Canonical Grothendieck polynomials with free fermions

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Clifford algebras in physics come from creation ψ_i and annihilation ψ_i^* operators satisfying the Canonical Anticommutator Relations

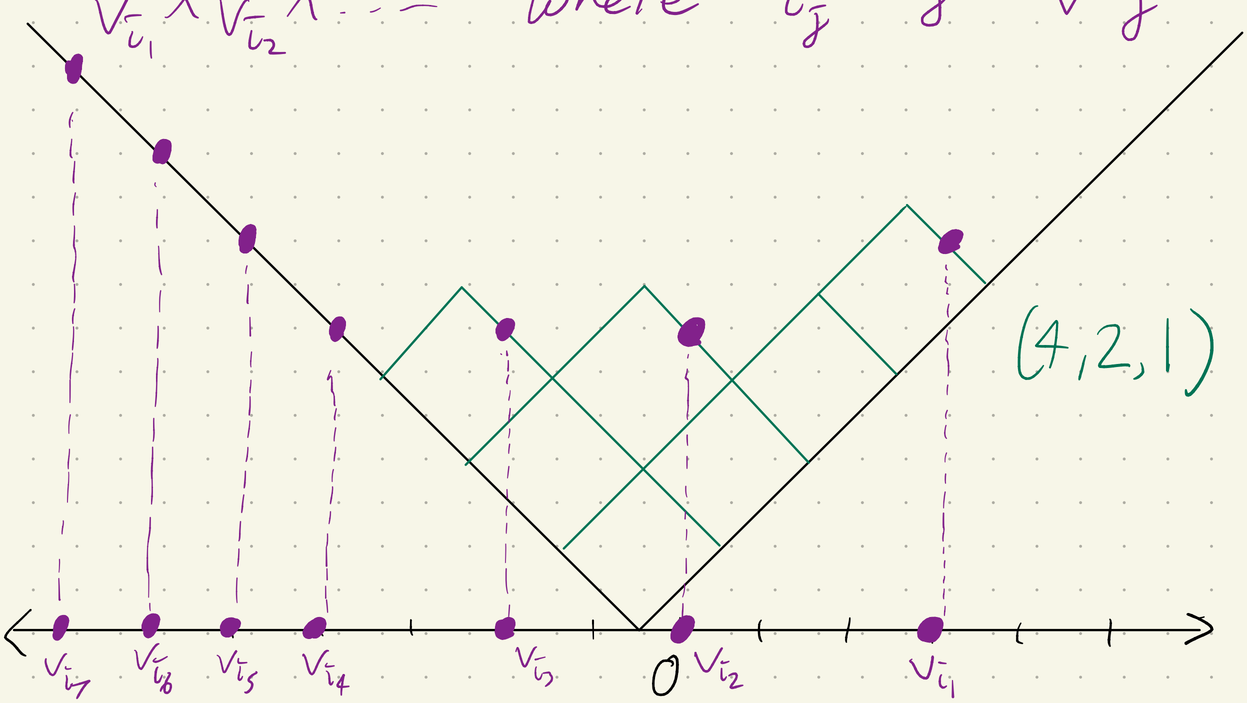
$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \quad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}$$

For finite rank, they act on $\Lambda\mathbb{C}^n$ as the spinor representation:

ψ_i adds $v_i \wedge \vec{v}$ $\qquad$ ψ_i^* removes $\widehat{v_i} \wedge \vec{v}$

We take $n \to \infty$, this becomes (fermionic) Fock space

$v_{i_1} \wedge v_{i_2} \wedge \cdots$ where $i_j = -j \quad \forall j >> 1$



Vacuum vector $|0\rangle = v_{-1} \wedge v_{-2} \wedge v_{-3} \wedge \cdots$

Shifted vacuum $|k\rangle = \begin{cases} \psi_{k-1} \cdots \psi_0 |0\rangle & \text{if } k \geq 0 \\ \psi^*_k \cdots \psi^*_{-1} |0\rangle & \text{if } k < 0 \end{cases}$

$$|\lambda\rangle = \psi_{\lambda_1 - 1} \cdots \psi_{\lambda_\ell - \ell} |-\ell\rangle$$

This is well-defined, i.e. independent of ℓ.

Dual space given by $*$ where $\psi_i \Longleftrightarrow \psi_i^*$

Natural bilinear form $\langle \mu | X | \lambda \rangle$ with

$$\langle \mu | \lambda \rangle = \delta_{\lambda\mu}$$

This also has $U(gl_\infty)$ action, in particular an infinite dimensional Heisenberg algebra action.

This is the Boson-Fermion Correspondence

Current operators

$$a_k = \sum_{i \in \mathbb{Z}} :\psi_i \psi^*_{i-k}: \qquad [a_m, a_k] = m\,\delta_{m,-k}$$

Half vertex operator $e^{H(X)}$

powersum $p_k(X/Y) = x_1^k + \cdots + x_n^k - y_1^k - \cdots - y_m^k$

Hamiltonian: $H(X/Y) = \sum_{k=1}^{\infty} \frac{p_k(X/Y)}{n} a_k$

Current operator a_k tries to move a particle k steps right in all possible ways.

a_0 is special, measures how "balanced" the # holes and # particles are.

$$a_k^* = a_{-k} \qquad H^*(X/Y) = \sum_{k=1}^{\infty} \frac{p_k(X/Y)}{n} a_{-k}$$

$$a_{-k}|0\rangle = 0 \quad (k>0) \qquad e^{H^*(X/Y)}|0\rangle = |0\rangle$$

$$\langle 0| a_k = 0 \quad (k>0) \qquad \langle 0| e^{H(X/Y)} = \langle 0|$$

$$a_0|\ell\rangle = \ell|\ell\rangle$$

$$e^{H(X/Y)} e^{H^*(A/B)} = \prod_{i,j,k,\ell} \frac{(1+y_k b_\ell)}{(1-x_i a_j)} e^{H^*(A/B)} e^{H(X/Y)}$$

$$e^{H(X/Y)} \psi_k e^{-H(X/Y)} = \sum_{i=0}^{\infty} h_i(X/Y) \psi_{k-i}, \qquad h_i(X/Y) = \sum_{k=0}^{i} (-1)^k \underbrace{h_{i-k}(X)}_{\text{homogeneous}} \underbrace{e_k(Y)}_{\text{elementary}}$$

$$e^{-H(X/Y)} \psi_k^* e^{H(X/Y)} = \sum_{i=0}^{\infty} h_i(X/Y) \psi^*_{k+i}$$

We want to evaluate $\langle \mu | e^{H(X/Y)} | \lambda \rangle$.

We use Wick's Theorem

$$\langle \mu | e^{H(X/Y)} | \lambda \rangle = \det \left[\langle 0 | \psi^*_{\mu_j - j} e^{H(X/Y)} \psi_{\lambda_i - i} | 0 \rangle \right]_{i,j=1}^{n}$$

and note each entry corresponds to

$$\sum_{m=0}^{\infty} \langle 0 | \psi^*_{\mu_j - j + m} h_m(X/Y) \psi_{\lambda_i - i} | 0 \rangle = h_{\lambda_i - \mu_j - i + j}(X/Y)$$

Jacobi-Trudi Formula says

$$\langle \mu | e^{H(X/Y)} | \lambda \rangle = s_{\lambda/\mu}(X/Y)$$

(super) symmetric Schur function.

Schur functions have geometric meaning

B = upper triangular matrices $\subseteq GL_n(\mathbb{C})$

$$P_k = \begin{array}{cc} \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) & \begin{array}{l} \}\,k \\ \}\,n-k \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{k} & \underbrace{\quad}_{n-k} \end{array} & \end{array} \subseteq GL_n(\mathbb{C})$$

Grassmannian $Gr(k,n) \cong GL_n / P_k = \{V \subseteq \mathbb{C}^n \mid \dim V = k\}$

Schubert variety X_λ = closure B orbit in $Gr(k,n)$

They are indexed by partitions $\lambda \subseteq \left.\underbrace{\boxed{}}_{n-k}\right\} k$

Give rise to basis of $H^*(Gr(k,n))$.

Polynomial representatives are $s_\lambda(X)$

We want a richer cohomology theory.

Move to K-theory, where polynomial representatives are (symmetric) Grothendieck polynomials $G_\lambda(X;\beta)$

There is a "weak" basis $J_\lambda(X;\alpha) = \omega G_\lambda(X;\alpha)$

and dual versions $g_\lambda(X;\beta)$ $j_\lambda(X;\alpha)$

Yeliussizov '17 combined regular and weak versions as canonical Grothendieck funcs.

Galashin–Grinberg–Liu '16 refined the parameter β for $g_\lambda(X;\beta)$.

Hwang et al. '21 combined these with combinatorial and Jacobi–Trudi formulas.

$$G_{\lambda/\mu}(X;\alpha,\beta)=C\det\left[h_{\lambda_i-\mu_j-i+j}\left(X/\!/\left(A_{(\mu_j,\lambda_i]}\sqcup B_{[i,j]}\right)\right)\right]_{i,j=1}^{\ell}$$

$$C=\prod_{i,j}^{\ell,n}(1-\beta_i x_j),\quad Y_I=(Y_{I_1},\dots,Y_{I_m}),\quad h_m(X/\!/Y)=\sum_{a-b=m}h_a(X)h_b(Y)$$

$$g_{\lambda/\mu}(X;\alpha,\beta)=\det\left[h_{\lambda_i-\mu_j-i+j}\left(X\sqcup A_{[\mu_j,\lambda_i]}\sqcup B_{[i,j)}\right)\right]_{i,j=1}^{\ell}$$

Thm [Iwao, Motegi, S., '22]

There exists a free fermion realization

$$G_{\lambda/\mu}(X;\alpha,\beta)={}^{[[\alpha,\beta]]}\langle\mu|e^{H(X)}|\lambda\rangle^{[\alpha,\beta]}$$

$$g_{\lambda/\mu}(X;\alpha,\beta)={}_{[\alpha,\beta]}\langle\mu|e^{H(X)}|\lambda\rangle_{[\alpha,\beta]}$$

$$|\lambda\rangle^{[\alpha,\beta]}=e^{-H^*(B_\ell)}\overrightarrow{\prod_{1\le i\le\ell}}\left(e^{-H^*(A_{\lambda_i}\sqcup B_{[i,\ell]})}\psi_{\lambda_i-i}e^{H^*(A_{\lambda_i}\sqcup B_{[i,\ell]})}\right)|-\ell\rangle$$

$$|\lambda\rangle_{[\alpha,\beta]}=\overrightarrow{\prod_{1\le i\le\ell}}\left(e^{H(B_{i-1}/A_{\lambda_i-1})}\psi_{\lambda_i-i}e^{-H(B_{i-1}/A_{\lambda_i-1})}\right)|-\ell\rangle$$

$$X_i=X_{[1,i]}$$

$${}^{[[\alpha,\beta]]}\langle\mu|=\langle-\ell|\overleftarrow{\prod_{1\le i\le\ell}}\left(e^{-H^*(B_i/A_{\mu_i})}\psi^*_{\mu_i-i}e^{H^*(B_i/A_{\mu_i})}\right)$$

Pf/ Wick's theorem and the Jacobi–Trudi formulas. $\square$

Thm [Iwao, Motegi, S, '22]/ Well-defined and

$$ {}^{[\alpha,\beta]}\langle \mu | \lambda \rangle^{[\alpha,\beta]} = {}_{[\alpha,\beta]}\langle \mu | \lambda \rangle_{[\alpha,\beta]} = \delta_{\lambda\mu}. $$

Cor [Iwao, Motegi, S., '22]/ New proof Branching rules

$$ G_{\lambda/\mu}(X,Y;\alpha,\beta) = \sum_{\nu \supseteq \mu} G_{\lambda/\nu}(X;\alpha,\beta)\, G_{\nu/\!/\mu}(Y;\alpha,\beta) $$

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where $G_{\nu/\!/\mu}(Y;\alpha,\beta) = {}^{[\alpha,\beta]}\langle \mu | e^{H(X)} | \nu \rangle^{[\alpha,\beta]}$

and has a Jacobi–Trudi formula.

This is a refined version of Yeliussizov and canonical version of Buch related to the coproduct.

Pf/ Use $\mathrm{id} = \sum_{\lambda} |\lambda\rangle_{[\alpha,\beta]}\, {}_{[\alpha,\beta]}\langle \lambda| = \sum_{\lambda} |\lambda\rangle^{[\alpha,\beta]}\, {}^{[\alpha,\beta]}\langle \lambda|$ $\blacksquare$

Cor [Iwao, Motegi, S., '22]/

$$ G_{\lambda/\mu}(X;\alpha,\beta) = \sum_{\nu \subseteq \mu} \prod_{(i,j) \in \mu/\nu} (\alpha_i + \beta_j)\, G_{\lambda/\!/\nu}(X;\alpha,\beta) $$

Thm [Iwao, Motegi, S.'22]/ New proof

$$\omega G_{\lambda/\mu}(X;\alpha,\beta) = G_{\lambda'/\mu'}(X;\beta,\alpha)$$

$$\omega g_{\lambda/\mu}(X;\alpha,\beta) = g_{\lambda'/\mu'}(X;\beta,\alpha)$$

Cor [Iwao, Motegi, S.'22]/

We can express $G_{\lambda/\mu}, G_{\lambda/\!/\mu}, g_{\lambda/\!/\mu}$ as Schur functions w/ certain determinants

Cor [Iwao, Motegi, S.'22]/ Skew Cauchy

$$\sum_{\lambda} G_{\lambda/\!/\mu}(X;\alpha,\beta)\, g_{\lambda/\nu}(Y;\alpha,\beta) = \prod_{i,j}(1-x_iy_j)^{-1} \sum_{\eta} G_{\nu/\!/\eta}(X;\alpha,\beta)\, g_{\mu/\eta}(Y;\alpha,\beta)$$

$$\sum_{\lambda} G_{\lambda'/\!/\mu'}(X;\beta,\alpha)\, g_{\lambda/\nu}(Y;\alpha,\beta) = \prod_{i,j}(1+x_iy_j) \sum_{\eta} G_{\nu'/\!/\eta'}(X;\beta,\alpha)\, g_{\mu/\eta}(Y;\alpha,\beta)$$

$$\sum_{\lambda} G_{\lambda/\!/\mu}(X;\alpha,\beta)\, g_{\lambda'/\nu'}(Y;\beta,\alpha) = \prod_{i,j}(1+x_iy_j) \sum_{\eta} G_{\nu/\!/\eta}(X;\alpha,\beta)\, g_{\mu'/\eta'}(Y;\beta,\alpha)$$

$$\sum_{\lambda} G_{\lambda'/\!/\mu'}(X;\alpha,\beta)\, g_{\lambda'/\nu'}(Y;\alpha,\beta) = \prod_{i,j}(1-x_iy_j)^{-1} \sum_{\eta} G_{\nu'/\!/\eta'}(X;\alpha,\beta)\, g_{\mu'/\eta'}(Y;\alpha,\beta)$$

Pf/ Evaluate ${}^{[\alpha,\beta]}\langle \mu | e^{H(X)} e^{H^*(Y)} | \nu \rangle^{[\alpha,\beta]}$ in two different ways. One inserts the identity, the other commutes $e^{H(X)} e^{H^*(X)}$ $\blacksquare$

Cor [Iwao, Motegi, S.'22]/

Gambelli-type Formulas

$= \det$ formula involving single row $G_{(k)}(X;0,\beta)$

Thm [Iwao, Motegi, S.,'22]/ Skew Pieri formulas

$$\sum_{\lambda,\eta} G_{\lambda'/\!/\mu'}(X;\alpha,\beta) G_{\nu'/\!/\eta'}(-X;\beta,\alpha)\, g_{\lambda/\eta}(Y;\alpha,\beta) = \prod_{i,j}(1-x_i y_j)^{-1} g_{\mu/\nu}(Y;\alpha,\beta)$$

$$\sum_{\lambda,\eta} G_{\lambda'/\!/\mu'}(X;\beta,\alpha) G_{\nu'/\!/\eta}(-X;\alpha,\beta)\, g_{\lambda/\eta}(Y;\alpha,\beta) = \prod_{i,j}(1+x_i y_j)\, g_{\mu/\nu}(Y;\alpha,\beta)$$

and similar interchanging $g_{\lambda/\mu} \longleftrightarrow G_{\lambda/\!/\mu}$

Cor [Iwao, Motegi, S.,'22]/ We can write $G_\lambda(X;\alpha,\beta)$ (resp. $g_\lambda(X;\alpha,\beta)$) in combinatorial terms of $G_\mu(X;0,\beta)$ (resp. $g_\mu(X;0,\beta)$). We answer problems of Yeliussizov about $g_\lambda(X;0,\beta)$ expansions negatively.

These expansions use certain flagged tableau given by Hwang et al.

Subsequent papers:

- Establish Schur operators for $G_\lambda(X;\alpha,\beta)$ and $g_\lambda(X;0,\beta)$ and $\bar{j}_\lambda(X;\alpha,0)$.
- Connection with discrete TASEP explored by [Dieker–Warren '08]
- Direct proof of combinatorial formula of Hwang et al by branching rule.

Thank You!