17/11/2023 Introducing string field theory from a geometrical perspective
- Harold ERBIN

SFT:
- 2nd quantization of string theory
- standard QFT with: ∞ # of fields, non-polynomial,
non-locality $\propto e^{-k^2}$
- constructive: makes clear geometry, symmetries, UV finiteness
- use QFT tools: off-shell, renormalization, analyticity...
- consistency properties: background independence, Cutkosky rules, unitarity, crossing symmetry, IR finiteness

Difficulties:
- non-locality (problems with position representation)
- interactions not constructed explicitly

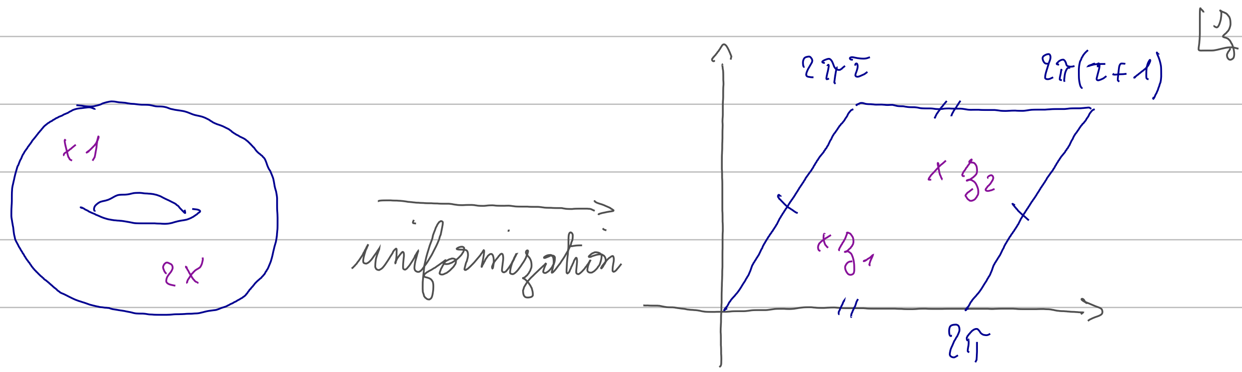
two directions with Atakan Hilmi Fırat
- machine learning [2211.09129]
- cubic theory [2308.08587]

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{Riemann surface}\\ \text{moduli spaces}\end{array} \arrow[r, "\begin{array}{c}\text{2d CFT}\\ =\text{background}\end{array}"] \arrow[d] & \begin{array}{c}\text{on-shell}\\ \text{string amplitudes}\end{array} \arrow[d] \\
\begin{array}{c}\text{Riemann surface}\\ \text{moduli spaces}\\ +\text{ local coordinates}\end{array} \arrow[r] \arrow[d, "\begin{array}{c}\text{decomposition}\\ \text{by sewing}\end{array}"'] & \begin{array}{c}\text{off-shell}\\ \text{string amplitudes}\end{array} \arrow[d, "\begin{array}{c}\text{decomposition}\\ \text{in Feynman graphs}\end{array}"'] \\
\underbrace{\begin{array}{c}\text{geometric}\\ \text{BV algebra}\end{array}}_{\begin{array}{c}\text{background}\\ \text{independent}\end{array}} \arrow[r, "\text{BV morphism}"'] & \begin{array}{c}\text{string field theory}\\ \text{(quantum BV)}\end{array}
\end{tikzcd}
$$

Goal: explain this diagram

1. Riemann surface moduli spaces

Ex: torus with two punctures, $\Sigma_{1,2}$
 = genus-1 Riemann surfaces with 2 marked points

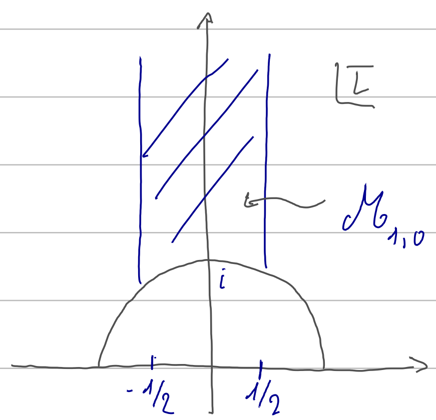


Uniformization coordinate:
$ds^2 = |dz|^2$ $\qquad z \sim z + 2\pi \sim z + 2\pi\tau$

todo: read more on torus, if $\Sigma_{1,2}$ in Ashoke

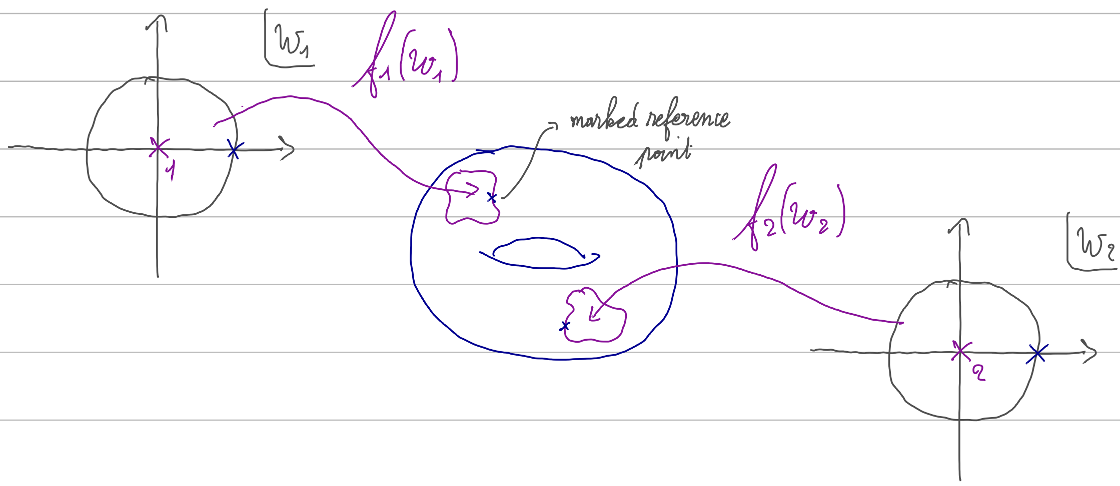
Symmetry (CKV): $U(1)$
$z \to z + \xi$
$\Rightarrow$ can fix the position of one puncture, say $z_2 = 1$

Moduli parameters $(\tau, z_1) \in \mathcal{M}_{1,2}$
 = parametrize surfaces which are topologically inequivalent



In general: moduli space $\mathcal{M}_{g,n}$ for n-punctured genus-g Riemann surfaces $\Sigma_{g,n}$

2. Riemann surface moduli spaces with local coordinates



$f_i(w_i)$: transition functions between surface coordinate patches and local coordinate patches w_i.

Puncture locations: $f_i(0) = z_i$

Motivations:

1. punctures are sources of negative curvature, so cannot work with flat metric: introduce local coordinates w_i s.t metric is locally flat $|dw_i|^2$
   (in fact, metric is $|dw_i|^2/|w_i|^2$, semi-infinite cylinder conformally equiv to $|dw_i|^2$)
2. decouple action of CKV ($SL(2,\mathbb{C})$ on sphere) from punctures:

$$f_i \longrightarrow \frac{a f_i + b}{c f_i + d} \qquad w_i \longrightarrow w_i$$

3. Geometric BV algebra

Plumbing fixture = sewing:

- pick two punctures on two surfaces $\Sigma^{(1)}_{g_1,n_1}$ and $\Sigma^{(2)}_{g_2,n_2}$
- identify coordinate patches by cutting and gluing

$$w^{(1)}_{n_1} w^{(2)}_{n_2} = q \qquad |q| \leq 1$$

new surface: $\Sigma_{g_1+g_2,\, n_1+n_2-2} := \Sigma_{g_1,n_1} \# \Sigma_{g_2,n_2}$

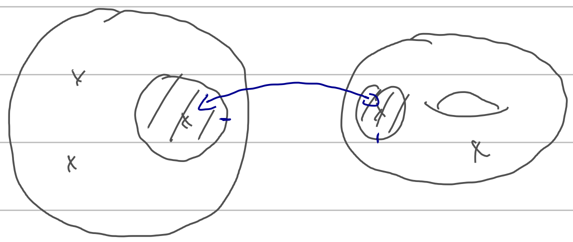
parametrization:

$q = e^{-s+i\theta}$ $\quad s \geq 0,\ \theta \in [0, 2\pi)$ serves as moduli

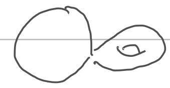
equivalent to connecting surfaces by a long tube

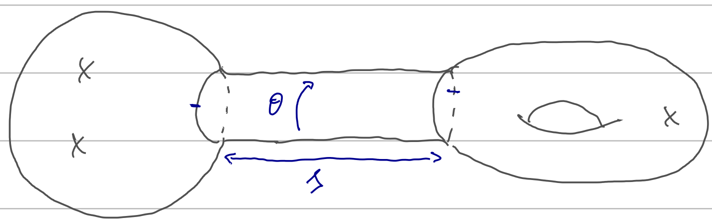
- s: length
- θ: twist

"separating degeneration" (can have non-sep. using punctures on same surface)



$s \to \infty$ limit:





But: not all surfaces can be obtained by gluing

→ decomposition of $\mathcal{M}_{g,n}$:

- $\tilde{F}_{g,n}$: Feynman region
  surfaces with tubes (close to degeneration)
- $V_{g,n}$: fundamental vertex region
  other surfaces

Note: decomposition depends on choice of local coord.

Given two subspaces $\mathcal{S}^{(1)}_{g_1,n_1}$ and $\mathcal{S}^{(2)}_{g_2,n_2}$ (separating):

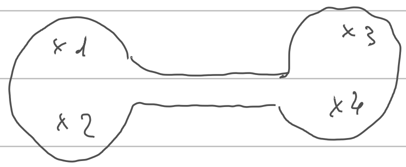
- $\mathcal{S}^{(1)}_{g_1,n_1} \# \mathcal{S}^{(2)}_{g_2,n_2}$: sewing between all surfaces in subspaces, with symmetrization over punctures
- $\{\mathcal{S}^{(1)}_{g_1,n_1}, \mathcal{S}^{(2)}_{g_2,n_2}\}$: twist sewing, same as $\#$ but keep $s=0$

Given a single subspace $\mathcal{S}_{g,n}$ (non-separating):

- $\# \mathcal{S}_{g,n}$: gluing
- $\Delta \mathcal{S}_{g,n}$: twist sewing

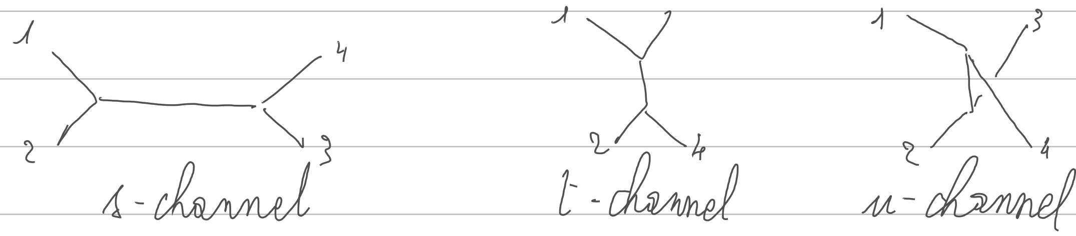
Recursive construction:

- $g=0$, $n=3$: $\mathcal{V}_{0,3} = \mathcal{M}_{0,3} = 1$
- $g=0$, $n=4$: 3 differents ways of gluing

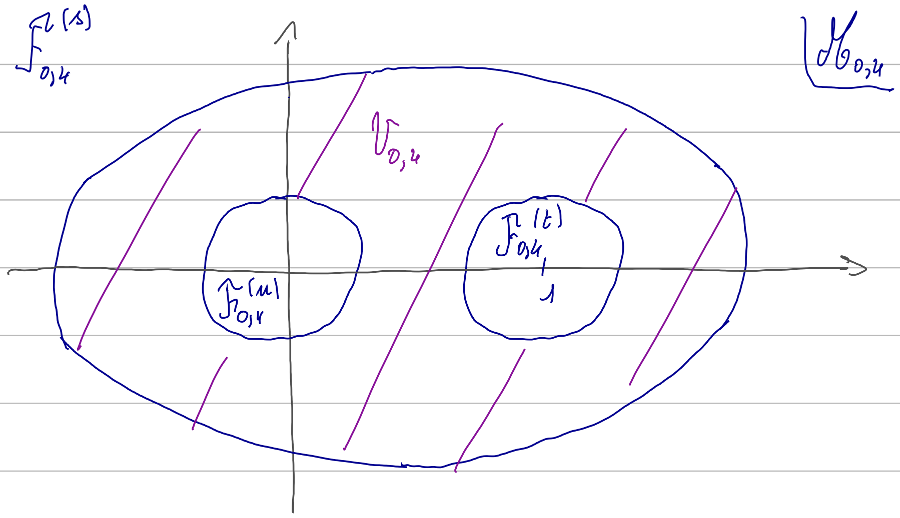


; $(2 \leftrightarrow 3)$; $(2 \leftrightarrow 4)$

$\sim$



$$\mathcal{M}_{0,4} = \mathcal{F}_{0,4} + \mathcal{V}_{0,4} \quad \text{where } \mathcal{F}_{0,4} = \mathcal{F}^{(s)}_{0,4} + \mathcal{F}^{(t)}_{0,4} + \mathcal{F}^{(u)}_{0,4}$$
$$= \mathcal{V}_{0,3} \# \mathcal{V}_{0,3}$$

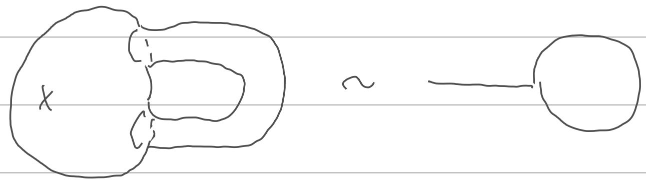


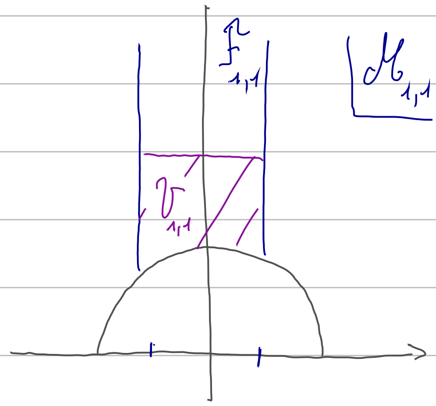
Esc: with $SL(2,\mathbb{C})$ local coordinates:

- s-boundary: ellipse
- t,u-boundaries: limaçons of Pascal

We have: $\partial V_4 + \}V_3, V_3\{ = 0$

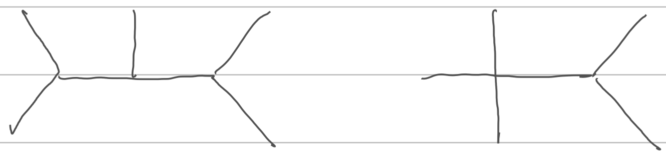
- $g=1,\ n=1$:





$$F_{1,1} = \# V_{0,3}$$

- $g=0,\ n=5$: $F_{0,5} = V_{0,3} \# V_{0,3} \# V_{0,3} + V_{0,3} \# V_{0,4}$



Define formal sum:

$$V = \sum_{g,n} V_{g,n}$$

Then V solves the geometric BV identity:

$$\partial V + \frac{1}{2}\}V, V\{ + \Delta V = 0$$

4. String theory

String theory spacetime background:

matter CFT_m	ghost CFT_{gh}	string coupling
$c_m = 26,\ h_i,\ C_{ijk}$	$c_{gh} = -26$ (universal)	g_s

[Bergman-Zwiebach, hep-th/9411047]

Note: $\{h_i, C_{ijk}\}$ constrained by: [Sonoda '88]
- $g=0,\ n=4$ crossing symmetry
- $g=1,\ n=1$ modular covariance

Define a map from moduli spaces to string amplitudes
Given a subspace $\mathcal{Y}_{g,n} \subset \mathcal{M}_{g,n}$ and n states $V_i \in \mathcal{H}_m$, define:

$$\mathcal{Y}_{g,n}(V_1,\dots,V_n) := \int_{\mathcal{Y}_{g,n}} d^{\dim \mathcal{M}_{g,n}} t \ \langle \dots \rangle_{gh} \times \left\langle \prod_{i=1}^{n} f_i \circ V_i(0) \right\rangle_m$$

Building blocks:
- ghosts: arise from gauge fixing 2d diffeo of worldsheet
  $\longrightarrow$ provides measure in $\mathcal{M}_{g,n}$
- $V_i \in \mathcal{H}_m$ (matter CFT Hilbert space): external states defined by quantum numbers (momentum, etc.)
  $\longrightarrow$ inserted at origin of local coordinate patches, $w_i = 0$
- $f_i \circ$: conformal transformation
  ex: for V_i primary (h_i, h_i):
  $$f_i \circ V_i(0) = |f_i'(0)|^{2h_i} V_i(f_i(0))$$

note: on-shell, $h_i = 0 \Rightarrow$ dependence in f_i drops

- CFT correlation functions on $\Sigma_{g,m}$: function of moduli

String theory (n-point, g-loop):

- $\mathcal{M}_{g,n}(V_1, \dots, V_n)$ with V_i: $(0,0)$-primary (on-shell) (no local coordinates)
  $\to$ on-shell amplitudes
- $\mathcal{M}_{g,n}(V_1, \dots, V_n)$ with V_i: generic
  $\to$ off-shell amplitudes
- $V_{g,n}(V_1, \dots, V_n)$ with V_i: generic
  $\to$ Feynman vertex
- $F_{g,n}(V_1, \dots, V_n)$ with V_i: generic
  $\to$ sum of Feynman graphs with propagators

5. String field theory

$\Psi \in \mathcal{H}$ general linear combination

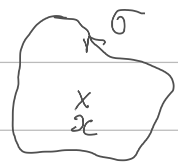
$$|\Psi\rangle = \sum_\alpha \int \frac{d^d k}{(2\pi)^d} \psi_\alpha(k)\, |k,\alpha\rangle$$

↳ basis of $\mathcal{H}$

↳ spacetime fields

Double Fourier expansion: spacetime and string extension

- k: continuous momentum
  centre of mass in spacetime
- α, discrete quantum members
  string direction σ, compact dimensions



BV quantum action:

$$S = \frac{1}{2}\langle \Psi, Q_B \Psi\rangle + \sum_{g,n} \frac{\hbar^{2g} g_s^{2g-2+n}}{n!} V_{g,n}(\Psi^{\otimes n})$$

↳ BPZ inner product

Q_B: CFT BRST operator, obtained from free eom
(physical state condition: $Q_B|\psi\rangle = 0$)

Siegel gauge $\Rightarrow$ plumbing fixture

$V_{g,n}: \mathcal{H}^{\otimes n} \to \mathbb{C}$ is multilinear, so define string products:

$$V_{g,n+1}(V_0, V_1, \dots, V_n) = \langle V_0, l_{g,n}(V_1, \dots, V_n) \rangle$$

$l_{g,1} := Q_B$

$\{l_{g,n}\}$ define quantum L_∞ algebra (from geometric BV)

$\Longrightarrow$ quantum BV equation

$$\underbrace{\frac{1}{2}\{S,S\}}_{\text{BV bracket}} + \hbar \underbrace{\Delta}_{\text{BV Laplacian}} S = 0$$

Note: also equivalent to covering moduli spaces by sum of Feynman diagrams

Esc: classical SFT ($g=0$)

eom: $$Q_B \Psi + \sum_{n \geq 2} \frac{1}{n!} l_{0,n}(\Psi^{\otimes n}) = 0$$

gauge symmetry:

$$\delta \Psi = Q_B \Lambda + \sum_{n \geq 1} \frac{1}{n!} l_{0,n+1}(\Psi^{\otimes n}, \Lambda)$$

L_∞ relations $\Longrightarrow$ $\delta S = 0$

$*$ $l_1(l_1(V_1)) = 0$

Q_B nilpotent

$*$ $l_1(l_2(V_1, V_2)) \pm l_2(l_1(V_1), V_2) \pm$ perms $= 0$

Q_B is a derivative for l_2

$*$ $l_1(l_3(V_1, V_2, V_3)) \pm l_3(l_1(V_1), V_2, V_3) \pm$ perms

$$= l_2(l_2(V_1, V_2), V_3) \pm \text{perms}$$

failure of Q_B being derivative of l_3 = failure of Jacobi identity

6. Machine learning for SFT

with: Atakan Hilmi Fırat

Summary: for SFT vertices, we need:
- vertex regions $\mathcal{V}_{g,n}$
- local coordinates $\{f_i^{(g,n)}\}_{i=1,\dots,n}$

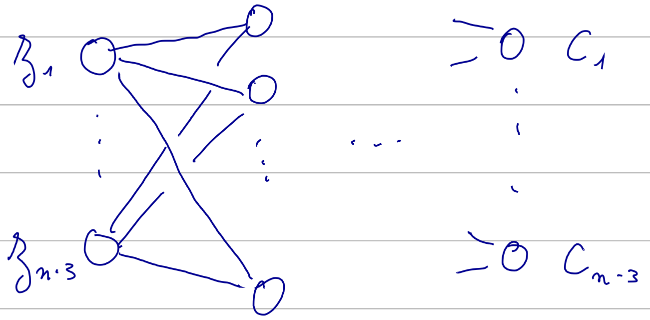
- Difficult to find set consistent with geometric BV
- Background-independent problem

Idea: use deep learning to build $\mathcal{V}_{g,n}$ and $f_i^{(g,n)}$

Solution to classical BV: contact vertices, characterized by Strebel differentials
(proof not constructive)

f_i and $\mathcal{V}_{g,n}$ are determined by set of moduli-dependent "accessory parameters" (limit from hyperbolic)
$\{c_1, \dots, c_{n-3}\}$
determined by solving complicated constraints $\mathcal{L}_{ij} = 0$
(unique solution)

Neural network:



Objective function: $\sum_{i,j} |\mathcal{L}_{i,j}|^2 = 0$

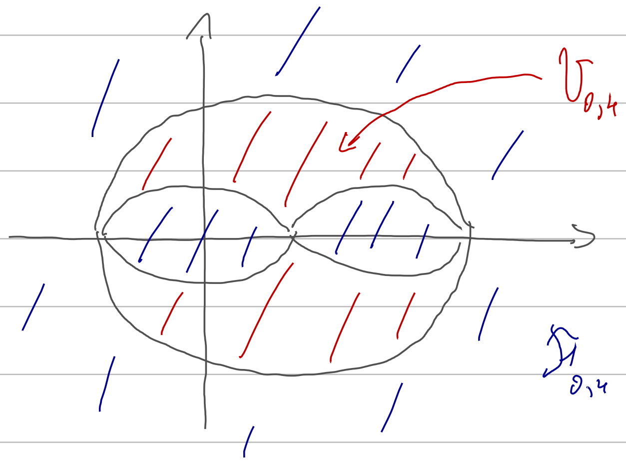
Ex: $g = 0$, $n = 4$: precision of 10^{-4} for c_1

$V_{g,n}$: determined by neural network with binary classification

$$\Theta(\xi_1, \dots, \xi_{n-3}) = \begin{cases} 1 & \xi_i \in V_{0,n} \\ 0 & \xi_i \notin V_{0,n} \end{cases}$$

network : outputs probability

$\longrightarrow$ boundary determined by confusion



Conclusion : deep learning is well-suited to solve geometrical problems.

Offshoot (Atakan) : new insights on vertices using hyperbolic geometry and Liouville theory

Backup

Strebel differential : $\varphi = \phi(z)\, dz^2$

$$\phi(z) = \sum_{i=1}^{n} \left[\frac{-1}{(z-\xi_i)^2} + \frac{c_i}{z-\xi_i} \right]$$

↳ punctures

c_i : "accessory" parameters (limit from Liouville acc. param.)

Metric :
$$ds = |\phi(z)|^2 |dz|^2 \sim \frac{|dw_i|^2}{|w_i|^2}$$

↳ semi-infinite cylinder

zeros : $\{ z_i \mid \phi(z_i) = 0 \}$

Horizontal trajectory : $\varphi > 0$

Critical graph : hor. traj. between zeros

Constraints :
$$\operatorname{Im} \int_{z_i}^{z_j} dz \sqrt{\phi(z)} = 0 \qquad \forall (z_i, z_j)$$

$\rightarrow$ system of equations for $\{c_i\}$