Graph complex action on Poisson structures: from theory to computation

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Universal deformations of Poisson structures arising from Kontsevich's graph complex

0. Background

1. Theory

2. Computation

Classical Hamiltonian mechanics

Hamilton (1833):

- Positions $q = (q^1, \dots, q^m) \in \mathbb{R}^m$ and momenta $p = (p_1, \dots, p_m) \in \mathbb{R}^m$ together define the state of a physical system at a point in time.
- Phase space $M = \{(q, p)\}$
- Energy $H : M \longrightarrow \mathbb{R}$
- Hamilton's equations of motion: $\dot{q}^i = +\dfrac{\partial H}{\partial p_i}$ $\quad \dot{p}_i = -\dfrac{\partial H}{\partial q^i}$

Example. Simple pendulum. $M = \mathbb{R}^2$.

$$H(q,p) = \frac{1}{2mL^2} p^2 + mgL(1 - \cos q) \qquad m, g, L \in \mathbb{R} \text{ const.}$$

$$\dot{q} = \frac{1}{mL^2} p \qquad \dot{p} = mgL \sin q$$

Standard Poisson bracket

How does the value of some $F: M \to \mathbb{R}$ evolve in time?

$$\dot{F} = \sum_{i=1}^{m} \left(\frac{\partial F}{\partial q^i} \cdot \dot{q}^i + \frac{\partial F}{\partial p_i} \cdot \dot{p}_i \right) = \sum_{i=1}^{m} \left(\frac{\partial F}{\partial q^i} \cdot \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \cdot \frac{\partial H}{\partial q^i} \right) =: \{F, H\}$$

Note $\dot{q}^i = \{q^i, H\}$ and $\dot{p}_i = \{p_i, H\}$ w.r.t. this Poisson bracket.

- Constants of motion: $F: M \to \mathbb{R}$ with $\dot{F} = \{F, H\} = 0$.
- Total energy H is conserved: $\dot{H} = \{H, H\} = 0$.
- Jacobi identity holds:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0.$$

- Hamiltonian vector field $X_H = \{-, H\}$.

Poisson brackets more generally

$M = \mathbb{R}^d$, coordinates $x^1, \dots, x^d$

$$\{f, g\} = \sum_{1 \le i < j \le d} p^{ij} \cdot \left(\frac{\partial f}{\partial x^i} \cdot \frac{\partial g}{\partial x^j} - \frac{\partial f}{\partial x^j} \cdot \frac{\partial g}{\partial x^i} \right) \qquad p^{ij} \in C^\infty(M)$$

such that $\displaystyle\sum_{\ell=1}^{d} \left(p^{i\ell} \cdot \frac{\partial p^{jk}}{\partial x^\ell} + p^{j\ell} \cdot \frac{\partial p^{ki}}{\partial x^\ell} + p^{k\ell} \frac{\partial p^{ij}}{\partial x^\ell} \right) = 0.$

Hamilton's equations of motion $\dot{x}^i = \{x^i, H\}$.

There may exist non-constant Casimirs $C : M \to \mathbb{R}$ such that $\dot{C} = \{C, H\} = 0$ for all $H : M \to \mathbb{R}$.

$\Rightarrow$ Motion happens in intersection of level sets of Casimirs.

Example $\qquad M = \mathbb{R}^3$, coordinates x, y, z

$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y, \qquad C = \frac{1}{2}(x^2 + y^2 + z^2)$

Basic objects: Multi-vector fields

Dual to differential k-forms $\omega \in \Gamma(\Lambda^k T^*M)$

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le d} \omega_{i_1 \cdots i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad \omega_{i_1 \cdots i_k} \in C^\infty(M)$$

are the k-vector fields $\alpha \in \Gamma(\Lambda^k TM)$

$$\alpha = \sum_{1 \le i_1 < \cdots < i_k \le d} \alpha^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_k}} \qquad \alpha^{i_1 \cdots i_k} \in C^\infty(M)$$

Vector fields $\quad A = \sum_{i=1}^{d} A^i \frac{\partial}{\partial x^i} \qquad$ e.g. $X_H = \{-, H\}$

Bi-vector fields $\quad B = \sum_{1 \le i < j \le d} B^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \qquad$ e.g. $\{-,-\}$

Tri-vector fields $\quad C = \sum_{1 \le i < j < k \le d} C^{ijk} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \qquad$ e.g. LHS(Jacobi)

Basic operation: Schouten bracket

After introducing formal anti-commuting variables $\xi_1, \dots, \xi_d$ $\left(\xi_j \cdot \xi_i = - \xi_i \cdot \xi_j \right)$ so that we can express k-vector fields as $\alpha = \sum_{1 \le i_1 < \dots < i_k \le d} \alpha^{i_1 \dots i_k} \xi_{i_1} \cdots \xi_{i_k}$,

define the natural extension of the Lie bracket of v.f.:

$$[\![A, B]\!] = \sum_{k=1}^{d} \left[(A) \frac{\overleftarrow{\partial}}{\partial \xi_k} \cdot \frac{\overrightarrow{\partial}}{\partial x^k} (B) - (A) \frac{\overleftarrow{\partial}}{\partial x^k} \cdot \frac{\overrightarrow{\partial}}{\partial \xi_k} (B) \right]$$

where e.g. $\frac{\overrightarrow{\partial}}{\partial \xi_k} (\xi_k \xi_\ell) = \xi_\ell$, $\frac{\overrightarrow{\partial}}{\partial \xi_k} (\xi_\ell \xi_k) = \frac{\partial}{\partial \xi_k} (- \xi_k \xi_\ell) = - \xi_\ell$

$(\xi_\ell \xi_k) \frac{\overleftarrow{\partial}}{\partial \xi_k} = \xi_\ell$, $(\xi_k \xi_\ell) \frac{\overleftarrow{\partial}}{\partial \xi_k} = (- \xi_\ell \xi_k) \frac{\overleftarrow{\partial}}{\partial \xi_k} = - \xi_\ell$

Schouten bracket properties

- It has degree -1 (eats one ξ).
- $[\![V, V]\!] = 0$ for vector fields $V = \sum_{i=1}^{d} V^i \xi_i$.
- For two bi-vector fields A, B:

$$[\![A, B]\!]^{ijk} = A^{li} \frac{\partial B^{jk}}{\partial x^l} + A^{lj} \frac{\partial B^{ki}}{\partial x^l} + A^{lk} \frac{\partial B^{ij}}{\partial x^l} + \\ + B^{li} \frac{\partial A^{jk}}{\partial x^l} + B^{lj} \frac{\partial A^{ki}}{\partial x^l} + A^{lk} \frac{\partial B^{ij}}{\partial x^l}$$

so $[\![B, B]\!] \neq 0$ in general.

- Graded Jacobi identity:

$$\pm [\![A, [\![B, C]\!]]\!] \pm [\![B, [\![C, A]\!]]\!] \pm [\![C, [\![A, B]\!]]\!] = 0.$$

Poisson structures (again)

Definition A Poisson structure on $\mathbb{R}^d$ is a bi-vector field P satisfying $\frac{1}{2}[\![P, P]\!] = 0$.

Examples

- On $\mathbb{R}^2$, $P = u \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ for any $u \in C^\infty(\mathbb{R}^2)$.
- On $\mathbb{R}^3$, $P = \rho [\![\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}, a]\!]$ for any $\rho, a \in C^\infty(\mathbb{R}^3)$ (rescaled Nambu – Poisson).
- On $\mathbb{R}^d$, $P = E \wedge V$ where $E = \sum_{i=1}^d x^i \frac{\partial}{\partial x^i}$ and V is a vector field with homogeneous polynomial coeffs of the same degree.
- Matrix Lie algebras $\rightsquigarrow$ R-matrix Poisson structures (linear, quadratic, cubic)

Infinitesimal deformations of Poisson structures

Let P be a Poisson structure. Want to deform P:

$$P(\varepsilon) = P + \varepsilon Q + O(\varepsilon^2)$$

By expanding $0 = [\![P(\varepsilon), P(\varepsilon)]\!]$ we obtain the condition $[\![P, Q]\!] = 0$.

From the Jacobi identity for $[\![\ ,\ ]\!]$ with $A = P$, $B = P$, $C = X$, it follows that

$[\![P, [\![P, X]\!]]\!] = 0$ for all vector fields X,

so we can <u>always</u> "deform" P with $Q = [\![P, X]\!]$.

Poisson cohomology in degree 2

In fact infinitesimal Poisson deformations of the form

$$P + \varepsilon [\![P, X]\!] + O(\varepsilon^2)$$

are the result of coordinate changes (time-ε flow of X).

$$H^2_P(\mathbb{R}^d) = \frac{\{ Q \text{ bi-vector field on } \mathbb{R}^d : [\![P, Q]\!] = 0 \}}{\{ Q \text{ bi-vector field on } \mathbb{R}^d : Q = [\![P, X]\!] \}}$$

Example. $M = \mathbb{R}^2$, $\quad P = xy \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$

$[\![P, X]\!]$ vanishes at $(0,0)$ for any smooth v.f. X

$Q = 1 \cdot \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ does not. $\quad \rightsquigarrow \quad H^2_P(\mathbb{R}^2) \neq \{0\}$.

Universal deformations of Poisson structures

Question. Could there exist a universal formula $Q(P)$ — in terms of P^{ij} and derivatives — yielding a deformation for any Poisson bi-vector field P?

That is, with $[\![P, Q(P)]\!] = 0$ for every bi-vector field P satisfying $[\![P, P]\!] = 0$?

And can it be done in such a way that $Q(P) \neq [\![P, X]\!]$ for some P?

M. Kontsevich's tetrahedral flow

Proposition (M. Kontsevich, 1996/2017)

For every Poisson bi-vector field $\mathcal{P} = \sum_{1\leq i<j\leq d} P^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$,

the bi-vector field

$$Q_{tetra}(\mathcal{P}) = \frac{\partial^3 P^{ij}}{\partial x^k \partial x^\ell \partial x^m} \cdot \frac{\partial P^{kk'}}{\partial x^{\ell'}} \cdot \frac{\partial P^{\ell\ell'}}{\partial x^{m'}} \cdot \frac{\partial P^{mm'}}{\partial x^{k'}} \, \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

$$+ 6 \cdot \frac{\partial^2 P^{ij}}{\partial x^k \partial x^\ell} \cdot \frac{\partial^2 P^{km}}{\partial x^{k'} \partial x^{\ell'}} \cdot \frac{\partial P^{k'\ell}}{\partial x^{m'}} \cdot \frac{\partial P^{m'\ell'}}{\partial x^j} \, \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^m}$$

is a Poisson 2-cocycle: $[\![\mathcal{P}, Q_{tetra}(\mathcal{P})]\!] = 0$.

[1] M.K. Formality Conjecture (1996).
[2] M.K. Seminaire Bourbaki, 69ème année, talk at IHP 14/01/17.
[3] A.B, R.B, A.V.K. The Kontsevich tetrahedral flow revisited (JGP, '17)

Universal deformations of Poisson structures arising from Kontsevich's graph complex

0. Background

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1. Theory

$\rightsquigarrow$

2. Computation

Universal deformations of Poisson structures arising from Kontsevich's graph complex

[1] M. Kontsevich. Formality Conjecture (Ascona, 1996).

[2] T. Willwacher. M. Kontsevich's graph complex and the Grothendieck-Teichmüller Lie algebra (arXiv 2010-2013, Invent. Math. 2015).

[3] C. Jost. Globalizing L_∞-automorphisms of the Schouten algebra of polyvector fields (arXiv 2012, Differ. Geom. Appl. 2013).

[4] R.B., A.V. Kiselev. The orientation morphism: from graph cocycles to deformations of Poisson structures (arXiv 2018, JPCS 2019)

Universal operations on multi-vector fields

$$S := \{\text{Multi-vector fields } \alpha = \sum_{1\le i_1<\dots<i_k\le d} \alpha^{i_1\dots i_k}\xi_{i_1}\cdot\ldots\cdot\xi_{i_k} \text{ on } \mathbb{R}^d\}$$

Operations $S\otimes\ldots\otimes S \longrightarrow S$ (multi-linear $S\times\ldots\times S\longrightarrow S$)

Natural examples

- $\mathrm{mult}_2 : S\otimes S \to S \qquad \alpha_1\otimes\alpha_2 \longmapsto \alpha_1\alpha_2$
- $\mathrm{mult}_3 : S\otimes S\otimes S \longrightarrow S \qquad \alpha_1\otimes\alpha_2\otimes\alpha_3 \longmapsto \alpha_1\alpha_2\alpha_3.$
- Schouten bracket $S\otimes S \longrightarrow S \qquad \alpha_1\otimes\alpha_2 \longmapsto \pm[\![\alpha_1,\alpha_2]\!]$

[(skew-)symmetrization]

$$\Pi_S = \mathrm{mult}_2 \circ \Delta_{12} \qquad \Delta_{12} = \sum_{k=1}^{d}\left(\frac{\vec\partial}{\partial\xi_k}\otimes\frac{\vec\partial}{\partial x^k} + \frac{\vec\partial}{\partial x^k}\otimes\frac{\vec\partial}{\partial\xi_k}\right)$$

$$\Delta_{12} : S\otimes S \longrightarrow S\otimes S$$

Universal operations on multi-vector fields, arising from graphs

Define analogously $\Delta_{ij} : S \otimes \cdots \otimes S \longrightarrow S \otimes \cdots \otimes S$ as acting like Δ_{12}, but on the i^{th} and j^{th} tensor factors, with Koszul sign.

Proposition. The operators Δ_{ij} anti-commute: $\Delta_{ij} \circ \Delta_{kl} = - \Delta_{kl} \circ \Delta_{ij}$.

Construction. Op : Graph with ordered edge set $\longrightarrow$ Operation

$Op($ 

$) = \mathrm{mult}_2 \circ \Delta_{12} = \pi_S$

$Op($ 

$) = \mathrm{mult}_3 \circ \Delta_{12} \circ \Delta_{13} \circ \Delta_{23}$

The natural bracket on (graph) operations

Nijenhuis-Richardson bracket = (graded) commutator of insertions.

$$[\Pi_S, Op(\gamma)]_{NR}(\_,\_,\_,\_) = \pm\, \Pi_S(Op(\gamma)(\_,\_,\_),\_)$$
$$\pm\, \Pi_S(\_, Op(\gamma)(\_,\_,\_))$$
$$\pm\, Op(\gamma)(\Pi_S(\_,\_),\_,\_)$$
$$\pm\, \ldots$$
$$\pm\, Op(\gamma)(\_,\_,\Pi_S(\_,\_))$$

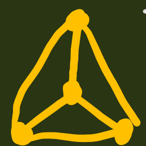
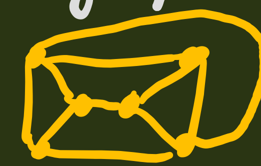
Bracket on sums of graphs $[\ ,\ ]$ = commutator of insertions

such that $Op([\beta,\gamma]) = [Op(\beta), Op(\gamma)]$

and $Op(\bullet\!\!-\!\!\bullet) = \Pi_S$ $\Rightarrow$ .... (next slides)

Graph cocycles $\rightsquigarrow$ Poisson cocycles

$d = [\bullet\!-\!\bullet, -]$ differential on sums of graphs

$d\Big($  $\Big) = 0$, $d\Big($  $+ \frac{5}{2}$  $\Big) = 0$, ...

$[\pi_S, -]_{NR}$ differential on operations of multi-vector fields

Theorem (Kontsevich, 1996). If γ is a graph cocycle on n vertices and $2n-2$ edges, then $[\![P, \mathrm{Op}(\gamma)(P, \dots, P)]\!] = 0$

Proof. If $d(\gamma) = 0$ then

$$0 = \mathrm{Op}(0) = \mathrm{Op}(d(\gamma)) = \mathrm{Op}([\bullet\!-\!\bullet, \gamma]) = [\overset{\pi_S}{\overset{\parallel}{\mathrm{Op}(\bullet\!-\!\bullet)}}, \mathrm{Op}(\gamma)]_{NR}$$

Evaluate at n copies of P (previous slide with $\beta = \bullet\!-\!\bullet$)

Graph coboundaries $\rightsquigarrow$ Poisson coboundaries

Proposition. If $\gamma = d(\delta)$ is a graph coboundary, then $Op(\gamma)(P, \ldots, P) = [\![P, X_\gamma]\!]$, where $X_\gamma \sim Or(\beta)(P, \ldots, P)$ is a Poisson coboundary.

Proof. Evaluate $Op(\gamma) = [\pi_S, Op(\delta)]_{NR}$ at $n-1$ copies of P.

(Non)triviality of Poisson flows

So if γ is a nonzero graph cohomology class, then $Q_\gamma(P) = \mathcal{O}_P(\gamma)(P, \dots, P)$ is "designed" to be a nonzero Poisson cohomology class. $Q_\gamma(P) \neq [\![P, X]\!]$.

Problem. Find γ and P such that $Q_\gamma(P) \neq [\![P, X]\!]$.

Open problem since 1996.

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Experimental mathematics ahead

I have developed the software package

gcaops (Graph Complex Action on Poisson Structures)

for SageMath, released under the MIT free software license, available from

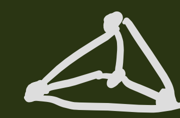
https://github.com/cburing/gcaops

implementing the theory that was just described, and used it jointly with my collaborators to calculate new examples, verify known identities, teach tutorial classes.

Highlights of experimental results

- Explicit representatives of graph cohomology classes

  ($\gamma_3, \gamma_5,$) γ_7, γ_9 Found using sparse Gaussian elimination implemented by Berenike Dieterle, student of Claus Fieker (RPTU Kaiserslautern).

- Explicit formulas & graph realizations of vector fields X_γ such that $Q_\gamma(P) = [\![P, X_\gamma]\!]$ for
  - P = generic 2D, $\gamma \in \{\gamma_3, \gamma_5, \delta_6, \gamma_7\}$
  - P = rescaled Nambu-Poisson, $\gamma = \gamma_3 =$ 
  - P = some R-matrix Poisson, $\gamma = \gamma_3$.

- Work in progress by students of AVK $\left\{\begin{array}{l}\text{Mollie Jagoe Brown} \\ \text{Floor Schipper}\end{array}\right\}$ 2D & 4D using gcaops