Positivity of Meijer G-functions

(Talk at IHES, Nov. 29, 2022)

Notation: For $a = (a_1, \dots, a_m)$, $a_i > 0$, set

$$(a)_s = \begin{cases} 1 & \text{if } m = 0 \\ (a_1)_s \times \dots \times (a_m)_s & \text{if } m > 0 \end{cases}$$

where $(a_i)_s = \dfrac{\Gamma(a_i + s)}{\Gamma(a_i)}$ $\quad s > -a_i$ (Pochhammer's symbol)

Consider
$$\mathcal{M}(s) = \frac{(a)_s\,(c)_{-s}}{(b)_s\,(d)_{-s}} \qquad -\inf\{a_i\} < s < \inf\{c_i\}$$
(definition strip)

with $a = (a_1 - a_m)$ $\quad b = (b_1 \dots b_p)$ $\quad c = (c_1 \dots c_n)$ $\quad d = (d_1 \dots d_q)$

The Meijer G-function on the half-line

$$f(x) = G^{m,n}_{n+p,\, m+q}\left(\begin{matrix} 1-c_1, \dots, 1-c_n;\ b_1, \dots, b_p \\ a_1, \dots, a_m;\ 1-d_1, \dots, 1-d_q \end{matrix} \,\middle|\, x \right), \quad x \geq 0$$

has a Mellin transform given by

$$\frac{\Gamma(b)\,\Gamma(d)}{\Gamma(a)\,\Gamma(c)} \int_0^\infty x^{s-1} f(x)\,dx = \mathcal{M}(s)$$

with $\Gamma(a) = \Gamma(a_1) \times \dots \times \Gamma(a_m) \dots \Gamma(d) = \Gamma(d_1) \dots \Gamma(d_q)$.

Beware that f might not be a true function. If $c = d = \emptyset$, $a = (1, 3)$, $b = (2, 2)$, then

$$f(x) = x\left(1_{(0,1)}(x) + \delta_1(dx)\right)$$

The situations where $\begin{cases} \operatorname{Supp} f = [0,1],\ [1, \infty) \text{ or } \mathbb{R}^+ \\ f \text{ has a singularity at } 1 \end{cases}$ can be characterized in terms of (a, b, c, d).

Problem: Non-negativity of f (and of its possible singularity) over the support

This amounts to the existence of a certain positive r.v. X such that

$$\mathbb{E}(X^s) = \mathcal{M}(s)$$

Basic framework : • Independent product and quotient of β and γ r.v.'s

Recall indeed $\mathbb{E}\left[\beta_{a,b}^{s}\right] = \dfrac{(a)_s}{(a+b)_s}$ $\qquad \mathbb{E}\left[\gamma_c^s\right] = (c)_s$

• Hypergeometric r.v.'s (see Dufresne (2010), Chamayou & Letac (2006))

A (non-exhaustive) list of such r.v.'s has been given by Janson (2010) and coined as having "Moments of Gamma type". Examples :

ⓐ $\{B_t, t \geq 0\}$ BM $\qquad S_t = \operatorname{Sup}\{B_s, s \leq t\}$ $\quad \mathcal{A} = \int_0^1 S_t \, dt$

(Brownian Supremum Area)

$$\frac{3\mathcal{A}^2}{2} \sim \begin{cases} c = d = \emptyset \\ a = (1/2, 5/6)\ ; \ b = (2/3) \end{cases}$$

In other words $\dfrac{3\mathcal{A}^2}{2} \overset{d}{=} \beta_{1/2, 1/6} \times \gamma_{5/6}$ (Confluent hypergeometric density)

ⓑ $X_1, \dots, X_n \sim \beta_{a,b}$ i.i.d. $\qquad X = \prod_{1 \leq i < j \leq n} |X_i - X_j|$ (Vandermonde determinant)

$$X^2 \sim \begin{cases} c = d = \emptyset \\ \# a = \# b = \dfrac{n(3n-1)}{2} \end{cases} \quad \text{(Selberg's integral)}$$

See also Dunkl (2013) for related results.

$\boxed{\text{Case } c = d = \emptyset :}$ By the so-called Malmsten formula

$$\Gamma(1+s) = \exp\left[-\gamma s + \int_{-\infty}^{0} (e^{sx} - 1 - sx) \frac{dx}{|x|(e^{|x|}-1)}\right], \quad s > -1$$

we obtain

$$-\operatorname{Log} \mathcal{M}(s) = \int_0^\infty (1 - e^{-sx}) \left(\varphi_a(e^{-x}) - \varphi_b(e^{-x})\right) \frac{dx}{x(1-e^{-x})} \qquad (*)$$

with $\varphi_a(t) = \sum_{i=1}^{m} t^{a_i}$ and $\varphi_b(t) = \sum_{j=1}^{p} t^{b_j}$, $t \in (0,1)$.

Setting $\psi_{a,b}(t) = \varphi_a(t) - \varphi_b(t)$ and differentiating $(*)$ we get an integro-differential equation governed by $\psi_{a,b}$ for the underlying G-function.

$\boxed{\text{If } \# a = \# b = p}$ and if $\phi(x) = G^{p,0}_{p,p}\left(\begin{smallmatrix} b \\ a \end{smallmatrix} \middle| x\right)$, $x \in [0,1]$, is the Meijer G-function associated to $\mathcal{M}(s)$, we have

$$\mathrm{Log}(1/x)\, \phi(x) = \int_x^1 \phi(t)\, \psi_{a,b}\left(\frac{x}{t}\right) \frac{dt}{t-x}$$

This shows that $\psi_{a,b}$ never vanishes on $(0,1)$ (is ≥ 0) $\Longrightarrow$ ϕ never vanishes on $(0,1)$ (is ≥ 0)

$\boxed{\text{Conjecture :}}$ This is an equivalence

By the Lévy-Khintchine formula, the conjecture amounts to

$$X \text{ exists} \Longrightarrow \mathrm{Log}\, X \text{ is infinitely divisible} \quad (*)$$

Conjecture true for $p = 1$ (obvious), $p = 2$ (hypergeometric r.v.'s), $p = 3$ (tech. computations with Selberg's integral). For $p > 3$ there is no counterexample.

We rephrase the conjecture as $G^{p,0}_{p,p}\left(\begin{smallmatrix} b \\ a \end{smallmatrix} \middle| x\right) \geq 0$ on $(0,1)$ $\Longleftarrow$ $\underbrace{\sum_{i=1}^{p} u^{a_i} - u^{b_i} \geq 0 \text{ on } (0,1)}_{(\#)}$

Sufficient conditions for (#) date back to Schur (Schur convexity): (#) is true if $\sum_{i=1}^{j} a_i \leq \sum_{i=1}^{j} b_i$ for $j = 1 \dots p$. (having previously ordered a and b)

For $p = 1,2$ this is an equivalence (easy) but not for $p > 2$.

See Ismail and his collaborators for relaxed conditions ensuring (#).

$\boxed{\text{Case } a \neq \emptyset \text{ and } c \neq \emptyset:}$ Then $(*)$ is not necessarily true anymore

A formula by Weber-Schafheitlin on the Bessel shows indeed

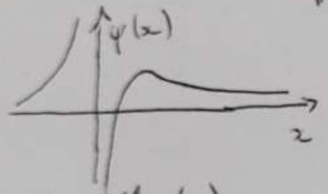
$$(4) \qquad \frac{2}{\Gamma(\alpha)} \int_0^\infty z^{-1-2s} J_\alpha^2(z)\, dz = \frac{(1/2)_s\, (\alpha)_{-s}}{(1)_s\, (\alpha+1)_s} \qquad \alpha > 0,\ -1/2 < s < \alpha$$

as a consequence of Selberg's integral.

Remarks:

- $\dfrac{(1/2)_s}{(1)_s\,(\alpha+1)_s}$ is not the Mellin transform of a positive r.v.
- The factor $(\alpha)_{-s}$ has a regularizing effect
- The underlying density $\dfrac{2}{\Gamma(\alpha)} z^{-1} J_\alpha^2(z)$ vanishes an infinite number of times.
- Malmsten's formula implies that $\mathcal{M}(s)$ in (4) is such that

$$\operatorname{Log} \mathcal{M}(s) = \int_{\mathbb{R}^*} \left(e^{sx} - 1 - sx\right) \frac{\psi(x)}{|x|\,(1 - e^{-|x|})}\, dx$$

with 

Hence $\mathcal{M}(s) = \dfrac{\mathcal{M}_1(s)}{\mathcal{M}_2(s)}$ where $\mathcal{M}_1(s)$ and $\mathcal{M}_2(s)$ are Mellin transforms of Log ID r.v.'s. This means that X with $\mathbb{E}(X^s) = \mathcal{M}(s)$ in (4) is such that $\operatorname{Log} X$ is "quasi infinitely divisible". See Lindner & Sato (2018).

The structure of quasi ID laws, which date back to Lévy, is still not well understood.

Fox H-functions: Same problem with $a = \begin{cases} a_1 \ldots a_m \\ A_1 \ldots A_m \\ \alpha_1 \ldots \alpha_m \end{cases}$, b, c, d

and $\mathcal{M}(s) = \dfrac{F_a(s)\, F_c(-s)}{F_b(s)\, F_d(-s)}$ $\qquad F_a(s) = (a_1)_{A_1 s}^{\alpha_1} \cdots\cdots (a_m)_{A_m s}^{\alpha_m}$

Partial results by Berg, Ismail, Karp and others on the non-negativity of the inverse Mellin transform

Example of a characterization (myself, 2021):

The inverse Mellin of $\dfrac{(a)^{\alpha}_{As}}{(b)^{\beta}_{Bs}}$ is non-negative $\iff$ $\begin{cases} \alpha A = \beta B \\ Ab \geq Ba \\ B^{B\beta} \geq A^{A\alpha} \\ \beta(2b-1) \geq \alpha(2a-1) \end{cases}$

The corresponding r.v. is log ID.

$\boxed{\text{Applications of Meijer G functions to other special functions}}$

- Wright function $\quad \phi(\rho, \beta, z) = \sum_{n \geq 0} \dfrac{z^n}{n!\,\Gamma(\beta + \rho n)} \quad \rho > -1 \quad \beta, z \in \mathbb{C}$

$\begin{cases} \rho = 0 & \text{Exponential} \\ \rho > 0 & \text{First kind (generalizes Bessel)} \\ \rho \in (-1,0) & \text{Second kind (links with Mittag-Leffler functions, see Erdelyi 1953)} \end{cases}$

In all cases the positivity problem is solved. Example for the second kind:

$$\phi(-\rho, \beta, -x) \geq 0 \text{ on } \mathbb{R}^+ \iff \beta \geq 0$$

Let $M_{\alpha,\beta}$ be the r.v. corresponding to $\alpha = -\rho \in (0,1)$ and $\beta \geq 0$.

One has $\quad \mathbb{E}\left(e^{x M_{\alpha,\beta}}\right) = \sum_{n \geq 0} \dfrac{\Gamma(\beta)\, x^n}{\Gamma(\alpha + \beta + n\alpha)} \quad$ (Mittag-Leffler function)

$$\mathbb{E}\left(M^s_{\alpha,\beta}\right) = \frac{(1)_s}{(\alpha+\beta)_{\alpha s}} \quad (\text{Fox function})$$

Theorem: $M_{\alpha,\beta}$ is unimodal

$\beta \geq \alpha \quad$ Easy since $\dfrac{d}{dx}\phi(-\alpha, \beta, -x) = -\phi(-\alpha, \beta - \alpha, -x) < 0$

$\beta \in (0, \alpha)$ Trickier. Write $\dfrac{(1)_s}{(\alpha+\beta)_{\alpha s}} = \underbrace{\dfrac{(1)_s}{(\alpha+\beta)_{\alpha s}\,(2-\alpha)_{(1-\alpha)s}}}_{\text{Mlotkowski-Penson}} \times \underbrace{(2-\alpha)_{(1-\alpha)s}}_{\text{LsTP}_2\text{ kernel}}$

Using Meijer G function, it turns out that Mlotkowski-Penson laws have an absolutely monotonic density on their bounded support.

Implies that $\# \{x > 0 \,/\, \phi(-\rho, \beta, -x) = 0\} = 1 \quad \forall \beta \in (-\rho, 0)$.

A classification of the number of zeroes of $\phi(-\rho, \beta, x)$ for $\beta < 0$ remains to be done.

$\boxed{\text{Log-concavity:}}$ A natural generalization called "strong unimodality" in the probabilistic literature (Ibragimov's theorem).

Focus on $\beta = 1-\alpha$ (Mittag-Leffler distribution). Central case since $M_{\alpha,1-\alpha}$ appears in

$$\left\{\begin{array}{l}\text{Interpolation between heat equation and wave equation}\\ \text{Occupation times of Markov processes}\\ \text{Elephant random walks}\\ \text{Coalescents}\end{array}\right.$$

and many other topics.

We have, setting $M_\alpha = M_{\alpha,1-\alpha}$, the mgf $\mathbb{E}(e^{z M_\alpha}) = E_\alpha(z)$ where

$$E_\alpha(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma(1+\alpha n)} \quad \text{(Mittag-Leffler function)}$$

Theorem: M_α is log-concave $\iff \dfrac{1}{E_\alpha}$ is convex $\iff \alpha \leq \alpha^*$

where α^* is the unique solution on $(0,1)$ to $\dfrac{1}{\Gamma(1-2\alpha)^2} = \dfrac{1}{\Gamma(1-\alpha)\Gamma(1-3\alpha)}$

Main arguments: Infinite divisibility, Meijer G-functions and "Bell-shape".

Ferreira & myself (2022). To be posted on the arxiv.