Lazard Elimination on Arbitrary Alphabets, Lyndon Words and Iterated Smash-Products.

From combinatorics of universal problems
to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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Combinatorics and Arithmetic for Physics, CAP 23.
Jubillee, Tenth Anniversary Edition,
IHES, 16 Nov. 2023.

Overture

Let me thank the organisers (minus one) for letting me deliver the following talk about filtrations of alphabets and their combinatorial counterparts.

Special words of gratitude are due to Darij Grinberg, Jean-Gabriel Luque and Pierre Simonnet who carefully read parts of this work, made fruitful remarks and asked constructive questions.

- Last time (CAP'22) we spoke about Lazard Elimination (LE) and $\mathbb{B}$-gradings.
- And yesterday Pr. Nakamura wrote a word $w \in \{x, y\}^*$ under the normal form

$$w = \underbrace{x^{k_1} y x^{k_2} y \cdots x^{k_d} y}_{\textit{regular part}} \underbrace{x^{k_\infty}}_{\textit{tail}}$$

- We will interpret this as the elimination of $\{y\}$ among the alphabet $X = \{x, y\}$ and the Magnus basis as the image of the code $x^* y$ under the isomorphism of Lazard's elimination.

Overture (cont'd)

- Today, I would like to call your attention to the result of iterating such a dichotomic process leading to a filtration on the alphabet of generators.
- And on the combinatorial couterpart of this phenomenon (Hilbert series, Indexed computation, Normal forms).
- Examples will be taken from
  1. Free structures for simplicity.
  2. Free partially commutative structures for the visual and mnemotechnic representations with heaps.
  3. The Drinfeld-Kohno Lie algebras and their enveloping algebras.
- The process is however general and rather simple to implement.
- From time to time categories will be used as a way to understand similarities and unify the exposition.
- The process is however general and rather simple to implement. We will end with iterated crossed-products allowing for deformations and perturbations (see [7]).

Part one :

Preamble and generalities.

Which sort of elimination will we consider here ?

$$STRUCT\langle x_1, x_2, \ldots, x_n\rangle \cong NICE\langle x_1, x_2, \ldots, x_n\rangle \diamond STRUCT_1\langle x_1, \ldots, x_{n-1}\rangle \tag{1}$$

where *NICE* et $STRUCT_1$ stand for algebraic structures generated (sometimes freely) by generators x_i. The diamond symbol being, according to the situation, a tensor product, a semi-direct product or a plain (unique) factorisation. For example, with the symmetric group $\mathfrak{S}_n$ and the pure braid group P_n [1] :

$$\mathfrak{S}_{\mathfrak{n}} \cong {}^{\mathbb{Z}}/_{n\mathbb{Z}} \diamond \mathfrak{S}_{\mathfrak{n}-1} \quad \text{and} \quad P_n \cong F_{n-1} \diamond P_{n-1}.$$

Here, in the first case, $\diamond$ is only a product and the iterated decomposition helps to construct a basis of $\mathbb{Q}[\mathfrak{S}_{\mathfrak{n}}]$ adapted to the calculation needs of *Dynkin*'s projector [6]. In the second case we have a semi-direct product (where F_{n-1} is the Free Group with $n-1$ generators.

Rewriting the factors

We recall the pattern with colors

$$STRUCT\langle x_1, x_2, \ldots, x_n\rangle \cong NICE\langle x_1, x_2, \ldots, x_n\rangle \diamond STRUCT_1\langle x_1, \ldots, x_{n-1}\rangle$$

(when $STRUCT_1 = STRUCT$ the process can be iterated).
Let us firstly see the case of two permutable subgroups (where the $\diamond$ is multiplicative), we have $G = G_1 G_2 = G_2 G_1$ (and it is required that $G = G_1.G_2$ be of unique factorisation). Then, at the level of the terms, the rewriting reads

$$g_2 g_1 \longrightarrow l(g_1, g_2) r(g_1, g_2) \tag{2}$$

and, in the case when $r(g_1, g_2) = g_2$, we have a semidirect product i.e. for every $(g_1, g_2) \in G_1 \times G_2$, $g_2 g_1 {g_2}^{-1} \in G_1$, so that we only need to know the factor $l(g_1, g_2)$.

Categories of this talk.

1. These categories are as follows
   1. **Set** the category of sets.
   2. **Mon**, the category of monoids.
   3. $\mathbf{k}-\mathbf{Lie}$, the category of $\mathbf{k}$-Lie algebras.
   4. **Grp**, the category of groups.
   5. $\mathbf{k}-\mathbf{AAU}$, the category of $\mathbf{k}$-associative algebras with unit.
2. Functors are as follows

$$
\begin{tikzcd}
 & \mathbf{Mon}(2) \arrow[dl, "F_{12}"'] & \mathbf{Grp}(5) \arrow[l, "F_{25}"'] \\
\mathbf{Set}(1) & & \mathbf{k}-\mathbf{AAU}(4) \arrow[ul, "F_{24}"'] \arrow[dl, "F_{34}"] \\
 & \mathbf{k}-\mathbf{Lie}(3) \arrow[ul, "F_{13}"] &
\end{tikzcd}
$$

Figure 1: Rq: Similar lower diagram with algebras and $\mathbf{k}-\mathbf{Mod}$ replacing **Set**.

Free Objects: Adjunction "A la Samuel".

3. We recall here the mechanism of adjunction w.r.t. a functor.
Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two categories and $F_{12} : \mathcal{C}_2 \to \mathcal{C}_1$ a (covariant) functor between them

$$\begin{tikzcd}
\mathcal{C}_1 & & \mathcal{C}_2 \arrow[ll, dotted, "F_{12}"'] \\
X \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j_X"'] & & Y \\
& & \hat{X} = G_{21}(X) \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Figure 2: In natural language, the universal problem reads:
Does it exist a pair $(j_X, \hat{X})$ with the property that for any $\mathcal{C}_1$-theoretical morphism $f : X \to Y$, there exists a unique $\hat{f} : \hat{X} \to Y$ such that the diagram above commutes through F (when needed).
If it is the case for every object $X \in \mathcal{C}_1$, then the correspondence $X \to \hat{X}, \; f \to \hat{f}$ between $\mathcal{C}_1$ and $\mathcal{C}_2$ turns out to be a (covariant) functor G_{21}.

Combinatorics of Free objects and their gradings (fine and coarse).

Category	Abbv.	Free Gen. by X
Monoids	**Mon**	X^*
Groups	**Grp**	$F(X)$ ($\to FG(X)$)
k unital associative algebras	**k** $-$ **AAU**	$\mathbf{k}\langle X\rangle$ $(= \mathbf{k}[X^*])$
k-Lie algebras	**k** $-$ **Lie**	$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \subset \mathbf{k}\langle X\rangle$

- $X^* = \sqcup_{\alpha\in\mathbb{N}^{(X)}} X^\alpha = \sqcup_{n\in\mathbb{N}} X^n$
- $\mathbf{k}\langle X\rangle = \oplus_{\alpha\in\mathbb{N}^{(X)}} \mathbf{k}\langle X\rangle^\alpha = \oplus_{n\in\mathbb{N}} \mathbf{k}\langle X\rangle^n$
- $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle = \oplus_{\alpha\in\mathbb{N}^{(X)}} \mathcal{L}ie_{\mathbf{k}}\langle X\rangle^\alpha = \oplus_{n\in\mathbb{N}} \mathcal{L}ie_{\mathbf{k}}\langle X\rangle^n$

Words and their gradings

Example with $X = \{a, b\}$ and $Z = \{a\}, B = \{b\}$

Length	*words*
0	1_{X^*}
1	$a,\ b$
2	$aa,\ ab,\ ba,\ bb$
3	$aaa,\ aab,\ aba,\ abb,\ baa,\ bab,\ bba,\ bbb$
4	$a^4,\ a^3b,\ a^2ba,\ a^2b^2,\ aba^2,\ abab,\ ab^2a,\ ab^3$
	$ba^3,\ ba^2b,\ baba,\ babb,\ b^2a^2,\ b^2ab,\ b^3a,\ b^4$

In red words of $(X^*)_{BZ}$ and in blue words of $(X^*)_B = B^*$.

Words and Lyndon words

Although words be strictly equivalent to lists (and in obvious one-to-one correspondence with them), coding by words gives access to a welter of structures, studies, relations and results (algebra, geometry, topology, probability, combinatorics on words, on polynomials and series). We will use in particular their complete factorisation by Lyndon words.

The data structure

Finite lists of symbols taken within a set (called alphabet) including the void one.

Algebraic structure

- Concatenation: Words concatenate by shifting and union of domains, this law is noted *conc*
- With the empty word as neutral, the set of words is the free monoid $(X^*, conc, 1_{X^*})$

Words and Lyndon words/2

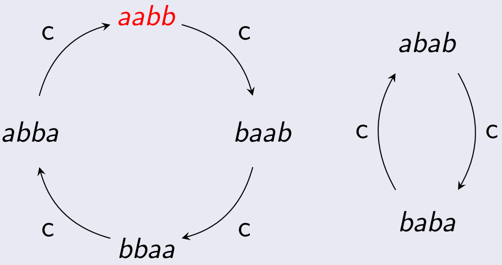
Words and classes

Example with $X = \{a, b\}$

Length	*words*
0	1_{X^*}
1	$a,\ b$
2	$aa,\ ab,\ ba,\ bb$
3	$aaa,\ aab,\ aba,\ abb,\ baa,\ bab,\ bba,\ bbb$
4	$a^4,\ a^3b,\ a^2ba,\ \underline{a^2b^2},\ aba^2,\ abab,\ ab^2a,\ ab^3$
	$ba^3,\ ba^2b,\ baba,\ babb,\ b^2a^2,\ b^2ab,\ b^3a,\ b^4$

In red Lyndon words (for the ordering $a < b$), in blue and (brown+underlined) two conjugacy classes (that of $abab$ and $\underline{aabb}$).

Conjugacy & Lyndon words



Two conjugacy classes, one of them is primitive and contains, as its minimum, a Lyndon word.

Words and Lyndon words/4

The word w, $|w| \geq 1$ is Lyndon iff, for each (non trivial) decomposition $w = uv$, $u, v \neq 1_{X^*}$, one has $u \prec_{lex} v$.

Factorisation properties and series

Free monoid

Each word w factorizes uniquely as $w = l_1^{\alpha_1} \cdots l_n^{\alpha_n}$ with $l_i \in \mathcal{L}yn(X)$ and $l_1 \succ \cdots \succ l_n$ (strict). We have (Schützenberger, MPS) $X^* = \prod^{\searrow}_{l \in \mathcal{L}yn(X)} l^*$.

$$\chi = \prod_{l \in \mathcal{L}yn(X)}^{\searrow} e^{\chi(S_l)\, P_l} \qquad (MRS)$$

Towards series

Series are functions $X^* \to R$ where R is a semiring (i.e. a ring without the "minus" operation). We have different ways to consider a series, namely:
Math: Functions, elements of a dual (total, restricted, Sweedler's &c.)
Computer Science: Behaviour of a system (automaton, transducer, grammar &c.)
Physics: Non commutative differential equations, evaluation of paths, normal orderings &c.

Classical Lazard elimination theorem

Theorem (Lazard elimination theorem)

Let $X = B \sqcup Z$ be a set partitioned in two blocks. We have an isomorphism of split short exact sequences (see [5] Ch II §2.9 Props 9 and 10])

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \arrow[r, hook, "rn"] \arrow[d, "rn"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \arrow[r, "p_{B|Z}"] \arrow[d, "Id"] & \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \arrow[r] \arrow[d, "\overline{j_B}"] & 0 \\ 0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \arrow[r, hook, "j"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \arrow[r, "p"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{B} \arrow[r] & 0 \end{tikzcd} \tag{3}$$

This non-trivial result is proved by "semi-direct recomposition" (of Lie algebras). Semi-direct products of groups have been evoked in slide "Rewriting the factors". The mechanism for Lie algebras is analogue replacing "action by automorphisms" by "action by derivations".

Classical Lazard elimination theorem/2

Theorem (Lazard elimination theorem)

Let $X = B \sqcup Z$ be a set partitioned in two blocks. We have an isomorphism of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \arrow[r, hook, "rn"] \arrow[d, "rn"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \arrow[r, "p_{B|Z}"] \arrow[d, "Id"] & \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \arrow[r] \arrow[d, "\overline{j_B}"] & 0 \\
0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \arrow[r, hook, "j"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \arrow[r, "p"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{B} \arrow[r] & 0
\end{tikzcd} \tag{4}$$

Remark

The bottom row is trivial, it is nevertheless the prototype of all semi-direct product of Lie algebras in the following sense : every semi-direct product is the homomorphic image of a Lazard elimination scheme.

Classical Lazard elimination theorem/3

Theorem (Lazard elimination theorem)

Let $X = B \sqcup Z$ be a set partitioned in two blocks. We have an isomorphism of split short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \arrow[r, hook, "rn"] \arrow[d, "rn"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \arrow[r, "p_{B|Z}"] \arrow[d, "Id"] & \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \arrow[r] \arrow[d, "\overline{j_B}"] & 0 \\ 0 \arrow[r] & \mathfrak{h} \arrow[r, hook, "j"] & \mathfrak{g} \arrow[r, "p"] & \mathfrak{b} \arrow[r] & 0 \end{tikzcd} \tag{5}$$

Remark

Every semi-direct product is the homomorphic image of a Lazard elimination scheme. Here $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$.

The Drinfeld-Kohno Lie algebra $\mathrm{DK}_{\mathbf{k},n}$.

$$\mathcal{T}_n = \begin{array}{cccccc} T_2 & T_3 & T_4 & \dots & \dots & \color{red}{T_n} \\ \hline t_{1,2} & t_{1,3} & t_{1,4} & \dots & \dots & \color{red}{t_{1,n}} \\ & t_{2,3} & t_{2,4} & \dots & \dots & \dots \\ & & t_{3,4} & \dots & \dots & \dots \\ & & & \dots & \dots & \dots \\ & & & & \dots & \dots \\ & & & & & \color{red}{t_{n-1,n}} \end{array}$$

$$\mathrm{DK}_{\mathbf{k},n} = \langle\, \mathcal{T}_n \,|\, \mathbf{R[n]} \,\rangle_{\mathbf{k-Lie}} = \mathcal{L}ie_{\mathbf{k}}\langle \mathcal{T}_n \rangle \Big/ \mathcal{J}_{\mathbf{R[n]}} \cong \mathcal{L}ie_{\mathbf{k}}\langle \color{red}{T_n} \rangle \rtimes \mathrm{DK}_{\mathbf{k},n-1} \quad (6)$$

Free objects, partition of alphabets and eliminations.

Category	Abbv.	Free Gen. by X
Monoids	**Mon**	X^*
Groups	**Grp**	$F(X)$ $(\to FG(X))$
k unital associative algebras	$\mathbf{k}-\mathbf{AAU}$	$\mathbf{k}\langle X\rangle$ $(=\mathbf{k}[X^*])$
k-Lie algebras	$\mathbf{k}-\mathbf{Lie}$	$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle\subset\mathbf{k}\langle X\rangle$

Category	Abbv.	Elimination formula (free case)
Monoids	**Mon**	$X^*=(B^*Z)^*B^*$
Groups	**Grp**	$F(X)=F(C_B(Z))\rtimes F(B)$
k AAU	$\mathbf{k}-\mathbf{AAU}$	$\mathbf{k}\langle X\rangle=\mathbf{k}\langle B^*Z\rangle\otimes\mathbf{k}\langle B\rangle$
k-Lie algebras	$\mathbf{k}-\mathbf{Lie}$	$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle\cong\mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle\rtimes\mathcal{L}ie_{\mathbf{k}}\langle B\rangle$

Part two :

Partially commutative structures.

Partially Commutative structures: between commutative and non commutative worlds as first example.

4. As, today, we consider four categories:

$$\textbf{Mon},\ \textbf{Grp},\ \textbf{k-Lie},\ \textbf{k-AAU} \tag{7}$$

   In each of these categories, there is a notion of "What are two commuting elements"

   - in **Mon**, **Grp**, **k-AAU**, it is $xy = yx$
   - in **k-Lie** it is $[x, y] = 0$

   but, for all of them, this relation is *reflexive* and *symmetric*.
   This leads us to the following questions

5. What is elimination in these categories ?

6. What is the best system or category of formal generators ? i.e. the category $\mathcal{C}_1$ (if possible) in order to consider these objects as freely generated over $\mathcal{C}_1$.

7. We will begin by the "partially commutative monoid" so called "Cartier-Foata monoid".

Partially Commutative monoids

8. This monoid was introduced by P. Cartier and D. Foata in (1969) for combinatorial problems of commutations and rerrangements [4]. It ignited a considerable literature in combinatorics and computer science.
9. A partially commutative alphabet (X, θ) is a set endowed with a commutation relation $\theta \subset X \times X$, reflexive and symmetric. As follows (loops are on every vertex and not represented).

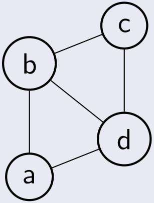


Figure 3:

Partially Commutative monoids

10. A partially commutative alphabet (X, θ) is a set endowed with a commutation relation $\theta \subset X \times X$, reflexive and symmetric.
11. The partially commutative monoid $M(X, \theta)$ is

$$M(X,\theta) := \langle X;\ (xy, yx)_{(x,y)\in\theta} \rangle_{\mathbf{Mon}} \tag{8}$$

12. For example, with the graph above we have

$$M(X,\theta) := \{a, c\}^* \times \{b\}^* \times \{d\}^* = \{w\, b^p\, d^q\}_{\substack{w\in\{a,c\}^*\\ p,q\in\mathbb{N}}} \tag{9}$$

and the first words of this monoid (i.e. traces, elements) are

$$1, \underbrace{a, b, c, d}_{length\ 1}, \underbrace{a^2, c^2, ac, {\color{red}ca}, ab, ad, cb, cd, b^2, d^2, bd}_{l=2,\ 11\ words}, \cdots$$

13. For length 2, if it were free, we would have 16 words and 10 if it were completely commutative.

Partially Commutative monoids: Hilbert series

14. In general, let us denote $M(X,\theta)^{(n)}$ the set of words of $M(X,\theta)$ of length n.
15. If the alphabet is finite, we have

$$Hilb(M(X,\theta),t) = \sum_{n\in\mathbb{N}} \left|M(X,\theta)^{(n)}\right| t^n = \frac{1}{\sum_{n\geq 0}(-1)^n c_n t^n} \tag{10}$$

    where c_n is the number of n-cliques of θ. This is a consequence of a more general theorem of Cartier and Foata [4] about the Möbius function of the Cartier-Foata monoid.
16. With the the graph above, we have

$$Hilb(M(X,\theta),t) = \frac{1}{1-4t+5t^2-2t^3} = \frac{1}{(1-2t)}\frac{1}{(1-t)^2} \tag{11}$$

Another commutation graph.

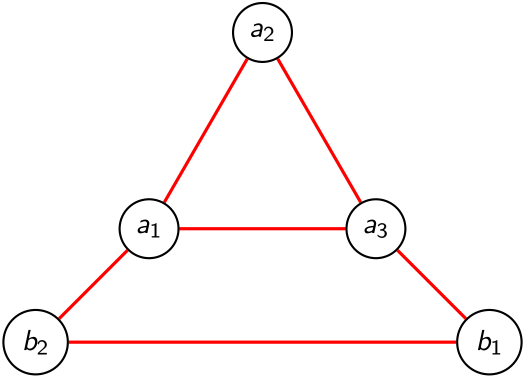


Figure 4: Commutation graph G1. The Hilbert series as in Eq. (10) is

$$\sum_{n\in\mathbb{N}} |M(X,\theta)^{(n)}|\, t^n = \frac{1}{1-6t+6t^2-t^3}$$

The model of heaps (of pieces)

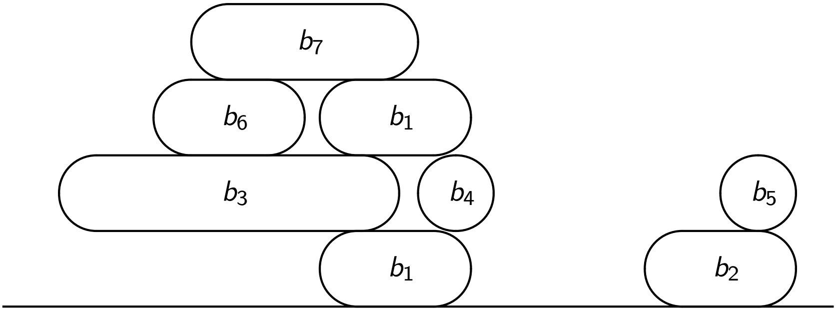


Figure 5: A heap of pieces H_t corresponding to the word $t = b_1 b_2 b_3 b_4 b_5 b_6 b_1 b_7 \in M(X, \theta)$ here $\mathrm{TAlph}(t) = \{b_1, b_2\}$ even if b_1 and b_6 commute.

From Krattenthaler [15], order of the letters has been reversed.

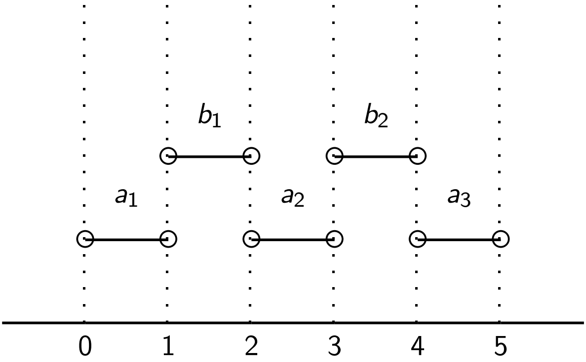


Figure 6: Dimers for the commutation graph G1.

17 We recall our research here

$$STRUCT\langle x_1, x_2, \ldots, x_n\rangle \cong NICE\langle x_1, x_2, \ldots, x_n\rangle \diamond STRUCT_1\langle x_1, \ldots, x_{n-1}\rangle$$

18 It turns out that, for partially commutative structures, eliminating totally noncommutative alphabets (i.e. Z is a set of vertices such that for all distinct $x, y \in Z$, we have $(x, y) \notin \theta$)) liberates a free structure.

Category	Abbv.	Elim. formula (part. comm.)
Monoids	**Mon**	$M(X,\theta) = C_B(Z)^* M(B,\theta_B)$
Groups	**Grp**	$F(X,\theta) = F(C_B^{Grp}(Z)) \rtimes F(B,\theta_B)$
k AAU	**k − AAU**	$\mathbf{k}\langle X,\theta\rangle = \mathbf{k}\langle C_B^{AAU}(Z)\rangle \otimes \mathbf{k}\langle B,\theta_B\rangle$
k-Lie algebras	**k − Lie**	$\mathcal{L}ie_{\mathbf{k}}\langle X,\theta\rangle \cong \mathcal{L}ie_{\mathbf{k}}\langle C_B(Z)\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B,\theta_B\rangle$

Free and partially commutative eliminations: comparison.

Category	Abbv.	Elimination formula (free)
Monoids	**Mon**	$X^* = (B^*Z)^*B^*$
Groups	**Grp**	$F(X) = F(C_B^{Grp}(Z)) \rtimes F(B)$
$\mathbf{k}$ AAU	$\mathbf{k} - \mathbf{AAU}$	$\mathbf{k}\langle X\rangle = \mathbf{k}\langle B^*Z\rangle \otimes \mathbf{k}\langle B\rangle$
$\mathbf{k}$-Lie algebras	$\mathbf{k} - \mathbf{Lie}$	$\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \cong \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B\rangle$

With free partially commutative structures (Z totally non-commutative and $X = B + Z$).

Category	Abbv.	Elim. formula (part. comm.)
Monoids	**Mon**	$M(X,\theta) = C_B(Z)^*M(B,\theta_B)$
Groups	**Grp**	$F(X,\theta) = F(C_B^{Grp}(Z)) \rtimes F(B,\theta_B)$
$\mathbf{k}$ AAU	$\mathbf{k} - \mathbf{AAU}$	$\mathbf{k}\langle X,\theta\rangle = \mathbf{k}\langle C_B^{AAU}(Z)\rangle \otimes \mathbf{k}\langle B,\theta_B\rangle$
$\mathbf{k}$-Lie algebras	$\mathbf{k} - \mathbf{Lie}$	$\mathcal{L}ie_{\mathbf{k}}\langle X,\theta\rangle \cong \mathcal{L}ie_{\mathbf{k}}\langle C_B(Z)\rangle \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B,\theta_B\rangle$

free partially commutative structures Lie algebra: Ladder

Theorem (Lazard elimination theorem)

Let $X = B \sqcup Z$ be a set partitioned in two blocks. We have an isomorphism of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \arrow[r, hook, "rn"] \arrow[d, "rn"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \arrow[r, "p_{B|Z}"] \arrow[d, "Id"] & \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \arrow[r] \arrow[d, "\overline{j_B}"] & 0 \\
0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle C_B(Z)\rangle \arrow[r, hook, "j"] & \mathcal{L}ie_{\mathbf{k}}\langle X, \theta\rangle \arrow[r, "p"] & \mathcal{L}ie_{\mathbf{k}}\langle B, \theta_B\rangle \arrow[r] & 0
\end{tikzcd} \tag{12}$$

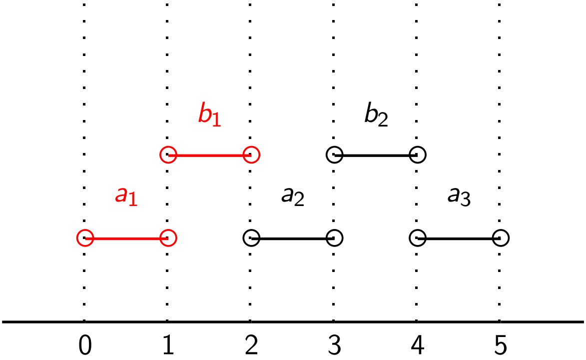


Figure 7: Example with the dimers of the commutation graph G1.
Here $Z = \{a_1, b_1\}$, $B = \{a_2, b_2, a_3\}$
and

$$C_B(Z) = a_1 + C_B(b_1)$$

.

Where the (forgetful) functor comes: Monoids.

19. Def **CAlph** be the category of alphabets with commutation i.e. reflexive and symmetric graphs (X, θ) with $f : (X_1, \theta_1) \to (X_2, \theta_2)$ such that $f : X_1 \to X_2$, set-theoretical such that $(u, v) \in \theta_1 \Longrightarrow (f(u), f(v)) \in \theta_2$ and **Mon** the category of monoids. Now a monoid M being given $\theta_M = F(M) = \{(u, v) \in M \mid uv = vu\}$ can be checked to be a functor $F : \mathbf{Mon} \to \mathbf{CAlph}$

$$\begin{tikzcd}
\mathbf{CAlph} & & \mathbf{Mon} \arrow[ll, dotted, "F"'] \\
(X,\theta) \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j"'] & & M \\
& & M(X,\theta) \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Figure 8: $M(X, \theta)$ is the monoid freely generated by (X, θ) w.r.t. F. Then

$$M(X, \theta) := \langle X;\ (xy, yx)_{(x,y)\in\theta} \rangle_{\mathbf{Mon}}$$

Functor/2: **k**-Lie algebras.

20. Let **k-Lie** be the category of **k**-Lie algebras (**k** is a ring). Now $L \in \mathbf{k\text{-}Lie}$ being given $\theta_L = F(L) = \{(u,v) \in L \,|\, [u,v] = 0\}$ can be checked to be a functor $F:\ \mathbf{k\text{-}Lie} \to \mathbf{CAlph}$

$$\begin{tikzcd}
\mathbf{CAlph} & & \mathbf{k\text{-}Lie} \arrow[ll, dotted, "F"'] \\
(X,\theta) \arrow[rr, dashed, "f"] \arrow[drr, dashed, "j"'] & & L \\
& & \mathcal{L}ie_k(X,\theta) \arrow[u, "\hat{f}"']
\end{tikzcd}$$

Figure 9: $\mathcal{L}ie_k(X,\theta)$ is the **k**-Lie algebra freely generated by (X,θ) w.r.t. F.

Then

$$\mathcal{L}ie_k(X,\theta := \langle X;\ [x,y] = 0_{(x,y)\in\theta}\rangle_{\mathbf{k}-\mathbf{Lie}}$$

Part three :

General case.

Main result: Elimination for presented Lie algebras/1.

21. Let $\mathbf{k}$ be a ring. Let $X = B + Z$ be a set partitioned in two blocks. We suppose given a relator $\mathbf{r} = \{r_j\}_{j\in J} \subset \mathcal{L}ie_{\mathbf{k}}\langle X\rangle$ (cf. [5] Ch II §2.3[a]) which is compatible with the alphabet partition i.e. there exists a partition of the set of indices $J = J_Z \sqcup J_B$ such that

   - $\mathbf{r}_B = \{r_j\}_{j\in J_B} = \mathbf{r} \cap \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B$ and $\mathbf{r}_Z = \{r_j\}_{j\in J_Z} = \mathbf{r} \cap \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ}$.

   The notations being as above, we construct the ideals

   - $\mathcal{J}_B$ is the Lie ideal of $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B$ generated by $\{r_j\}_{j\in J_B}$
   - $\mathcal{J}, \mathcal{J}_Z$ and $\mathcal{J}_{BZ}$ are the Lie ideals of $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle$ generated respectively by $\mathbf{r}, \mathbf{r}_Z$ and $\mathbf{r}_{BZ} := \{\mathrm{ad}_Q\, z\}_{Q\in\mathcal{J}_B, z\in Z}$.

[a] The set I there being replaced by X.

Elimination for presented Lie algebras/2

When we have such a type of relator, we can state the following theorem.

Theorem (Th 2)

With our constructions above, we get the following properties:

i) *we have* $(\mathcal{J}_Z + \mathcal{J}_{BZ}) \subset \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ}$ *(and then* $(\mathcal{J}_Z + \mathcal{J}_{BZ}) \cap \mathcal{J}_B = \{0\}$*). Moreover,* $(\mathcal{J}_Z + \mathcal{J}_{BZ})$ *is a Lie ideal of* $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ}$ *(and even, by definition, of* $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle$*).*

ii) *the action of* $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B$ *on* $\mathfrak{Der}(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ}$ *(by internal* ad*) passes to quotients as an action*

$$\alpha : \ \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \to \mathfrak{Der}\Big(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ})\Big) \tag{13}$$

such that $\mathbf{r}_B \subset \ker(\alpha)$ *and then, we get an action*

$$\overline{\alpha} : \ \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \Big/ \mathcal{J}_B \to \mathfrak{Der}\Big(\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ})\Big) \tag{14}$$

Elimination for presented Lie algebras/3

Th 2 cont'd

iii) We can construct an isomorphism (and its inverse) from presented Lie algebra $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle \big/ \mathcal{J}$ by the set $\mathbf{r} = \{r_j\}_{j\in J}$ of relators onto the semidirect product of Lie algebras $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) \rtimes \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \big/ \mathcal{J}_B$ which will be denoted by

$$\beta_{25} : \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \big/ \mathcal{J} \xrightarrow{\simeq} \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) \rtimes \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \big/ \mathcal{J}_B \tag{15}$$

iv) In fact, one has a commutative diagram of Lie algebras with split short exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \arrow[r, hook, "j"] \arrow[d, "s_{\mathcal{J}_Z+\mathcal{J}_{BZ}}"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \arrow[r, "p"] \arrow[d, "s_{\mathcal{J}}"] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \arrow[r] \arrow[d, "s_{\mathcal{J}_B}"] & 0 \\
0 \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_{BZ} \Big/ (\mathcal{J}_Z + \mathcal{J}_{BZ}) \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle \big/ \mathcal{J} \arrow[r] & \mathcal{L}ie_{\mathbf{k}}\langle X\rangle_B \big/ \mathcal{J}_B \arrow[r] & 0
\end{tikzcd}$$

Example 1

Infinitesimal Pure Braids Relations (n strands).

22. We consider the alphabet $\mathcal{T}_n = \{t_{ij}\}_{1\leq i<j\leq n}$ and the infinitesimal pure braid relator $\mathbf{R[n]}$ in the free Lie algebra

$$\mathbf{R[n]} = \begin{cases} \mathbf{R_1[n]} & [t_{i,j}, t_{i,k} + t_{j,k}] & \text{for } 1 \leq i < j < k \leq n, \\ \mathbf{R_2[n]} & [t_{i,j} + t_{i,k}, t_{j,k}] & \text{for } 1 \leq i < j < k \leq n, \\ \mathbf{R_3[n]} & [t_{i,j}, t_{k,l}] & \text{for } {}^{1\leq i<j\leq n,}_{1\leq k<l\leq n,} \text{ and } |\{i,j,k,l\}| = 4 \end{cases}$$

23. This is a typical example of relator compatible with the partition

$$X := \mathcal{T}_n = \mathcal{T}_{n-1} \sqcup T_n := B \sqcup Z$$

where $T_n = \{t_{i,n}\}_{1\leq i\leq n-1}$ and the infinitesimal pure braid relator $\mathbf{r} := \mathbf{R[n]} \subset \mathcal{L}ie_{\mathbf{k}}\langle\mathcal{T}_n\rangle = \mathrm{DK}_{\mathbf{k},n}$ the Drinfel'd-Kohno Lie algebra.

24. Applying the theorem, we get a semi-direct decomposition. One can prove that the first (i.e. "acted") factor is free.

$$\mathcal{T}_n = \begin{array}{cccccc} T_2 & T_3 & T_4 & \dots & \dots & \color{red}{T_n} \\ \hline t_{1,2} & t_{1,3} & t_{1,4} & \dots & \dots & \color{red}{t_{1,n}} \\ & t_{2,3} & t_{2,4} & \dots & \dots & \dots \\ & & t_{3,4} & \dots & \dots & \dots \\ & & & \dots & \dots & \dots \\ & & & & \dots & \dots \\ & & & & & \color{red}{t_{n-1,n}} \end{array}$$

$$\mathrm{DK}_{\mathbf{k},n} = \langle\, \mathcal{T}_n \,|\, \mathbf{R[n]} \,\rangle_{\mathbf{k}-\mathbf{Lie}} = \mathcal{L}ie_{\mathbf{k}}\langle \mathcal{T}_n \rangle \Big/ \mathcal{J}_{\mathbf{R[n]}} \cong \mathcal{L}ie_{\mathbf{k}}\langle \color{red}{T_n} \rangle \rtimes \mathrm{DK}_{\mathbf{k},n-1} \quad (16)$$

Example 2

Infinitesimal Pure Braids Relations (infinitely many strands).

25. For $\mathrm{DK}_{\mathbf{k},\infty}$, we consider the alphabet $\mathcal{T}_\infty = \{t_{ij}\}_{1\le i<j}$ and the infinitesimal pure braid relator $\mathbf{R}[\infty]$ in the free Lie algebra $\mathcal{L}ie_{\mathbf{k}}\langle \mathcal{T}_\infty \rangle$

$$\mathbf{R}[\infty] = \begin{cases} \mathbf{R_1}[\infty] & [t_{i,j}, t_{i,k} + t_{j,k}] & \text{for } 1 \le i < j < k, \\ \mathbf{R_2}[\infty] & [t_{i,j} + t_{i,k}, t_{j,k}] & \text{for } 1 \le i < j < k, \\ \mathbf{R_3}[\infty] & [t_{i,j}, t_{k,l}] & \text{for } \substack{1\le i<j,\\ 1\le k<l,} \text{ and } |\{i,j,k,l\}| = 4 \end{cases}$$

26. With the embeddings $\mathrm{DK}_{\mathbf{k},n} \hookrightarrow \mathrm{DK}_{\mathbf{k},n+1}$, the Lie algebra $\mathrm{DK}_{\mathbf{k},\infty}$ can be proved to be the inductive (direct) limit of all $\mathrm{DK}_{\mathbf{k},n}$.

Part four :

Sup gradings and tensor indexed computations.

Generalized gradings

27. We will take the families $(\mathrm{DK}_{\mathbf{k},n})_{n\in\mathbb{N}\cup\{\infty\}}$, $\mathcal{L}ie_{\mathbf{k}}\langle X,\theta\rangle$ as guiding examples.
28. Remark that $\mathrm{DK}_{\mathbf{k},n}$ (resp. $\mathrm{DK}_{\mathbf{k},\infty}$) is $([2,\cdots,n],\vee)\times\mathbb{N}_{\geq 1}$-graded (resp. $(\mathbb{N}_{\geq 2},\vee)\times\mathbb{N}_{\geq 1}$-graded.
29. On the other hand when a Lie algebra is a semi-direct product $\mathfrak{g}=\mathfrak{h}\rtimes\mathfrak{b}$ we can endow it with a $(\mathbb{B},+)$-grading where $(\mathbb{B},+)$ is the additive part of the boolean semiring whose law reads

$$\begin{array}{c|c|c} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 1 \end{array} \tag{17}$$

with $\mathfrak{g}_0=\mathfrak{b}$ and $\mathfrak{g}_1=\mathfrak{h}$.

30. Therefore iterated semi-direct products can be seen and $(I,\vee)$-graded (where "the low" acts on "the high").

Example of the Drinfeld-Kohno Lie algebras

31. As an example we see that $\mathsf{DK}_{\mathbf{k},n+1}$ is an iterated semi-direct product of free Lie algebras as follows

$$\begin{aligned} \mathsf{DK}_{\mathbf{k},n+1} &= \mathsf{DK}^{(n+1)}_{\mathbf{k},n+1} \rtimes \Big(\mathsf{DK}^{(n)}_{\mathbf{k},n+1} \rtimes (\cdots \rtimes \mathsf{DK}^{(2)}_{\mathbf{k},n+1})\cdots\Big) \\ &\cong \mathcal{L}ie_{\mathbf{k}}\langle X_n\rangle \rtimes (\mathcal{L}ie_{\mathbf{k}}\langle X_{n-1}\rangle \rtimes (\cdots \rtimes \mathcal{L}ie_{\mathbf{k}}\langle X_1\rangle)\cdots) \end{aligned} \tag{18}$$

32. Now when $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$, we have a canonical morphism (of modules)

$$\mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{b}) \to \mathcal{U}(\mathfrak{g})$$

which is one-to-one when $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$.

33. Therefore, formula 18 entails

$$\mathcal{U}(\mathsf{DK}_{\mathbf{k},n+1}) \cong \mathbf{k}\langle X_n\rangle \otimes \mathbf{k}\langle X_{n-1}\rangle \otimes \cdots \otimes \mathbf{k}\langle X_1\rangle. \tag{19}$$

A rewriting result (setting).

34 In order to formulate a theorem about iterated smash products, we start with $(A,<)$ a totally ordered alphabet. Let $S_A := \{A, \vee\}$ be the corresponding max-semigroup (i.e. $a \vee b = \max\{a,b\}$ for all $a,b \in A$) and $\mathfrak{g} = \bigoplus_{a\in A} \mathfrak{g}_a$ a S_A-graded Lie algebra. Let us consider

1. the formal direct sum $M = \bigoplus_{a\in A} \mathcal{U}_+(\mathfrak{g}_a)$ (where $\mathcal{U}_+(\mathfrak{g}_a)$ is the augmentation ideal of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_a)$)
2. the language of strictly increasing words $SI(A) \subset A^*$, formally

$$SI(A) := \{w \in A^* \mid \text{for all } j < |w|,\ w[j] < w[j+1]\}$$

3. the decomposition $T(M) = \bigoplus_{w\in A^*} T_w(M)$
4. the space $T_{SI(A)} := \bigoplus_{w\in SI(A)} T_w(M)$ where $SI(A) \subset A^*$ is the language of strictly increasing words
5. the language of (weakly) increasing words $WI(A) \subset A^*$, formally

$$WI(A) := \{w \in A^* \mid \text{for all } 1 \le j < |w|,\ w[j] \le w[j+1]\}.$$

A rewriting result (Theorem).

The following theorem states that $T_{SI(A)}$ is a section of the natural morphism $T(M) \to \mathcal{U}(\mathfrak{g})$.

Theorem (A)

We consider the canonical morphism defined by multiplication of factors

$$\begin{aligned} &\mathrm{can}: \ T(M) \to \mathcal{U}(\mathfrak{g}) \ \textit{i.e.}\ldots \\ &x_{a_1} \otimes \cdots \otimes x_{a_k} \mapsto x_{a_1} \cdots x_{a_k}. \end{aligned} \tag{20}$$

Then

$$T(M) = T_{SI(A)} \oplus \ker(\mathrm{can}) \tag{21}$$

Computation scheme/0

1. In order to prove Thm (A) we must construct a "word driven" way of rearranging the tensors in increasing form which converges towards the projector on $T_{SI(A)}$ parallel to the kernel of the natural morphism.
2. To this end, we must define what is "rearranging the tensors" and will use the structure of paths of computations through appropriate labeled graphs in the spirit of Hopcroft and Ullmann [13]. For a modern version (with R. Motwani), see [14]).

Computation scheme/1

We define

The graph of transitions Γ_{trans}

(a) **Vertices**: All finite sets of words $2^{(A^*)}$.

(b) **Elementary Steps**: Their set will be noted ES. These steps are of three types:

First type (Reduction of inversions) $\alpha = (\{ubav\}, \varphi_\alpha, \{uabv, ubv\})$ with $a < b$ and

$$\varphi_\alpha \quad : \quad x_u \otimes x_b \otimes x_a \otimes x_v \ \rightarrow \ x_u \otimes \tau(x_b \otimes x_a) \otimes x_v \qquad (22)$$

where τ_0 is the "twist" of the smash product (see Remark **??**). It can be shown that

$$\tau_0(\mathcal{U}_+(\mathfrak{g}_b) \otimes \mathcal{U}_+(\mathfrak{g}_a)) \subset \mathcal{U}_+(\mathfrak{g}_a) \otimes \mathcal{U}_+(\mathfrak{g}_b) + 1_{\mathbf{k}} \otimes \mathcal{U}_+(\mathfrak{g}_b) \qquad (23)$$

Computation scheme/2

1. therefore the result of the preceding reduction process belongs to $T_{uabv}(M) \oplus T_{ubv}(M)$.
   Second type (Reduction of powers) $\alpha = (\{ua^p v\}, \varphi_\alpha, \{uav\})$ with $p \geq 2$, by

$$\varphi_\alpha \quad : \quad x_u \otimes \overbrace{x_a^{(1)} \otimes \cdots \otimes x_a^{(p)}}^{p \text{ factors in } \mathcal{U}_+(\mathfrak{g}_a)} \otimes x_v \ \to \ x_u \otimes \underbrace{x_a^{(1)} \cdots x_a^{(p)}}_{\text{multiplication}} \otimes x_v \tag{24}$$

   the result of this reduction process is in $T_{uav}(M)$.
   Third type (Loops) $\alpha = (\{w\}, \varphi_\alpha, \{w\})$ for $w \in SI(A)$ with $\varphi_\alpha = \mathrm{Id}_{T_w}$.
   All the preceding (linear) maps φ_α (of first, second and third types) are extended by 0 outside of their definition domains ($T_{ubav}(M)$ for the first type $T_{ua^p v}(M)$ for the second and $T_w(M)$, $w \in SI(A)$ for the third).

Computation scheme/3

2. Summarizing, all φ_α belong to $\mathrm{End}(T(M))$.

(c) **General arrows** i.e. all arrows of Γ_{trans}. Their set is denoted GA. It is the set of triplets (F_1, Φ, F_2), with $F_i \in 2^{(A^*)}$, $\Phi \in 2^{(ES)}$ (finite sets of elementary steps) such that

1. for all $w \in F_1$ exists one and only one elementary step in $\alpha \in \Phi$ with $t(\alpha) = \{w\}$ (its tail).
2. $F_2 = \cup_{\alpha\in\Phi} h(\alpha)$ (union of their heads).

(d) **Tail and Head**: For every general arrow $\alpha = (F_1, \Phi, F_2)$, we set $t(\alpha) = F_1$ and $h(\alpha) = F_2$. This definition is extended for elementary arrows by (for $\alpha = (F_1, \varphi, F_2)$) the same projections (i.e. $t(\alpha) = F_1$ and $h(\alpha) = F_2$).

Computation scheme/4

(e) **Composition of Arrows**: Composition of (F_1, Φ_1, F_2) and (F_2, Φ_2, F_3) is $(F_1, \Phi_2 \circ \Phi_1, F_3)$ where

$$\Phi_2 \circ \Phi_1 = \{pr_2(\beta) \circ pr_2(\alpha) \,|\, \beta \in \Phi_2, \alpha \in \Phi_1, t(\beta) \subseteq h(\alpha)\}.$$

(f) **Paths**: A path in Γ_{trans} is a word $P = \alpha_1 \cdots \alpha_n \in GA^*$ such that, for all $j < |P| \, (= n)$, $h(\alpha_j) = t(\alpha_{j+1})$, we classically have $t(P) = t(\alpha_1)$ and $h(P) = h(\alpha_n)$. The evaluation of P, $Ev(P)$ is the composition of all the linear maps of its arrows i.e. with $P = \alpha_1 \cdots \alpha_n$,

$$Ev(P) = pr_2(\alpha_n) \circ \cdots \circ pr_2(\alpha_1) \tag{25}$$

Computation scheme/5

3. **Norm:** For all $w \in A^*$, we set $\mathrm{norm}(w) = |w| + Inv(w))$ (where $Inv(w) = \sharp\{(i,j) | 1 \leq i < j \leq |w| \text{ and } w[i] > w[j]\})$. This definition is at once extended to finite subsets of $F \subset A^*$ by $\mathrm{norm}(F) = \max_{w \in F \setminus SI(A)} \mathrm{norm}(w)$. We remark that, for all elementary arrow α of the two first types, $\mathrm{norm}(t(\alpha)) > \mathrm{norm}(h(\alpha))$ and equality is got for the third type. Hence, for any general arrow $\alpha = (F_1, \Phi, F_2)$, $\mathrm{norm}(t(\alpha)) > \mathrm{norm}(h(\alpha))$ unless $F_1 = F_2 \subset SI(A)$ in which case we have equality and all arrows of Φ are of third type.

(g) **Aperiodic paths**: An aperiodic path is a path whose last arrow has identical head and tail i.e. $\alpha_n = (F, \Phi, F)$, this entails that $F \subset SI(A)$ and that all arrows of Φ are of third type.

(h) **Remark.** – Conditions (b.i) and (b.ii) above say respectively that there is no outgoing computation fork (i.e. two different elementary steps) from one $w \in F_1$ and that F_2 is the image of F_1 through the arrows of Φ.

Alternative with an algebra cross: Crossed products

See [3]

We consider augmented algebras $(\mathcal{A}, \epsilon_{\mathcal{A}})$ (resp. $(\mathcal{B}, \epsilon_{\mathcal{B}})$) with $\mathcal{A}^+ := \ker(\epsilon_{\mathcal{A}})$ (resp. $\mathcal{B}^+ := \ker(\epsilon_{\mathcal{B}})$) and an algebra cross (see below)

$$\tau_{\mathcal{B},\mathcal{A}} : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$$

Definition

Suppose given two objects $\mathcal{A}$ and $\mathcal{B}$ in $\mathbf{k} - \mathbf{AAU}$. A morphism $\tau : \mathcal{B} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{B}$ in $\mathbf{k} - \mathbf{Mod}$ is called an algebra cross if it satisfies the following conditions

c1) $\tau(1_{\mathcal{B}} \otimes a) = a \otimes 1_{\mathcal{B}}$,

c2) $\tau \circ (m_{\mathcal{B}} \otimes \mathrm{Id}_{\mathcal{A}}) = (\mathrm{Id}_{\mathcal{A}} \otimes m_{\mathcal{B}}) \circ (\tau \otimes \mathrm{Id}_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{B}} \otimes \tau)$,

d1) $\tau(b \otimes 1_{\mathcal{A}}) = 1_{\mathcal{A}} \otimes b$,

d2) $\tau \circ (\mathrm{Id}_{\mathcal{B}} \otimes m_{\mathcal{A}}) = (m_{\mathcal{A}} \otimes \mathrm{Id}_{\mathcal{B}}) \circ (\mathrm{Id}_{\mathcal{A}} \otimes \tau) \circ (\tau \otimes \mathrm{Id}_{\mathcal{A}})$.

Alternative with an algebra cross: Link with the smash product of enveloping algebras and iterated versions

4. When a Lie algebra is decomposed as a semi-direct product $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{b}$, one has an algebra cross

$$\tau : \mathcal{U}(\mathfrak{b}) \otimes \mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{b})$$

given by the action of $\mathfrak{b}$ on $\mathfrak{h}$ by derivations.

5. Let us call $\alpha : \mathfrak{b} \to \mathfrak{Der}(\mathfrak{h})$ the action by $\mathrm{ad}^{\mathfrak{g}}$ on $\mathfrak{h}$. It is a morphism in $\mathbf{k}-\mathbf{Lie}$. We first extend (classically) α from $\mathfrak{b}$ to $\mathfrak{Der}(\mathcal{U}(\mathfrak{h})) \subset \mathrm{End}(\mathcal{U}(\mathfrak{h}))$. Moreover, we can also extend α as a morphism $\alpha_{\mathcal{U}} : \mathcal{U}(\mathfrak{b}) \to \mathrm{End}(\mathcal{U}(\mathfrak{h}))$ in $\mathbf{k}-\mathbf{AAU}$ by the universal property. Together with a bialgebra structure $(\mathcal{U}(\mathfrak{b}), \mu_{\mathcal{U}}, 1_{\mathbf{k}}, \Delta_{\mathcal{U}}, \epsilon_{\mathcal{U}})$, we then obtain a left $\mathcal{U}(\mathfrak{b})$-module algebra action $\triangleright : \mathcal{U}(\mathfrak{b}) \otimes \mathcal{U}(\mathfrak{g}_1) \to \mathcal{U}(\mathfrak{g}_1), b \otimes a \mapsto b \triangleright a = \alpha_{\mathcal{U}}(b)(a)$.. Now, the $\mathbf{k}$-module $\mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{b})$ can be endowed with a smash (cross) product structure $\mathcal{U}(\mathfrak{h}) \sharp \mathcal{U}(\mathfrak{b}) = (\mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{b}), 1_{\mathbf{k}} \otimes 1_{\mathbf{k}})$. The multiplication being

$$m_{\sharp}[(u_1 \otimes u_2) \otimes (v_1 \otimes v_2)] = \sum_{(1)(2)} u_1 \alpha_{\mathcal{U}}(u_2^{(1)})(v_1) \otimes u_2^{(2)} v_2. \qquad (26)$$

Alternative with an algebra cross/2.

6. We start with $(A,<)$ a totally ordered alphabet. Let $(\mathcal{A}_a)_{a\in A}$ be a family of augmented algebras and, for $b>a$, an algebra cross
$$\tau_{b,a}:\mathcal{A}_b\otimes\mathcal{A}_a\to\mathcal{A}_a\otimes\mathcal{A}_b$$
The limit of the finite iterated cross-products can be realized by the quotient $T(M)/_{\mathcal{J}}$ where M is the formal direct sum $M=\bigoplus_{a\in A}\mathcal{A}_a^+$ and $\mathcal{J}$ be the two-sided ideal generated by the elements $m_b\otimes m_a-\tau_{b,a}(m_b\otimes m_a)$. Let us consider:
   1. the language of strictly increasing words $SI(A)\subset A^*$, formally
   $$SI(A):=\{w\in A^*\mid \text{for all } j<|w|,\ w[j]<w[j+1]\}$$
   2. the decomposition $T(M)=\bigoplus_{w\in A^*}T_w(M)$
   3. the space $T_{SI(A)}:=\bigoplus_{w\in SI(A)}T_w(M)$ where $SI(A)\subset A^*$ is the language of strictly increasing words
   4. the language of (weakly) increasing words $WI(A)\subset A^*$, formally
   $$WI(A):=\{w\in A^*\mid \text{for all } 1\le j<|w|,\ w[j]\le w[j+1]\}.$$

Alternative with an algebra cross/3.

We define

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(a) **Vertices**: All finite sets of words $2^{(A^*)}$.

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First type (Reduction of inversions) $\alpha = (\{ubav\}, \varphi_\alpha, \{uabv, ubv\})$ with $a < b$ and

$$\varphi_\alpha \quad : \quad x_u \otimes x_b \otimes x_a \otimes x_v \; \rightarrow \; x_u \otimes \tau(x_b \otimes x_a) \otimes x_v \qquad (27)$$

where τ is the "twist" of the an algebra cross. We have

$$\tau(\mathcal{A}_b^+ \otimes \mathcal{A}_a^+) \subset \mathcal{A}_a^+ \otimes \mathcal{A}_b^+ + 1_{\mathbf{k}} \otimes \mathcal{A}_b^+ + \mathcal{A}_a^+ \otimes 1_{\mathbf{k}} + 1_{\mathbf{k}} \otimes 1_{\mathbf{k}} \qquad (28)$$

Alternative with an algebra cross/4.

1. therefore the result of the preceding reduction process belongs to $T_{uabv}(M) \oplus T_{ubv}(M) \oplus T_{uav}(M) \oplus T_{uv}(M)$.
   Second type (Reduction of powers) $\alpha = (\{ua^pv\}, \varphi_\alpha, \{uav\})$ with $p \geq 2$, by

$$\varphi_\alpha \quad : \quad x_u \otimes \overbrace{x_a^{(1)} \otimes \cdots \otimes x_a^{(p)}}^{p \text{ factors in } \mathcal{U}_+(\mathfrak{g}_a)} \otimes x_v \quad \to \quad x_u \otimes \underbrace{x_a^{(1)} \cdots x_a^{(p)}}_{\text{multiplication}} \otimes x_v \tag{29}$$

   the result of this reduction process is in $T_{uav}(M)$.
   Third type (Loops) $\alpha = (\{w\}, \varphi_\alpha, \{w\})$ for $w \in SI(A)$ with $\varphi_\alpha = \mathrm{Id}_{T_w}$.
   All the preceding (linear) maps φ_α (of first, second and third types) are extended by 0 outside of their definition domains ($T_{ubav}(M)$ for the first type $T_{ua^pv}(M)$ for the second and $T_w(M)$, $w \in SI(A)$ for the third).

Alternative with an algebra cross/5.

2. Summarizing, all φ_α belong to $\mathrm{End}(T(M))$.

(c) **General arrows** i.e. all arrows of Γ_{trans}. Their set is denoted GA. It is the set of triplets (F_1, Φ, F_2), with $F_i \in 2^{(A^*)}$, $\Phi \in 2^{(ES)}$ (finite sets of elementary steps) such that

   1. for all $w \in F_1$ exists one and only one elementary step in $\alpha \in \Phi$ with $t(\alpha) = \{w\}$ (its tail).
   2. $F_2 = \cup_{\alpha\in\Phi} h(\alpha)$ (union of their heads).

(d) **Tail and Head**: For every general arrow $\alpha = (F_1, \Phi, F_2)$, we set $t(\alpha) = F_1$ and $h(\alpha) = F_2$. This definition is extended for elementary arrows by (for $\alpha = (F_1, \varphi, F_2)$) the same projections (i.e. $t(\alpha) = F_1$ and $h(\alpha) = F_2$).

Alternative with an algebra cross/6.

(e) **Composition of Arrows**: Composition of (F_1, Φ_1, F_2) and (F_2, Φ_2, F_3) is $(F_1, \Phi_2 \circ \Phi_1, F_3)$ where

$$\Phi_2 \circ \Phi_1 = \{pr_2(\beta) \circ pr_2(\alpha) \,|\, \beta \in \Phi_2, \alpha \in \Phi_1, t(\beta) \subseteq h(\alpha)\}.$$

(f) **Paths**: A path in Γ_{trans} is a word $P = \alpha_1 \cdots \alpha_n \in GA^*$ such that, for all $j < |P|\,(= n)$, $h(\alpha_j) = t(\alpha_{j+1})$, we classically have $t(P) = t(\alpha_1)$ and $h(P) = h(\alpha_n)$. The evaluation of P, $Ev(P)$ is the composition of all the linear maps of its arrows i.e. with $P = \alpha_1 \cdots \alpha_n$,

$$Ev(P) = pr_2(\alpha_n) \circ \cdots \circ pr_2(\alpha_1) \tag{30}$$

Alternative with an algebra cross/7.

3. **Norm:** For all $w \in A^*$, we set $\mathrm{norm}(w) = |w| + Inv(w))$ (where $Inv(w) = \sharp\{(i,j) | 1 \le i < j \le |w| \text{ and } w[i] > w[j]\})$. This definition is at once extended to finite subsets of $F \subset A^*$ by $\mathrm{norm}(F) = \max_{w \in F \setminus SI(A)} \mathrm{norm}(w)$. We remark that, for all elementary arrow α of the two first types, $\mathrm{norm}(t(\alpha)) > \mathrm{norm}(h(\alpha))$ and equality is got for the third type. Hence, for any general arrow $\alpha = (F_1, \Phi, F_2)$, $\mathrm{norm}(t(\alpha)) > \mathrm{norm}(h(\alpha))$ unless $F_1 = F_2 \subset SI(A)$ in which case we have equality and all arrows of Φ are of third type.

(g) **Aperiodic paths**: An aperiodic path is a path whose last arrow has identical head and tail i.e. $\alpha_n = (F, \Phi, F)$, this entails that $F \subset SI(A)$ and that all arrows of Φ are of third type.

(h) **Remark.** – Conditions (b.i) and (b.ii) above say respectively that there is no outgoing computation fork (i.e. two different elementary steps) from one $w \in F_1$ and that F_2 is the image of F_1 through the arrows of Φ.

Convergence result

4. The preceding computation scheme converges to **proj**. Indeed,
   1. every sufficiently long path is aperiodic, more precisely
   2. A path of Γ_{calc} originating from F_1

   $$F_1 \to F_2 \to \cdots\cdots \to F_n \to F_{n+1}$$

   (with arrows (F_i, Φ_i, F_{i+1})) with $n \geq \mathrm{norm}(t(F_1))$ is aperiodic
   3. If $t \in T(M)$ and $supp(t) \subset F_1$, then the evaluation of the path applied to t has value $\mathbf{proj}(t)$.

Concluding remarks and perspectives

1. Starting with a dichotomy of the alphabet of generators $X = B + Z$, we constructed an adapted semi-direct product in the free Lie algebra $\mathcal{L}ie_{\mathbf{k}}\langle X\rangle$ (classical LE).
2. This semi-direct product is the prototype of all other semi-direct products in the sense that any semi-direct product is the homomorphic image of a (LE)
3. Iterated (LE) lead to a filtration of the alphabet which accounts for repeated semi-direct products
4. When the ideal factors are free Lie algebras, we can get normal forms in terms of words with conditions.
5. It will be interesting to extend the work done with Drinfeld-Kohno Lie algebras to other configuration spaces taking into account that (a) central filtrations provide $\mathbb{Z}$-Lie algebras (b) our procedures preserve torsion phenomena and hence cyclic direct sums with no bases.

Concluding remarks and perspectives/2

6. Remains to carefully pave the way(s) of contact points with the schools who developed noncommutative Gröbner bases [2, 12], especially in the light of Lie algebras like

$$\left(\mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle \middle/ n.\mathcal{L}ie_{\mathbf{k}}\langle B^*Z\rangle\right) \rtimes \mathcal{L}ie_{\mathbf{k}}\langle B\rangle \text{ having images like}$$
$$\mathfrak{g}/n.\mathfrak{h} = \mathfrak{h}/n.\mathfrak{h} \rtimes \mathfrak{b} \tag{31}$$

7. This can be applied to p-adic approximation, for example, with

$$\mathrm{DK}_{\mathbf{k},n} \Big/ p^r.\mathcal{L}ie_{\mathbf{k}}\langle T_n\rangle = \mathcal{L}ie_{\mathbf{k}}\langle T_n\rangle \Big/ p^r.\mathcal{L}ie_{\mathbf{k}}\langle T_n\rangle \rtimes \mathrm{DK}_{\mathbf{k},n-1} \tag{32}$$

$\mathrm{DK}_{\mathbf{k},n+1}$ can be seen as a projective limit, but none of the factors have a basis (although they have implementable normal forms).

8. Passing to enveloping algebras and then their iterated smash products helps us understand what can be iterated crossed products a model that can be deformed.

Thank you for your presence (close or remote) ...
and your attention.

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group
8. Traces and hilbertian operators
   https://hal.archives-ouvertes.fr/hal-01015295/document
9. State on a star-algebra
   https://ncatlab.org/nlab/show/state+on+a+star-algebra
10. Hilbert module
    https://ncatlab.org/nlab/show/Hilbert+module

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