Macdonald Duality in genus 1 and 2

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Ⓐ Macdonald Theory

- SDAHA, operators & polynomials, duality
- universal formulation

Ⓑ Koornwinder Theory

Ⓒ Genus 2 Macdonald Theory

A. Macdonald Theory

(type $A_{N-1}^{(1)}$).

1. DAHA

[Cherednik 95]

Generators & Relations (A_r type)

(q, t) $\quad \theta = t^{1/2}$

Ⓐ Hecke Algebra

$$\boxed{\begin{array}{c} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ (T_i - \theta)(T_i + \theta^{-1}) = 1 \end{array}} \quad (i = 1, \dots, r)$$

Ⓑ X_i: $i = 1 \dots r+1$ commuting variables $\quad \boxed{T_i X_i T_i = X_{i+1}}$

Ⓒ Y_i $\quad i = 1 \dots r+1$ commuting variables $\quad \boxed{T_i^{-1} Y_i T_i^{-1} = Y_{i+1}}$ $\quad (i = 1, \dots r)$

and commutations:

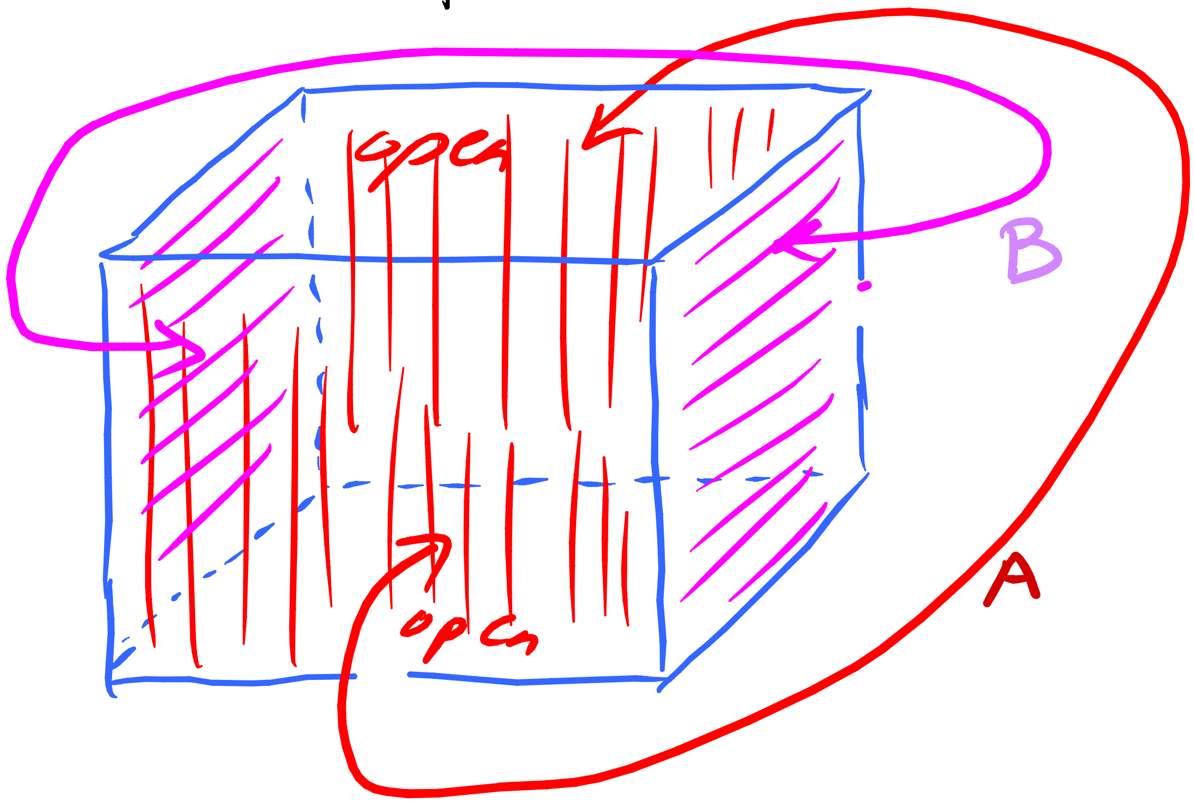
$$\boxed{\begin{cases} T_i X_j = X_j T_i \\ T_i Y_j = Y_j T_i \end{cases} \quad (j \neq i, i+1)} \quad (i = 1 \dots r)$$

$$\boxed{X_1 Y_2 = Y_2 T_1^2 X_1} \quad \boxed{\left(\prod_1^{r+1} X_i\right) Y_j = q^{-1} Y_j \left(\prod_1^{r+1} X_i\right)} \quad \boxed{\left(\prod_{i=1}^{r+1} Y_i\right) X_j = q X_j \left(\prod_1^{r+1} Y_i\right)}$$

$(j = 1 \dots r+1)$ $\qquad (j = 1 \dots r+1)$

Applications: Torus knot invariants, integrable systems...

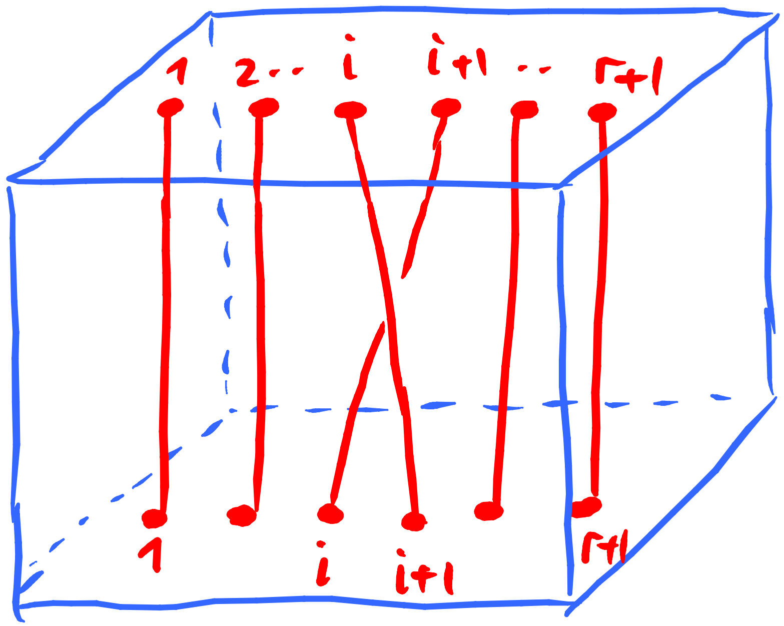
Pictorial Representation

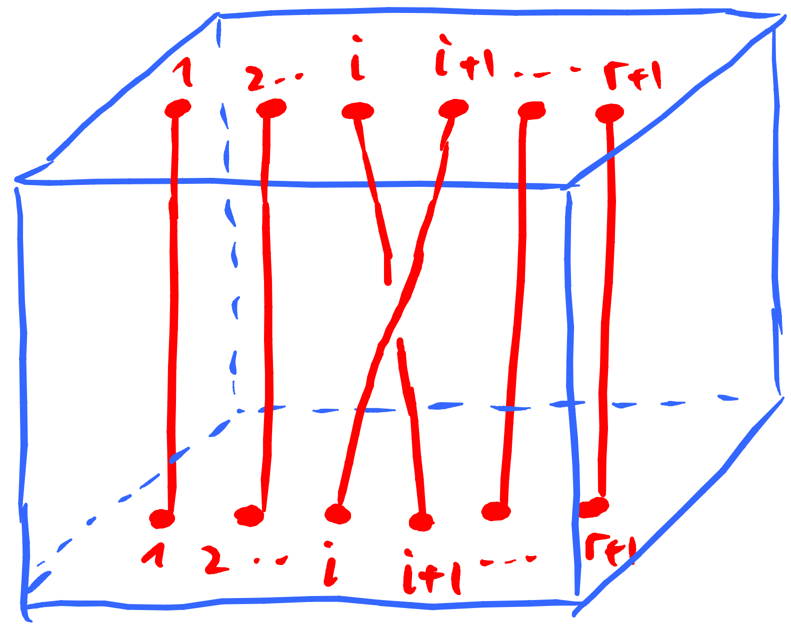


solid Torus

[Burella, Watts, Pasquier, Vala: 13]

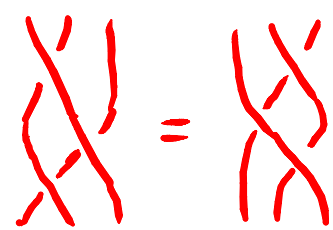
Product = "Matriochka" of genus 1

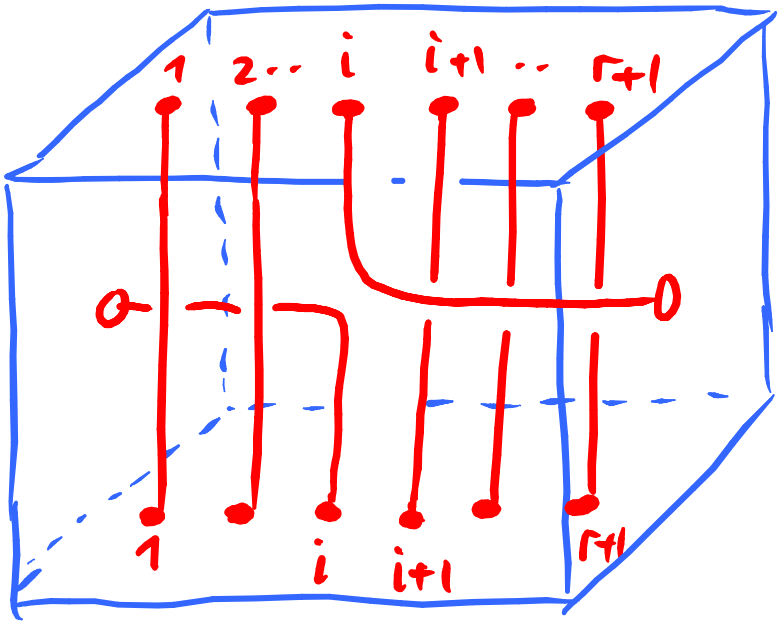




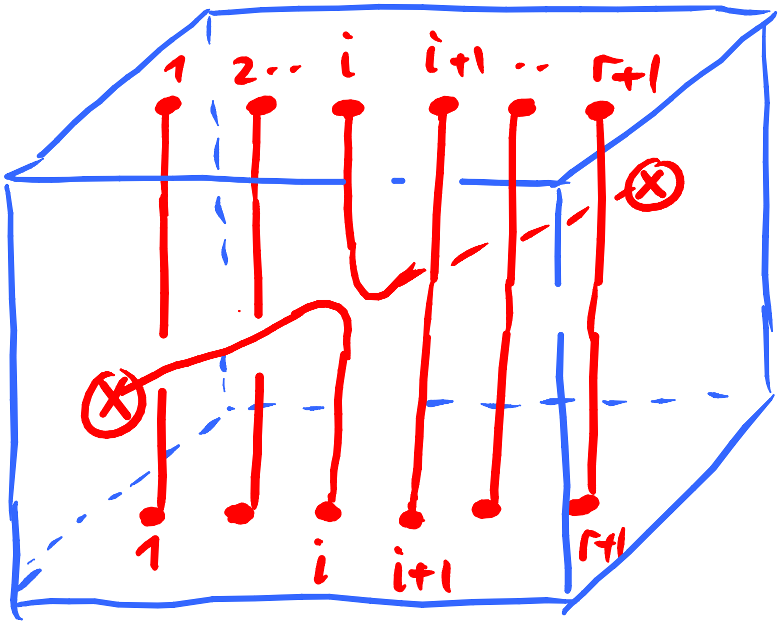
T_i $\qquad\qquad$ T_i^{-1} $\qquad\qquad$ $i \in [1, r]$

- Action : top → bottom
- Braid relation = pull strings

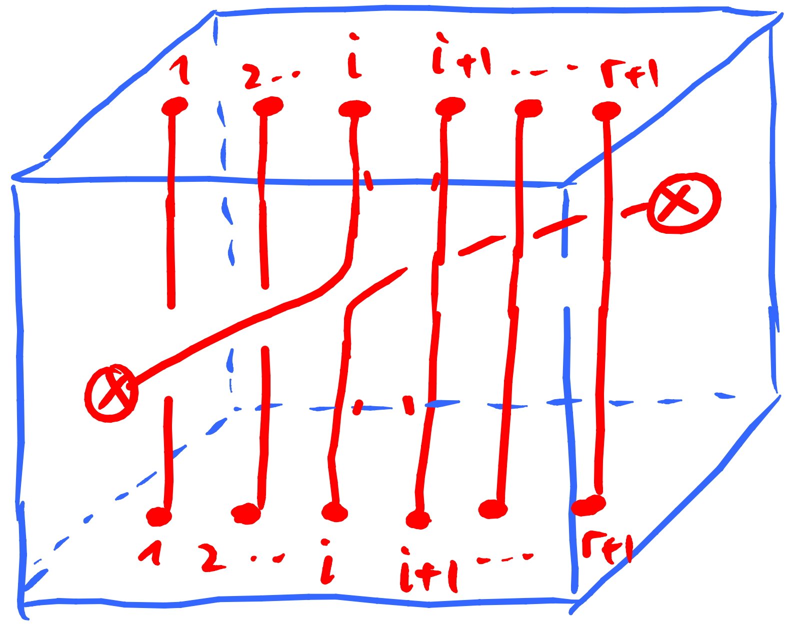




Y_i $\qquad$ Y_i^{-1} $\qquad$ $i \in [1, r]$



X_i



X_i^{-1} $i \in [1, r]$

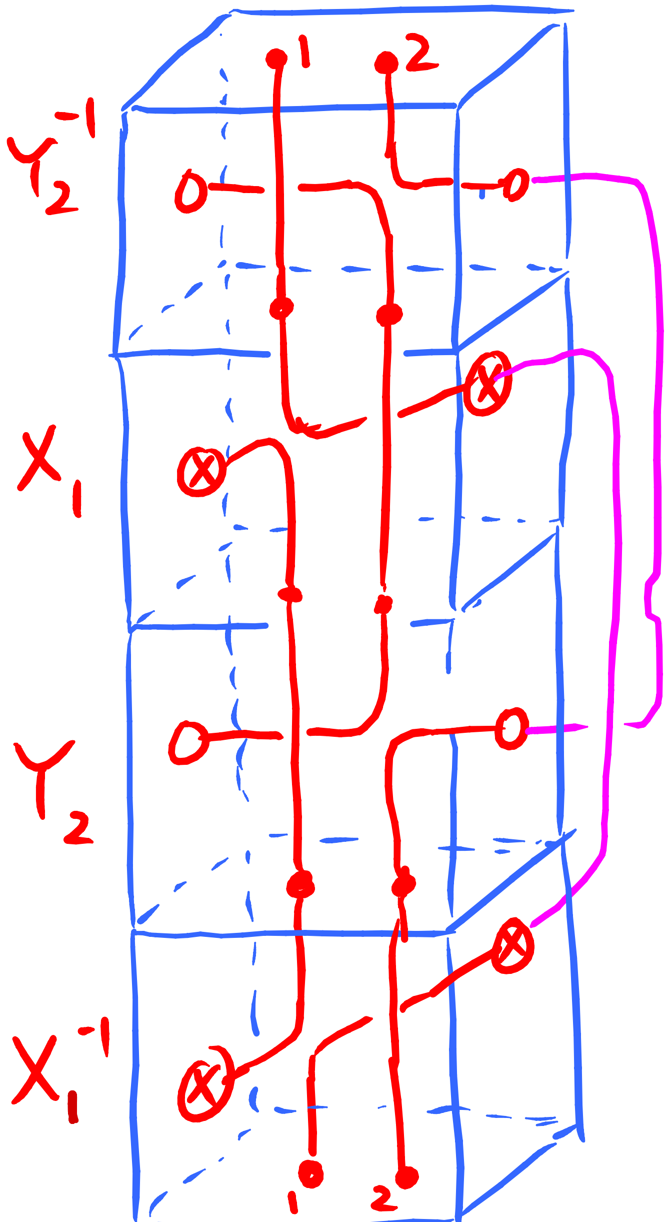
and Π = cyclic permutation of the points.

then: $\boxed{\text{All the relations are pictorial !}}$

$$Y_2 T_1^2 X_1 = X_1 Y_2$$

or

$$Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^2$$



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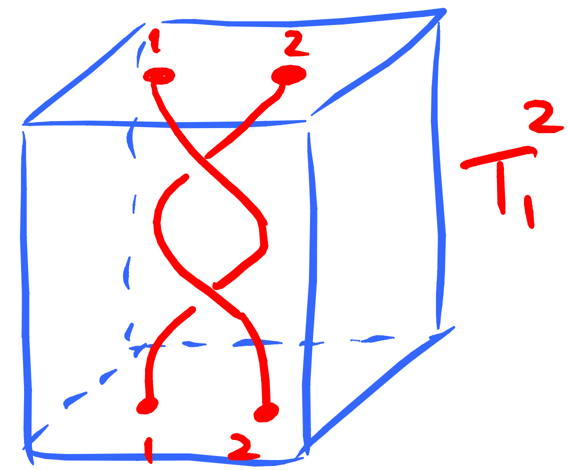


2. spherical DAHA : functional representation

generated by elementary symmetric fctns of X_i and of Y_i :

- $e_a(\{X_i\})\, f(x_1 \cdots x_N) = e_a(\{x_i\})\, f(x_1, \dots, x_n)$

- $e_a(\{Y_i\})\, f(x_1 \dots x_N) = \sum_{\substack{I \\ |I| = a}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t x_i - x_j}{x_i - x_j} \Gamma_I = t^{\binom{a}{2}} D_a$

$$\Gamma_{\{i_1, i_2, \dots, i_k\}} = \Gamma_{i_1} \Gamma_{i_2} \cdots \Gamma_{i_k} , \quad \Gamma_i f = f|_{x_i \to q x_i}$$

- D_a = Macdonald Operators (type $A^{(1)}_{N-1}$)

$$[D_a, D_b] = 0 \quad \forall a, b = 1, 2, \dots, N .$$

3. Macdonald Polynomials

- common eigenfunctions to D_a, monic.

$$\boxed{D_a \, P_\lambda(x) = e_a(s) \, P_\lambda(x)}$$

$$\begin{cases} s = t^\rho q^\lambda \\ s_i = t^{N-i} q^{\lambda_i} \end{cases} \qquad \begin{cases} \lambda \vdash n \\ \ell(\lambda) \leq N \end{cases}$$

- Duality:

[Macdonald Cherednik]

$$\boxed{\frac{P_\lambda(x = t^\rho q^\mu)}{P_\lambda(t^\rho)} = \frac{P_\mu(s = t^\rho q^\lambda)}{P_\mu(t^\rho)}}$$

$\lambda \vdash n$, $\mu \vdash n$ $\ell(\lambda), \ell(\mu) \leq N$

4. Duality $\Rightarrow$ Pieri rules

Pieri rules: $\boxed{e_a(x)\,P_\lambda(x) = H_a(s)\,P_\lambda(x)}$

difference op acting on λ.

Macdo eigenvalue $\quad D_a(x)\,P_\lambda(x) = e_a(s)\,P_\lambda(x) \qquad x = t^s q^\mu$

$\downarrow$

Duality $\quad D_a(t^s q^\mu)\,\dfrac{P_\mu(s)\,\cancel{P_\lambda(t^s)}}{P_\mu(t^s)} = e_a(s)\,\dfrac{P_\mu(s)\,\cancel{P_\lambda(t^s)}}{P_\mu(t^s)}$

$$\left[P_\mu(t^s)\,D_a(x)\,P_\mu(t^s)^{-1}\right]P_\mu(s) = e_a(s)\,P_\mu(s)$$

$\downarrow$

$x \leftrightarrow s \quad \underbrace{P_\mu(t^s)\,D_a(s)\,P_\mu(t^s)^{-1}}_{H_a(s)}\,P_\mu(x) = e_a(x)\,P_\mu(x)$

Explicitly:

$$\boxed{H_a(s) = \sum_{\substack{I\subset[1,N]\\ |I|=a}} \prod_{\substack{i\in I\\ j\notin I\\ j<i}} \frac{t^{-1}s_j - s_i}{s_j - s_i} \, \frac{t s_j - q s_i}{s_j - q s_i} \, T_I}$$

$$T_{\{i_1, i_2 \dots i_k\}} = T_{i_1} \cdots T_{i_k} \; ; \quad T_i f = f|_{s_i \to q s_i}$$

$$[H_a, H_b] = 0 \quad \forall a, b \in [1,N] .$$

5. Reformulation: universal solution

look for solutions of the Macdo eigenvalue eqn ((x,s) generic)

$$\boxed{D_a(x)\ x^{\lambda} f(x|s) = e_a(s) x^{\lambda} f(x|s)}$$

f = Series $\qquad f(x|s) = \sum_{\beta\in Q_+} c_\beta(s)\, x^{-\beta}$

Q_+ = positive root cone $= \bigoplus_{i=1}^{N-1} \mathbb{Z}\alpha_i$; $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$

Possible because coeffs in $D_a(x)$ = series of same form

$\Rightarrow$ triangular system for $c_\beta(s)$, unique up to $c_0(s)=1$

(our choice)

Example of sl_2

$$D_1(x) = \frac{t x_1 - x_2}{x_1 - x_2}\Gamma_1 + \frac{t x_2 - x_1}{x_2 - x_1}\Gamma_2 \qquad \Gamma_i : x_i \to q x_i$$

$$\boxed{\left(t - \frac{x_2}{x_1}\right) F(qx_1, x_2) - \left(t\frac{x_2}{x_1} - 1\right) F(x_1, qx_2) = (s_1 + s_2)\left(1 - \frac{x_2}{x_1}\right) F(x_1, x_2)}$$

$$F(x_1, x_2) \doteq x_1^{\lambda_1} x_2^{\lambda_2} \sum_{n=0}^{\infty} c_n(s_1, s_2)\left(\frac{x_2}{x_1}\right)^n \qquad \left(s_1 = t q^{\lambda_1} ; s_2 = q^{\lambda_2}\right)$$

coeff of $\left(\frac{x_2}{x_1}\right)^n$:

$$(t q^{\lambda_1} c_n - c_{n-1} q^{\lambda_1+1}) q^{-n} + (q^{\lambda_2} c_n - t q^{\lambda_2-1} c_{n-1}) q^n = (s_1 + s_2)(c_n - c_{n-1})$$

$$\left[s_1(1 - q^{-n}) + s_2(1 - q^n)\right] c_n = \left[s_1\left(1 - \frac{q}{t}^{1-n}\right) + s_2(1 - t q^{n-1})\right] c_{n-1}$$

$$\boxed{c_m = \prod_{n=1}^{m}\left(\frac{1 - t q^{n-1}}{1 - q^n}\,\frac{1 - t q^{n-1} s_2/s_1}{1 - q^n s_2/s_1}\right)}$$

General Case: Path model (sl_N)

- $D_1(x)\, F(x|s) = e_1(s)\, F(x|s)$

  $\hookrightarrow$ rational in $x^{-\alpha}$ $\quad \alpha \in R_+ = \{\varepsilon_i - \varepsilon_j \,;\, 1 \leq i < j \leq N\}$

- Write $F(x|s) = x^{\lambda} \sum_{\beta \in Q_+} c_\beta(s)\, x^{-\beta}$ $\quad Q_+ =$ posit. root cone

- recursion relation for c_β:

$$\boxed{\begin{aligned} c_\beta(s) &= \sum_{\alpha \in R_+} u_{\alpha,\beta}(s)\, c_{\beta-\alpha}(s) \\ &= \sum_{\substack{\text{paths} \\ 0 \to \beta \\ \text{in } Q_+}} \prod_{\text{steps } \bullet} u_\bullet(s) \end{aligned}}$$

(triangular)

Why universal?

- $f(x|s)$ specializes to Macdo polynomials

$$f(x|\, s=q^{\lambda}t^{\delta}) = x^{-\lambda} P_{\lambda}(x) \qquad \left(\begin{array}{l}\forall\ \lambda \vdash n \\ \ell(\lambda) \leq N\end{array}\right)$$

series truncates

polynomial

cf. Previous example (sl_2):

$$\boxed{C_m = \prod_{n=1}^{m} \left(\frac{1-tq^{n-1}}{1-q^{n}} \, \frac{1-tq^{n-1}s_2/s_1}{1-q^{n}s_2/s_1} \right)}$$

$$\frac{s_2}{s_1} = \frac{q^{\lambda_2}}{tq^{\lambda_1}} \Rightarrow$$

$$c_m = 0 \quad \forall\, m > \lambda_1 - \lambda_2$$

Similarly for Pieri equation:

$$\boxed{H_a(s)\, x^\lambda \varphi(s|x) = e_a(x)\, x^\lambda \varphi(s|x)}$$

$$\varphi(s|x) = \sum_{\beta \in Q_+} \tilde{c}_\beta(x)\, s^{-\beta} \quad \text{unique up to } \tilde{c}_0(x) = 1$$

THM ① $f(x|s) = \Delta(x)\,\varphi(s|x)$

[Koden-DF]

positive roots

$$\Delta(x) = \prod_{1\le i<j\le N} \prod_{n=1}^{+\infty} \frac{1 - \frac{x_j}{x_i} q^n}{1 - \frac{x_j}{x_i} \frac{q^n}{t}} = \prod_{1\le i<j\le N} \frac{\left(q\frac{x_j}{x_i}; q\right)_\infty}{\left(\frac{q}{t}\frac{x_j}{x_i}; q\right)_\infty}$$

② Duality is:

$$\boxed{\begin{aligned} &\varphi(x|s) = \varphi(s|x) \\ \Leftrightarrow\ &f(x|s)\,\Delta(s) = f(s|x)\,\Delta(x) \end{aligned}}$$

③ Norm conjecture (thm)

$$\boxed{P_\lambda(t^\rho) = t^{\rho\cdot\lambda} \frac{\Delta(t^\rho)}{\Delta(t^\rho q^\lambda)}}$$

finite product

cancellations for $\lambda \vdash n$

$$\Rightarrow \quad H_a(s) = \Delta(s)^{-1} D_a(s) \Delta(s)$$

Why?

- we can write

$$\boxed{D_a(x) = \operatorname*{Sym}_{\{x_1 \cdots x_N\}} \left\{ \Delta(x) \Gamma_1 \cdots \Gamma_a \Delta(x)^{-1} \right\}}$$

- Conjugation with $\Delta(s)^{-1} D_a(s) \Delta(s)$ undoes the first term $H_a(s) = T_1 T_2 \cdots T_a$ + other terms.

B. Koornwinder Theory

1. Koornwinder operator(s)

$$\mathcal{D}_1^{(a,b,c,d,q,t)} = \sum_{i=1}^{N} \sum_{\varepsilon=\pm 1} \phi_{i,\varepsilon}^{(abcdqt)}(x) \left(\Gamma_i^{\varepsilon} - 1 \right) + K_0$$

- $$\phi_{i,\varepsilon}^{(abcdqt)}(x) = \frac{\prod_{u \in \{a,b,c,d\}} 1 - u x_i^{\varepsilon}}{\sigma t^{N-1} (1 - x_i^{2\varepsilon})(1 - q x_i^{2\varepsilon})} \prod_{\substack{j \neq i \\ \eta = \pm 1}} \frac{t x_i^{\varepsilon} - x_j^{\eta}}{x_i^{\varepsilon} - x_j^{\eta}}$$

- $K_0 = \hat{e}_1(\sigma t^{\delta})$, $\hat{e}_1(x_1 \ldots x_N) = \sum_{i=1}^{N} x_i + x_i^{-1}$, $\sigma = \sqrt{\frac{abcd}{q}}$

Weyl-elementary sym. fctn $\mathfrak{S}_N \ltimes \mathbb{Z}_2$ $\quad x_i \to x_i^{-1}$

addition of K_0 = non-standard, but makes eigenvalues nicer!

$$\boxed{\mathcal{D}_1 \, P_\lambda^{(abcdqt)}(x) = \hat{e}_1(s) \, P_\lambda^{(abcdqt)}(x)}$$

Koornwinder (monic) polynomial

$s = \sigma t^{\delta} q^{\lambda}$

$\lambda \vdash n, \ell(\lambda) \le N$.

- Rk 1 = Askey-Wilson
- major application: specializations give Macdo theories for classical/twisted types.

$\mathfrak{g}$	$\mathfrak{g}^*$	a	b	c	d	R	S	R^*	σ
$D_N^{(1)}$	$D_N^{(1)}$	1	-1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	D_N	D_N	D_N	1
$B_N^{(1)}$	$C_N^{(1)}$	t	-1	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	B_N	B_N	C_N	$t^{\frac{1}{2}}$
$C_N^{(1)}$	$B_N^{(1)}$	$t^{\frac{1}{2}}$	$-t^{\frac{1}{2}}$	$t^{\frac{1}{2}} q^{\frac{1}{2}}$	$-t^{\frac{1}{2}} q^{\frac{1}{2}}$	C_N	C_N	B_N	t
$A_{2N-1}^{(2)}$	$A_{2N-1}^{(2)}$	$t^{\frac{1}{2}}$	$-t^{\frac{1}{2}}$	$q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	C_N	B_N	C_N	$t^{\frac{1}{2}}$
$D_{N+1}^{(2)}$	$D_{N+1}^{(2)}$	t	-1	$t\, q^{\frac{1}{2}}$	$-q^{\frac{1}{2}}$	B_N	C_N	B_N	t
$A_{2N}^{(2)}$	$A_{2N}^{(2)}$	t	-1	$t^{\frac{1}{2}} q^{\frac{1}{2}}$	$-t^{\frac{1}{2}} q^{\frac{1}{2}}$	BC_N	–	BC_N	t

2. Duality

transformation $(a,b,c,d) \longmapsto (a^*, b^*, c^*, d^*)$

$$q, t \longmapsto q, t$$

$$a^* = \sqrt{\frac{abcd}{q}} = \sigma \ , \ b^* = -\sqrt{q\frac{ab}{cd}}, \ c^* = \sqrt{q\frac{ac}{bd}}, \ d^* = -\sqrt{q\frac{ad}{bc}}.$$

then

[van Diejen Sahi]

$$\boxed{\frac{P_\lambda(x = \sigma^* t^\rho q^\mu)}{P_\lambda(\sigma^* t^\rho)} = \frac{P^*_\mu(s = \sigma t^\rho q^\lambda)}{P^*_\mu(\sigma t^\rho)}}$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$(a,b,c,d) \qquad\qquad (a^*, b^*, c^*, d^*)$$

3. Pieri rule(s):

Same calculation as in Macdo case $\Longrightarrow$

$$\mathcal{H}_1(s)\, P_\lambda(x) = \hat{e}_1(x)\, P_\lambda(x)$$

$$\boxed{\mathcal{H}_1(s) = P^*_\lambda(\sigma t^s)\, \mathcal{D}^*_1(s)\, P^*_\lambda(\sigma t^s)^{-1}}$$

(a,b,c,d)

dual theory's operator

(a^*,b^*,c^*,d^*)

4. Universal Solution(s)

Same story

- $\mathcal{D}_1(x)\, x^{\lambda} f^{abcd}(x|s) = \hat{e}_1(s)\, x^{\lambda} f^{abcd}(x|s)$
- $f^{abcd}(x|s) = \sum_{\beta \in Q_+} c_\beta^{abcd}(s)\, x^{-\beta}$; $c_0 = 1$

$Q_+ \leftarrow$ B-type root cone

(path solution)

specialization: $f^{abcd}(x|s = \sigma t^{\rho} q^{\lambda}) = x^{-\lambda} P_\lambda^{abcd}(x)$

f contains information on all Koornwinder polynomials.

Even better: for the various specializations f^{abcd} gives rise to all $\mathfrak{g}$-type Macdonald polynomials ($\lambda = \mathfrak{g}$-dominant weight)

eg • type $D_N^{(1)}$: $\lambda_1 \geq \cdots \lambda_{N-1} \geq |\lambda_N| \geq 0$

$\lambda_N \in \mathbb{Z}$

• type $B_N^{(1)}$: $\frac{1}{2}$-integer λ's. etc...

5. Duality revisited

universal Pieri solution : $\varphi^{abcd}(s,x) = \sum_{\beta\in Q_+} \tilde{c}^{abcd}_\beta(x)\, s^{-\beta}$ $(\tilde{c}_0 = 1)$

$$\mathcal{H}^{abcd}_1(s)\, x^\lambda \varphi^{abcd}(s|x) = e^\lambda_1(x)\, x^\lambda \varphi^{abcd}(x|s)$$

THM

[Kedem-DF]

① $f(x|s) = \Delta(x)\,\varphi(s|x)$

$$\Delta(x) = \prod_{i=1}^{N} \frac{(q x_i^{-2};q)_\infty}{\prod_{u\in abcd}(\frac{q}{u}x_i^{-1};q)_\infty} \prod_{1\le i<j\le N} \frac{(q\frac{x_j}{x_i};q)_\infty(\frac{q}{x_i x_j};q)_\infty}{(\frac{q}{t}\frac{x_j}{x_i};q)_\infty(\frac{q/t}{x_i x_j};q)_\infty}$$

② Duality is $\varphi(s|x) = \varphi^*(x|s)$

$$\Longleftrightarrow \quad f(x|s)\,\Delta^*(s) = f^*(s|x)\,\Delta(x)$$

③ Norms:

$$\boxed{P_\lambda(\sigma^* t^\rho) = (\sigma^* t^\rho)^\lambda \frac{\Delta^*(\sigma t^\rho)}{\Delta^*(\sigma t^\rho q^\lambda)}}$$

$$\Rightarrow \quad \mathcal{H}_1(s) = \Delta^*(s)^{-1}\,\mathcal{D}^*_1(s)\,\Delta^*(s)$$

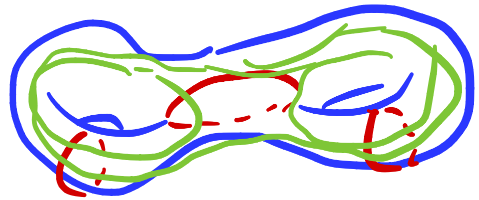
(a, b, c, d) $\qquad\qquad$ (a^*, b^*, c^*, d^*)

C. Genus 2 Macdonald Theory

1. Genus 2 Macdonald Operators

[Arthamonov-Shakirov 19] • Refined Chern-Simons theory on a genus 2 surface → 6j coefficients → Pieri rules with non-monic normalization.

• defined genus 2 sDAHA (rank 1 only).



  • 3 sets of X_i, X_i^{-1} (A cycles)
  • 3 sets of Y_i, Y_i^{-1} (B cycles)

→ 3 Macdonald operators (commuting)

→ Macdonald polynomials (eigenfunctions).

Genus 2 Macdonald operators:

$$\boxed{D_{i,j}(x) = \frac{1}{t} \sum_{\varepsilon_i, \varepsilon_j = \pm 1} \frac{(t x_i^{\varepsilon_i} x_j^{\varepsilon_j} x_k - 1)(t x_i^{\varepsilon_i} x_j^{\varepsilon_j} x_k^{-1} - 1)}{(x_i^{2\varepsilon_i} - 1)(x_j^{2\varepsilon_j} - 1)} \Gamma_i^{\varepsilon_i} \Gamma_j^{\varepsilon_j}}$$

acts on functions of $x = (x_1, x_2, x_3)$; $1 \leq i < j \leq 3$.

THM $[D_{i,j}, D_{k,l}] = 0 \quad \forall\, i<j,\ k<l$ in $\{1,2,3\}$.

- Common eigenfunctions are labeled by "genus 2 partitions" λ

"Roots": in $\mathbb{C}^3$: $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\alpha_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\alpha_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

"Root systems": $R_\pm$: $\rho = \frac{1}{2}(\alpha_1+\alpha_2+\alpha_3) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$-R_- = R_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1+\alpha_2, \alpha_1+\alpha_3, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3\}$

"Positive cone": $Q_+ = \bigoplus \mathbb{Z}_+ \alpha_i$

genus 2 partitions:

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3; \ (\alpha_i, \lambda) \in 2\mathbb{Z}_+$$

$i = 1, 2, 3$

"Weyl group": $W = (\mathbb{Z}_2)^3$

$x_i \to x_i^{-1}$

$$\boxed{D_{ij}(x)\, P_\lambda(x) = \hat{e}_1(s_k)\, P_\lambda(x)}$$

- $\alpha_i + \alpha_j = 2\varepsilon_k \quad \left(\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right)$
- $\hat{e}_1(x) = x + x^{-1}$, $\quad s_k = tq^{\frac{1}{2}(\alpha_i+\alpha_j,\lambda)} \quad (\alpha_i + \alpha_j = 2\varepsilon_k)$
- λ = genus 2 partition
- P_λ monic $\quad P_\lambda(x) = \prod_{i=1}^{3} x_i^{\frac{1}{2}(\alpha_i,\lambda)} \left(1 + O(x^{-\beta})\right)$

  $\beta \in Q_+^*$

2. Duality

THM [Kedem DF]

$$\boxed{\frac{P_\lambda(t^\rho q^\mu)}{P_\lambda(t^\rho)} = \frac{P_\mu(t^\rho q^\lambda)}{P_\mu(t^\rho)}}$$

$\forall \lambda, \mu$ genus 2 partitions.

As before $\Longrightarrow$ Pieri rules.

$$H_m(s)\, P_\lambda(x) = \hat{e}_1(x_m)\, P_\lambda(x)$$

3. Universal Solutions

- $D_{ij}(x)\ x^{\lambda} f(x|s) = \hat{e}_1(s_k)\ x^{\lambda} f(x|s)$

$$f(x|s) = \sum_{\beta \in Q_+} c_\beta(s)\, x^{-\beta} \quad ; \quad c_0 = 1 .$$

(path model)

- $H_m(s)\ x^{\lambda} \varphi(s|x) = \hat{e}_1(x_m)\ x^{\lambda} \varphi(s|x)$

$$\varphi(s|x) = \sum_{\beta \in Q_+} \tilde{c}_\beta(x)\, s^{-\beta}$$

THM

[Kedem DF]

$$\boxed{f(x|s = t^{\rho} q^{\lambda}) = x^{-\lambda} P_\lambda(x)}$$

$\forall\ \lambda$ genus 2 partition

4. Duality

THM

[Kedem DF]

① $f(x|s) = \Delta(x)\,\varphi(s|x)$

(R_+)

$$\Delta(x) = \frac{\prod_{1\le i<j\le 3} \left(q^2 x^{-\alpha_i-\alpha_j}; q^2\right)_\infty}{\left(\frac{q^2}{t} x^{-\alpha_1-\alpha_2-\alpha_3}; q^2\right)_\infty \prod_{i=1}^{3} \left(\frac{q^2}{t} x^{-\alpha_i}; q^2\right)_\infty}$$

② Duality is: $\varphi(s|x) = \varphi(x|s)$

$\iff \quad f(x|s)\,\Delta(s) = f(s|x)\,\Delta(x)$

③ Norms

$$\boxed{P_\lambda(t^\rho) = t^{\frac{1}{2}(\rho,\lambda)} \frac{\Delta(t^\rho)}{\Delta(q^\lambda t^\rho)}}$$

$\Rightarrow$ Pieri operators are

$$\boxed{H_m(s) = \Delta^{-1}(s)\, D_{i,j}(s)\, \Delta(s)}$$

$1 \le i < j \le 3 \;;\; \alpha_i + \alpha_j = 2\varepsilon_m$

Example

$$H_1(s) = T_1 T_2 + \frac{s_1 s_2}{s_3} \frac{\left(1 - t^{-1}\frac{s_2 s_3}{s_1}\right)\left(1 - \frac{t}{q^2}\frac{s_2 s_3}{s_1}\right)}{(1 - s_2^2)\left(1 - \frac{1}{q^2}s_2^2\right)} \frac{T_1}{T_2}$$

$$+ \frac{s_1 s_2}{s_3} \frac{\left(1 - t^{-1}\frac{s_1 s_3}{s_2}\right)\left(1 - \frac{t}{q^2}\frac{s_1 s_3}{s_2}\right)}{(1 - s_1^2)\left(1 - \frac{1}{q^2}s_1^2\right)} \frac{T_2}{T_1}$$

$$+ \frac{\left(1 - t^{-1}\frac{s_1 s_2}{s_3}\right)\left(1 - \frac{t}{q^2}\frac{s_1 s_2}{s_3}\right)\left(1 - t^{-1}s_1 s_2 s_3\right)\left(1 - \frac{t}{q^2}s_1 s_2 s_3\right)}{(1 - s_1^2)(1 - s_2^2)\left(1 - \frac{1}{q^2}s_1^2\right)\left(1 - \frac{1}{q^2}s_2^2\right)} \frac{1}{T_1 T_2}$$

$$R_+ : \left\{ \begin{array}{llll} s^{\alpha_1} = \frac{s_1 s_2}{s_3} & s^{\alpha_2} = \frac{s_1 s_3}{s_2} & s^{\alpha_3} = \frac{s_2 s_3}{s_1} & \\ s^{\alpha_1+\alpha_2} = s_1^2 & s^{\alpha_2+\alpha_3} = s_3^2 & s^{\alpha_1+\alpha_3} = s_2^2 & s^{\alpha_1+\alpha_2+\alpha_3} = s_1 s_2 s_3 \end{array} \right.$$

(D) Applications / Conclusion / Perspectives

- connection to cluster algebras / quantum Q-systems in the q-Whittaker limit $t \to \infty$

  finite t? Koornwinder? genus 2 yes

  no higher rank!

- new dualities from geometry

  [Shiraishi - Braverman - Finkelberg for $A_{N-1}^{(1)}$].

- Action of the mapping class group
- Fourier transform
- To do: combinatorics / Plethistic transformations.

Merci!

P. Di Francesco and R. Kedem "Duality and Macdonald difference operators" [arXiv 2303.04276]

P. Di Francesco and R. Kedem "Macdonald duality and the proof of the quantum Q-system conjecture" [arXiv 2112.09798]
(To appear in Selecta)