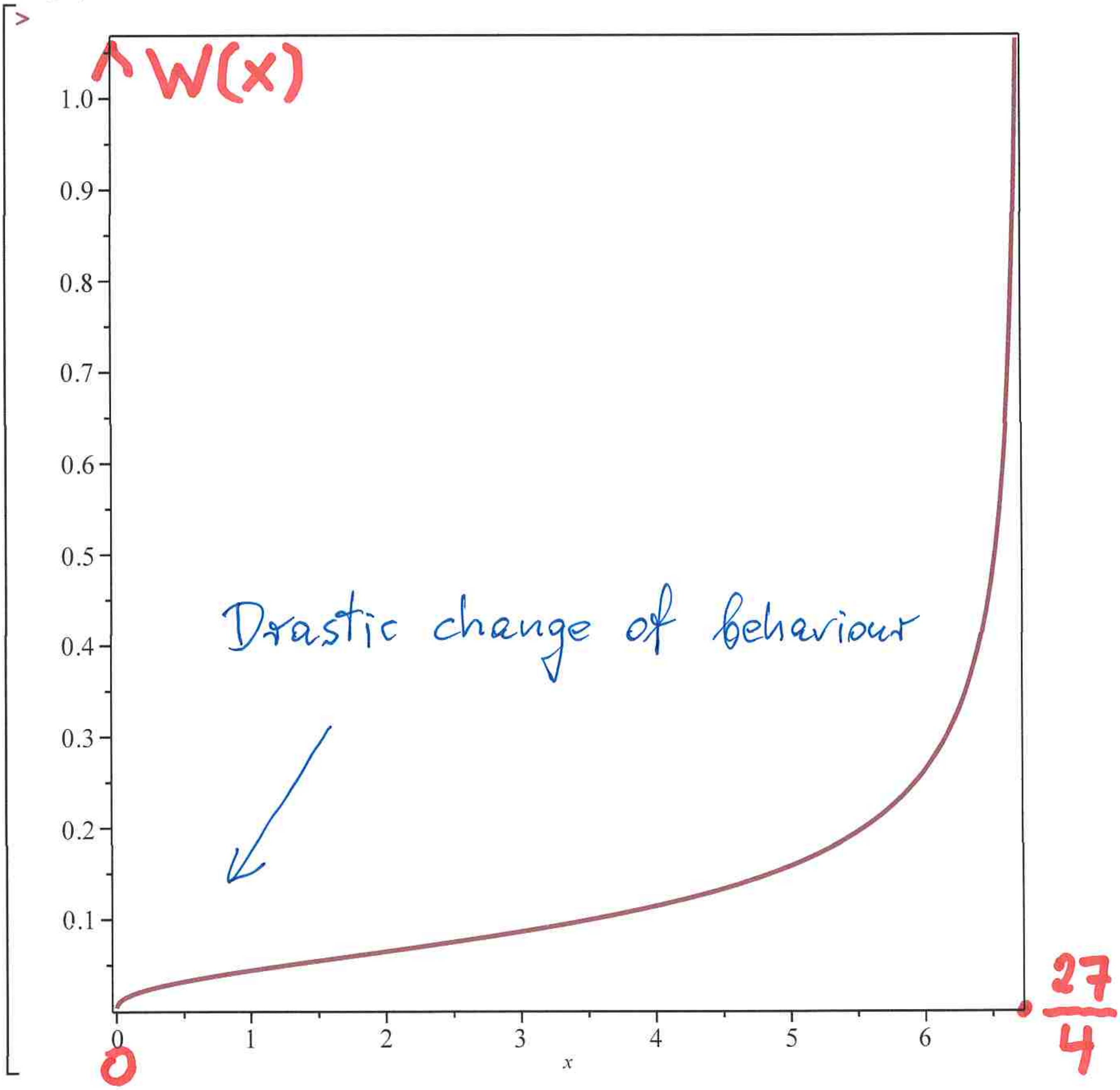
satisfies $\boxed{\text{cubic}}$ equation

(Kontsevich, CAP 2021)

this conference

$$W(x)=\frac{(6x)^{1/3}\left(2^{1/3}\left(9+\sqrt{81-12x}\right)^{2/3}+2\cdot(3x)^{1/3}\right)}{12\pi\left(9+\sqrt{81-12x}\right)^{1/3}\sqrt{27-4x}}$$

$W(0)=0.$



Moments

$\binom{3n+2}{n}$ = A025174 = 1, 5, 28, 165, 1001...

Moments: $\binom{5n}{2n} = 1, 10, 210, 5005, 125970, \ldots$

A001450

$$\text{ogf}(z) = {}_4F_3\left(\left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right], \left[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right], Rz\right)$$

where $R \cong 29$

$$W(x) \sim \mathcal{M}^{-1}\left[\frac{\Gamma(s-\frac{1}{5})\Gamma(s-\frac{2}{5})\Gamma(s-\frac{3}{5})\Gamma(s-\frac{4}{5})}{\Gamma(s)\Gamma(s-\frac{1}{3})\Gamma(s-\frac{1}{2})\Gamma(s-\frac{2}{3})} ; x\right]$$

Four Gamma Ratios

Observation:

- (*) ogf(z) is one hypergeometric function
- (**) W(x) $\begin{cases} \text{is sum of three hypergeometric f.} \\ \text{is MeijerG} \end{cases}$

Maybe things will be clearer when both quantities will be in MeijerG notation

???

$\Downarrow$

Use some properties of MeijerG

$$L1 = \left[-\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3}\right]$$

$$L2 = \left[-\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}, -\frac{4}{5}\right]$$

$$\frac{1}{z}\,\mathrm{ogf}\left(\frac{1}{z}\right) \cong \mathrm{MeijerG}\left(\left[[0],[L1]\right],\left[[L2],[-]\right], -\frac{z}{R}\right)$$

$$W(x) \cong \mathrm{MeijerG}\left(\left[[-],[0,L1]\right],\left[[L2],[-]\right], \frac{x}{R}\right)$$

In:

$$\boxed{\frac{1}{z}\,\mathrm{ogf}\left(\frac{1}{z}\right) = \int_0^R \frac{W(x)}{z-x}\,dx, \quad z > R}$$

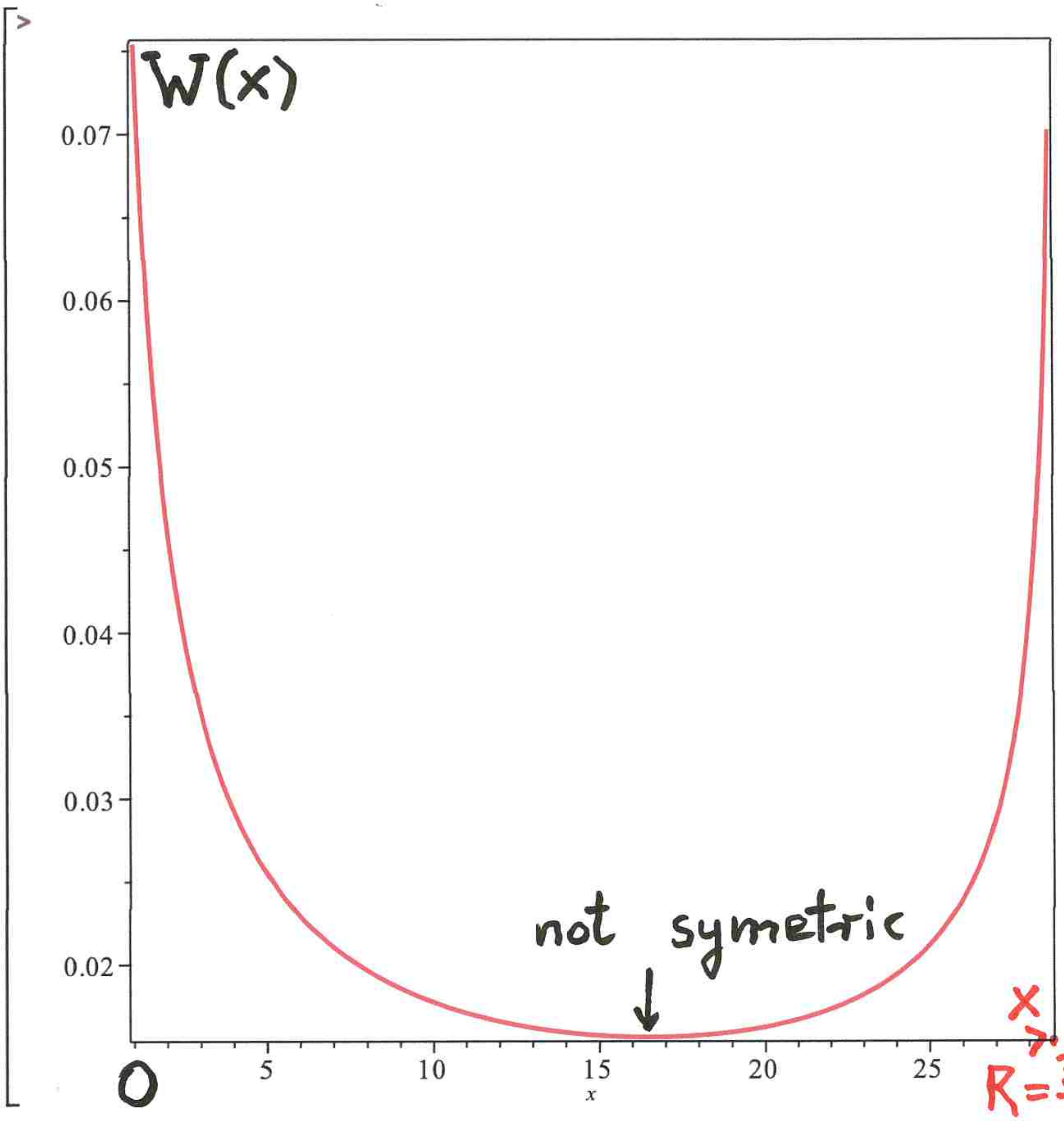
to obtain $W(x)$ = solution of Hausdorff moment problem

A) find MeijerG representation of $\frac{1}{z}\,\mathrm{ogf}\left(\frac{1}{z}\right)$; find L1, L2

B) Move one "0" from one bracket to another without changing L1 and L2

— * —

Moments: $\binom{5n}{2n}$ OEIS: A001450



$R = \frac{3125}{108} \approx 29.$

$\text{ogf}(z) = {}_4F_3\left(\left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right], \left[\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right], Rz\right)$

satisfies alg. eq. of 10 th order

Parametrized series of moments
for which the ogf's are algebraic :
(J. Bober, 2009)

a, b – mutually prime integers:

I: $\dfrac{[(a+b)n]!}{(an)!\,(bn)!}$,

II: $\dfrac{(2an)!\,(bn)!}{(an)!\,(2b)!\,[(a-b)n]!}$, $a>b$

III: $\dfrac{(2an)!\,(2bn)!}{(an)!\,(bn)!\,[(a+b)n]!}$,

Series I + II + III solved for W(x) analytically

+

52 "sporadic" = (non-parametrized) cases,

examples:

• III'rd series : $a=2$, $b=3$: $\dfrac{(4n)!\,(6n)!}{(2n)!\,(3n)!\,(5n)!}$

= 1, 12, 308, 8976, 276276, 8767512...

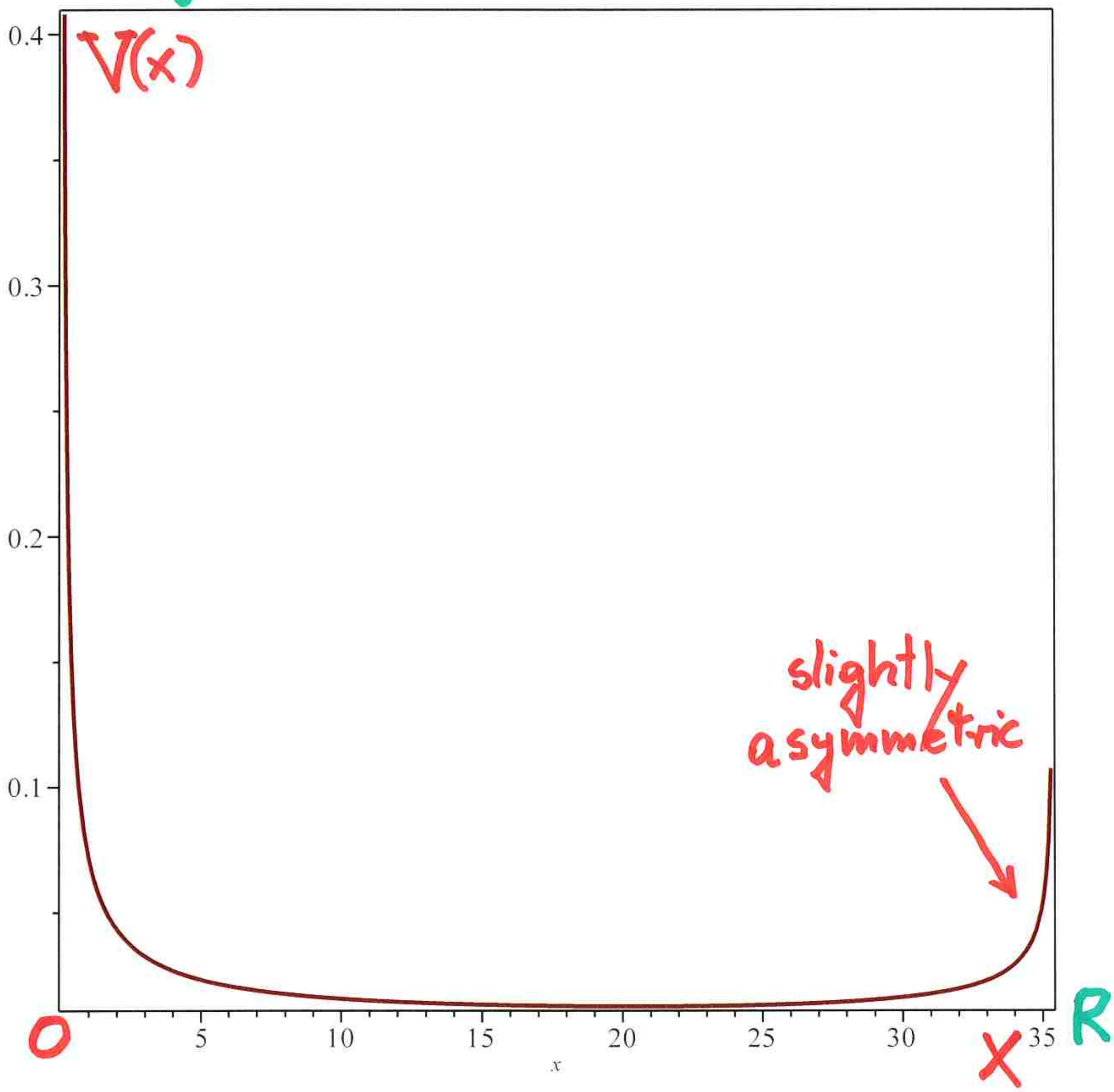
= OEIS A304126

$$\mathrm{Ogf}(z) = {}_5F_4\left(\left[\tfrac{1}{6},\tfrac{1}{4},\tfrac{1}{2},\tfrac{3}{4},\tfrac{5}{6}\right],\left[\tfrac{1}{5},\tfrac{2}{5},\tfrac{3}{5},\tfrac{4}{5}\right], R\cdot z\right)$$

$$R = \frac{110592}{3125} = 2^{12}\cdot 3^3/5^5$$

Ogf(z) satisfies Alg. Eq. of $\boxed{\text{10th degree}}$ (Explicit !)

Weight



Moments: $\dfrac{(4n)!(6n)!}{(2n)!(3n)!(5n)!}$

A304126

V(x) contains 5 hypergeometric functions

```
> V(x);
```

$$\frac{2^{1/3}\,\mathrm{hypergeom}\left(\left[\frac{1}{6},\frac{11}{30},\frac{17}{30},\frac{23}{30},\frac{29}{30}\right],\left[\frac{1}{3},\frac{5}{12},\frac{2}{3},\frac{11}{12}\right],\frac{3125\,x}{110592}\right)}{12\,\pi\,x^{5/6}}$$

$$+\frac{\sqrt{2}\,\mathrm{hypergeom}\left(\left[\frac{1}{4},\frac{9}{20},\frac{13}{20},\frac{17}{20},\frac{21}{20}\right],\left[\frac{5}{12},\frac{1}{2},\frac{3}{4},\frac{13}{12}\right],\frac{3125\,x}{110592}\right)}{16\,\pi\,x^{3/4}}$$

$$+\frac{1}{16\,\pi\,\sqrt{x}}\left(\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\,\pi}{5}\right)\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{7}{10},\frac{9}{10},\frac{11}{10},\frac{13}{10}\right],\left[\frac{2}{3},\frac{3}{4},\frac{5}{4},\frac{4}{3}\right],\frac{3125\,x}{110592}\right)\right)$$

$$+\frac{11\sqrt{2}\,\mathrm{hypergeom}\left(\left[\frac{3}{4},\frac{19}{20},\frac{23}{20},\frac{27}{20},\frac{31}{20}\right],\left[\frac{11}{12},\frac{5}{4},\frac{3}{2},\frac{19}{12}\right],\frac{3125\,x}{110592}\right)}{2048\,\pi\,x^{1/4}}$$

$$+\frac{247\,2^{2/3}\,\mathrm{hypergeom}\left(\left[\frac{5}{6},\frac{31}{30},\frac{37}{30},\frac{43}{30},\frac{49}{30}\right],\left[\frac{13}{12},\frac{4}{3},\frac{19}{12},\frac{5}{3}\right],\frac{3125\,x}{110592}\right)}{110592\,\pi\,x^{1/6}}$$

308.0000012

R

```
> seq(round(evalf(int(x^kk*V(x),x=0..110592/3125))),kk=0..2);;
```

1, 12, 308

reproduces good moments

$$\text{Moments}: \frac{(4n)!\,(6n)!}{(2n)!\,(3n)!\,(5n)!} = \text{A304126}$$

$= 1, 1, 12, 308, 8976, 276276, \ldots$

$$\frac{(30n)!\,n!}{(6n)!\,(10n)!\,(15n)!}$$

In[1]:=

```
Table[(30*n)!*n!/( (6*n)!*(10*n)!*(15*n)! ), {n, 0, 4}]
```

table

$\text{HypergeometricPFQ}\left[\left\{\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right\},\left\{\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{4}{5}\right\},1\,007\,769\,600\,000\,z\right]$

PFQ hypergéométrique

R (support)

Out[1]= {1, 77 636 318 760, 53 837 289 804 317 953 893 960, 43 880 754 270 176 401 422 739 454 033 276 880, 38 113 558 705 192 522 309 151 157 825 210 540 422 513 019 720}

First 4 terms.

Out[2]= $\text{HypergeometricPFQ}\left[\left\{\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right\},\left\{\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{4}{5}\right\},1\,007\,769\,600\,000\,z\right]$ ← ogf(z)

$$\frac{G_{8,8}^{8,0}\left(\frac{x}{R}\,\middle|\,\begin{matrix}-\frac{4}{5},-\frac{2}{3},-\frac{3}{5},-\frac{1}{2},-\frac{2}{5},-\frac{1}{3},-\frac{1}{5},0\\ -\frac{29}{30},-\frac{23}{30},-\frac{19}{30},-\frac{17}{30},-\frac{13}{30},-\frac{11}{30},-\frac{7}{30},-\frac{1}{30}\end{matrix}\right)}{2\,015\,539\,200\,000\,\sqrt{15\pi}}$$ ← weight(x)

Chebyshev A211417

one of "sporadic" Baber cases

—— * ——

Many other "sporadic" cases solved analytically

Further applications

Tutte numbers (M, int. parameter) $\frac{2(2M+3)!\,(4n+2M+1)!}{(M+2)!\,M!\,n!\,(3n+2M+3)!}$

Intervals on M-Tamari Lattices (—— " ——) $\frac{M+1}{n(Mn+1)}\binom{(M+1)^2 n+M}{n-1}$

Noam Zeilberger

X. Viennot

E. Fusy,

G. Schaeffer

Fuss-Catalan numbers

Constellation numbers

$\vdots$

Etc.

F. Rodriguez Villegas

(2019)

A) $$\frac{(11n)!(2n)!}{n!(3n)!(4n)!(5n)!}$$

B) $$\frac{(63n)!(8n)!(2n)!}{n!(4n)!(16n)!(21n)!(31n)!}$$

A) and B) treated and solved by the same methods