Combinatorics and quantum invariant differential operators on Reflection Equation algebras

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First, I want to explain why from my viewpoint RE algebras are objects of special interest. The point is that these algebras are far going generalizations of the enveloping algebras $U(gl(N))$, they have many properties similar to these of the algebras $U(gl(N))$ (in particular, the representation theory). However, the trace entering all corresponding constructions should be different from the classical one and it should be coordinated with the braiding, which takes a part in the definition of the RE algebras.
A simplest example of such a trace is the super-trace, which is coordinated with super-flip

$$P_{m|n}(x \otimes y) = (-1)^{\overline{x}\,\overline{y}} y \otimes x,$$

where $\overline{x}$ stands for the parity of an element $x \in V$. The component V_0 is even and that V_1 is odd, $dim V_0 = m, dim V_1 = n$.

Let V be a vector space over the field $\mathbb{C}$. We say that a linear invertible operator $R: V^{\otimes 2} \to V^{\otimes 2}$ is a braiding, if it is subject to the following relation in $V^{\otimes 3}$

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

where $I: V \to V$ is the identity operator.
Particular examples are the usual flip $R = P$, which acts as follows $P(x \otimes y) = y \otimes x$ for any $x, y \in V$, and super-flips. They are subject to the relation $R^2 = I$. Such braidings are called involutive symmetries.
The braidings subject to the Hecke condition

$$(q\, I - R)(q^{-1}\, I + R) = 0,\ \ q \in \mathbb{C},\ \ q \neq 0,\ \ q \neq \pm 1$$

are called Hecke symmetries.
For any braiding R we denote $R_k : V^{\otimes p} \to V^{\otimes p},\ \ k = 1, ...p-1$ the operator R acting on the components numbers k and $k+1$. Thus,

$$R_k = I_{1...k-1} \otimes R \otimes I_{k+2...p}.$$

Observe that the braidings and Hecke symmetries are in fact representations of the braid groups and Hecke algebras. We call these representations R-matrix ones.
Recall that the Artin braid group B_N is the group generated by the unit e and $N-1$ invertible elements

$$\tau_1, \dots, \tau_{N-1}$$

subject to the following relations

$$\tau_i\,\tau_j = \tau_j\,\tau_i \text{ if } |i-j|\geq 2 \text{ and } \tau_i\,\tau_{i+1}\,\tau_i = \tau_{i+1}\,\tau_i\,\tau_{i+1},\;\; i\leq N-2.$$

The last relation is called braid one.
If we impose the following relation

$$(\tau_i - q\,e)(\tau_i + q^{-1}\,e) = 0,\;\; \forall i,\;\;\; q\in\mathbb{C},$$

we get an algebra $H_N(q)$ called Hecke (or Iwahori-Hecke) algebra.
Observe that for $q=\pm 1$ we get the algebra $\mathbb{C}[\mathbb{S}_N]$.
We deal with q generic: $q\notin\{0,\pm 1\}$ and q is not a root of unity.

By fixing in the space V a basis $\{x_1 \ldots x_N\}$ and the corresponding basis $\{x_i \otimes x_j\}$ in the space $V^{\otimes 2}$ we can represent the operators R_i by matrices. Let us exhibit two examples of Hecke symmetries

$$\left(\begin{array}{cccc} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{array}\right), \quad \left(\begin{array}{cccc} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{array}\right).$$

The first matrix tends to the matrix of the flip P as $q \to 1$. The second one tends to the super-flip $P_{1|1}$.

Note that there exist involutive and Hecke symmetries which are deformations neither of the usual flips nor of the super-flips. So, the following problem is of interest; what involutive and Hecke symmetries could be. Consider two related algebras: "R-symmetric" and "R-skew-symmetric" ones

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \;\; \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle,$$

where $T(V) = \bigoplus V^{\otimes k}$ is the free tensor algebra of V.
Also, consider the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \;\; P_-(t) = \sum_k dim \bigwedge\nolimits_R^{(k)}(V)t^k,$$

where the upper index (k) labels the homogenous components.
If R is involutive, we put $q = 1$ in the above formulae.

Examples. If R is a deformation of the usual flip P and $dimV = N$, then

$$P_-(t) = (1+t)^N.$$

If R is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $N \geq 2$ there exit involutive and Hecke symmetries such that

$$P_-(t) = 1 + Nt + t^2.$$

Here $dimV = N$.
If $P_-(t)$ is a polynomial, R is called *even*.
In general, $P_-(t)$ is a rational function.
By assuming it to be uncancellable we call the couple $(r|s)$ the bi-rank of R, where r (resp., s) is the degree of the numerator (resp., denominator) of this function.

Note that the generators of the algebras $Sym_R(V)$ and $\bigwedge_R(V)$ play the role of generalized bosons and fermions respectively. They can be used in order to get bosonization and fermionization of the RE algebras.

First, consider the classical case. Let l_i^j be a generator of the algebra $U(gl(N))$. Then its bosonization is $l_i^j = x_i \otimes x^j$, where x_i is a generator of the space V and x^j is a generator of the dual space V^*. Assume that $< x_i, x^j >= \delta_i^j$, i.e. the basis $\{x^j\}$ is the right dual to that $\{x_i\}$. However, the pairing in the opposite order $< x^j, x_i >$ is completely different. Denote it B_i^j. The $N \times N$ matrix (B_i^j) is completely determined by R.

In the classical case, as $R = P$, $B = I$. In the super-case, as $R = P_{m|n}$ the matrix B is $diag(1, 1, 1... - 1, -1)$. In general, B has not to be diagonal.

Note a very important property of the matrix B

$$R\, B_1\, B_2 = B_1\, B_2\, R.$$

Now, by assuming R to be an involutive symmetry or a Hecke symmetries, we introduce the corresponding RE algebra. Let $L = \|l_i^j\|$ be a $N \times N$ matrix. Consider the relation

$$RL_1RL_1 - L_1RL_1R = 0,\ L = (l_i^j),\ 1 \le i,j \le N,$$

where $L_1 = L \otimes I$. The algebra generated by the unity and the entries l_i^j, subject to the above relation, is called Reflection Equation (RE) one and denoted $\mathcal{L}(R)$.
If a $N \times N$ matrix $\hat{L} = \|\hat{l}_i^j\|$ is subject to

$$R\hat{L}_1R\hat{L}_1 - \hat{L}_1R\hat{L}_1R = R\hat{L}_1 - \hat{L}_1R,\ \hat{L} = (\hat{l}_i^j),\ 1 \le i,j \le N,$$

the algebra generated by the unity and the entries $\hat{l}_i^j$ is called modified RE algebra and denoted $\hat{\mathcal{L}}(R)$.

Note that these algebras are isomorphic to each other. Their isomorphism can be realised via the following relations between the generating matrices

$$L = I - (q - q^{-1})\,\hat{L}.$$

Observe that this isomorphism fails if $q = \pm 1$.
Nevertheless, if $R \to P$ as $q \to 1$, the RE algebras $\mathcal{L}(R)$ tends to the algebra $Sym(gl(N))$, whereas the modified RE algebra $\hat{\mathcal{L}}(R)$ tends to that $U(gl(N)))$.
However, in spite of the fact that the RE algebras $\mathcal{L}(R)$ tends to the algebra $Sym(gl(N))$, its properties are more similar to these of the enveloping algebra $U(gl(N))$.

Note that the algebra $Sym(gl(N))$ can be presented in the matrix form

$$PL_1PL_1 - L_1PL_1P = 0, \;\; L = (l_i^j), \;\; 1 \le i,j \le N,$$

whereas the algebra $U(gl(N)))$ can be presented by the system

$$P\hat{L}_1P\hat{L}_1 - \hat{L}_1P\hat{L}_1P = P\hat{L}_1 - \hat{L}_1P, \;\; \hat{L} = (\hat{l}_i^j), \;\; 1 \le i,j \le N,$$

Our next aim is to describe the center of the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ and to introduce analogs of the symmetric polynomials on them.

Observe that the center of the algebra $U(gl(N))$ is generated by the elements $Tr\hat{L}^k$, which are called power sums.
In a similar manner we introduce *power sums* in the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ but the trace must be *quantum*.
Let us introduce the so-called R-trace of matrices

$$Tr_R M = Tr C\, M,$$

where the matrix $C = (C_i^j)$ is completely defined by a given braiding R (it can be defined for all braidings, which are "skew-invertible").
Then the power sums in the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ are defined as follows

$$p_k(L) = Tr_R\, L^k = Tr\, C\, L^k,\;\; p_k(\hat{L}) = Tr_R\, \hat{L}^k = Tr\, C\, \hat{L}^k.$$

These elements are central in the corresponding algebras.
Note that the matrix C is almost inverse to that B, introduced above. Namely, we have

$$B\, C = q^{-2\,(r-s)}\, I.$$

Now, we want to describe other elements of the center of the algebra $\mathcal{L}(R)$.
Below, we use the following notations

$$L_{\overline{1}} = L_1,\; L_{\overline{2}} = R_{12}\, L_{\overline{1}}\, R_{12}^{-1},\; L_{\overline{3}} = R_{23}\, L_{\overline{2}}\, R_{23}^{-1} = R_{23}\, R_{12}\, L_{\overline{1}}\, R_{12}^{-1}\, R_{23}^{-1}, \dots$$

Also, we use the notation

$$L_{\overline{1\to k}} = L_{\overline{1}}\, L_{\overline{2}} \dots L_{\overline{k}}.$$

Example. If $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$L_{\overline{1}} = L_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix},\; L_{\overline{2}} = R \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} R^{-1}.$$

The following claim plays the main role in describing the center of the RE algebras (Isaev-Pyatov).

Proposition.

Let $z \in H_k(q)$ be an arbitrary element. Then the element

$$ch(z) := Tr_{R(1\dots k)}\, \rho_R(z) L_{\overline{1\to k}}$$

is central in the algebra $\mathcal{L}(R)$.

Here $\rho_R(z)$ is R-matrix representation of the Hecke algebra

$$\rho_R(\tau_k) = R_k.$$

The map

$$ch : H_k(q) \to \mathcal{L}(R), \;\; z \mapsto ch(z)$$

is called characteristic. The image of the characteristic map is called characteristic subalgebra of $\mathcal{L}(R)$. Hopefully, it coincides with the center of the RE algebra.

Observe that the power sums $p_k(L) = Tr_R L^k$ in the algebra $\mathcal{L}(R)$ can be written in this form.
Also, in this manner we introduce the Schur polynomials (functions) in this algebra $\mathcal{L}(R)$. In order to introduce them we put $z = E_{ii}^{\lambda}$, where E_{ii}^{λ} some idempotents belonging to the Hecke algebra. Thus, we put

$$s_\lambda(L) = Tr_{R(1\dots k)}\, \rho_R(z) L_{\overline{1\to k}}, \quad z = E_{ii}^{\lambda}.$$

Now, we exhibit the classical Frobenius formula and its "braided analog".

The classical Frobenius formula is

$$p_\nu(L) = \sum_{\lambda \vdash k} \chi_\nu^\lambda s_\lambda(L),$$

where $\nu = (\nu_1, \nu_2 ... \nu_k)$ and $\lambda = (\lambda_1 ... \lambda_k)$ are partitions, $p_\nu(L) = p_{\nu_1} ... p_{\nu_k}$ are the corresponding power sums, and $s_\lambda(L)$ are the Schur polynomials. (Here L is the generating matrix of the algebra $Sym(gl(N))$.) Also, χ_ν^λ is is the character of the symmetric groups in the representation, labeled by λ, evaluated on the element whose cyclic type is $\nu = (\nu_1 ... \nu_N)$.
The q-analog of the Frobenius formula looks like the classical one but L becomes the generating matrix of the algebra $\mathcal{L}(R)$ and χ_ν^λ is the character of the representation of the Hecke algebra.

Note that a similar formula has been obtained by Arum Ram but only in the case related to the QG $U_q(sl(N))$. In general the main problem is how to correctly define the symmetric function related to RE algebras.

Note that our definition of all symmetric functions is somewhat done in the Miwa way, i.e. via a matrix.
Now, we want to represent them also analogically to the Miwa approach via the "eigenvalues" of this matrix. Now, we introduce "eigenvalues" of the matrices L, generating the RE algebras $\mathcal{L}(R)$.

Observe that in the algebra $\mathcal{L}(R)$ there are analogs of the Newton identities

$$p_k - q p_{k-1}\, e_1 + (-q)^2 p_{k-2}\, e_2 + \ldots + (-q)^{k-1} p_1\, e_k + (-1)^k k_q e_k = 0,$$

$k = 1.2\ldots$ and the Cayley-Hamilton identity

$$L^m - q\, L^{m-1}\, e_1 + (-q)^2 L^{m-2}\, e_2 + \ldots + (-q)^{m-1} L\, e_{m-1} + (-q)^m\, I\, e_m = 0,$$

provided R is even of bi-rank $(m|0)$.
Here, $e_k = e_k(L)$ are elementary symmetric polynomials (functions), i.e. particular cases of the Schur polynomials, respective to one-column diagrams. Recall that all these elements are central in the RE algebra.

The Newton identities enable us to explicitly express the elementary symmetric functions via the power sums and vice versa:

$$p_k = det \begin{pmatrix} 1_q\, e_1 & 1 & 0 & \dots & 0 \\ 2_q\, e_2 & q\, e_1 & 1 & \dots & 0 \\ 3_q\, e_3 & q^2\, e_2 & q\, e_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ k_q\, e_k & q^{k-1}\, e_{k-1} & q^{k-2}\, e_{k-2} & \dots & q\, e_1 \end{pmatrix},$$

$$k_q!\, e_k = det \begin{pmatrix} p_1 & 1_q & 0 & \dots & 0 \\ p_2 & q\, p_1 & 2_q & \dots & 0 \\ p_3 & q\, p_2 & q^2\, p_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & (k-1)_q \\ p_k & q\, p_{k-1} & q^2\, p_{k-2} & \dots & q^{k-1}\, p_1 \end{pmatrix}.$$

In a similar manner it is possible to get the following formulae

$$p_k = (-1)^{k-1}\, det \begin{pmatrix} 1_q\, h_1 & 1 & 0 & \dots & 0 \\ 2_q\, h_2 & q^{-1}\, h_1 & 1 & \dots & 0 \\ 3_q\, h_3 & q^{-2}\, h_2 & q^{-1}\, h_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ k_q\, h_k & q^{-(k-1)}\, h_{k-1} & q^{-(k-2)}\, h_{k-2} & \dots & q^{-1}\, h_1 \end{pmatrix}.$$

$$k_q!\, h_k = det \begin{pmatrix} p_1 & -1_q & 0 & \dots & 0 \\ p_2 & q^{-1}\, p_1 & -2_q & \dots & 0 \\ p_3 & q^{-1}\, p_2 & q^{-2}, p_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & -(k-1)_q \\ p_k & q^{-1}\, p_{k-1} & q^{-2}\, p_{k-2} & \dots & q^{-(k-1)}\, p_1 \end{pmatrix}.$$

Let us observe that the modified RE algebra $\hat{\mathcal{L}}(R)$ can be treated as the enveloping algebra of a "braided Lie algebra" $gl(V_R)$. Moreover, this algebra possesses a pairing which can be deduced from the bosonization, described above

$$< l_i^j, l_k^l >=< x_i \otimes x^j, x_k \otimes x^l >= B_k^j \, \delta_i^l.$$

Thus, it is reasonable to use these algebras in order to define analogs of Vassiliev invariant of knots in the spirit of the Kontsevich approach. Lately, this approach was treated in terms of weight systems corresponding to transpositions from the symmetric group. Namely, for any $\sigma \in S_n$ one assigns the following element

$$h_{gl(n)}(\sigma) = \sum_{i_1 \, i_2 \ldots i_n} l_{i_1}^{i_{\sigma(1)}} \, l_{i_2}^{i_{\sigma(2)}} \ldots l_{i_n}^{i_{\sigma(n)}}.$$

Here, l_i^j are the standard generators of the Lie algebra $gl(N)$.

The element $h_{gl(n)}(\sigma)$ belongs to the center $Z(U(gl(N))$. One usually applies the Harish-Chandra map to this element. The final result is a polynomial $P(l_1^1 ... l_n^n)$ in the elements of the Cartan subalgebra.

Recently Zhuoke Yang generalized this scheme to the super-algebras $U(gl(m|n))$.

It is natural to use a similar scheme by using the RE algebras instead of $U(gl(m|n))$. This attempt was undertaken by A.Vaintrob in the 90's but he only used algebras related to involutive symmetries. Besides, it was not clear what is an analog of the HC map.

Now, I explain how it is possible to proceed by using the RE algebras and what could be an analog of the HC map.

Let us explain this scheme on an example. We set $\sigma = (1 \to 3, 2 \to 2, 3 \to 1)$. It is possible to write down this element as

$$Tr_{123}(\hat{L}_1\,\hat{L}_2\,\hat{L}_3\,P_1\,P_2\,P_1),$$

where $\hat{L}$ is the generating matrix of the algebra $U(gl(N))$. This element is central in the algebra $U(gl(N))$.

By passing to the braided case we replace the above element by this

$$Tr_{R(123)}(\hat{L}_1\,\hat{L}_2\,\hat{L}_3\,R_1\,R_2\,R_1),$$

where $\hat{L}$ is the generating matrix of the modified RE algebra.

The result of the computation, corresponding to the above permutation is

$$\langle L^2\rangle\,\langle L\rangle - (q-q^{-1})\,\langle L^3\rangle,$$

where $\langle X\rangle = Tr_R X$.

However, it is not clear whether a relation similar to the 4-term relation is valid in this case.

Now, I want to discuss what analog of the HC map could be.
Let $\mu_i,\ i = 1...m$ be indeterminates meeting the following system

$$\sum_i \mu_i = e_q(L),\ \sum_{i<j} \mu_i\,\mu_j = q^2\,e_2(L),\ \dots, \mu_1\,\mu_2...\mu_m = q^m\,e_m(L).$$

The elements in the r.h.s. are coefficients of the CH identity. We call μ_i (quantum) "eigenvalues" of the matrix L. These indeterminates are assumed to be central in the algebra $\mathcal{L}(R)[\mu_1...\mu_m]$.
If R is not even and its bi-rank is $(m|n)$, then the eigenvalues can be defined in a similar way. In this case we have two families of them $\mu_1, ..., \mu_m$ (even eigenvalues) and $\nu_1, ..., \nu_n$ (odd eigenvalues) such that any symmetric polynomial can be expressed via these quantities.

For the power sums we have the following parametrization via the "eigenvalues" μ_i and ν_i:

$$p_k(L) = Tr_R\, L^k = \sum_i^m \mu_i^k\, d_i + \sum_j^n \nu_j^k\, \tilde{d}_j,$$

$$d_i = q^{-1} \prod_{p=1,p\neq i}^{m} \frac{\mu_i - q^{-2}\mu_p}{\mu_i - \mu_p} \prod_{j=1}^{n} \frac{\mu_i - q^2\nu_j}{\mu_i - \nu_j},$$

$$\tilde{d}_j = -q \prod_{i=1}^{m} \frac{\nu_j - q^{-2}\mu_i}{\nu_j - \mu_i} \prod_{p=1,\, p\neq j}^{n} \frac{\nu_j - q^2\nu_p}{\nu_j - \nu_p},$$

In the limit $q = 1$ we get the formula, corresponding to the involutive symmetry R

$$p_k(L) = \sum_i^m \mu_i^k - \sum_j^n \nu_j^k.$$

In the even case these polynomials coincide with the Hall-Littlewood polynomials up to numerical factor q^{-1} and identification $t = q^{-2}$.

I want to observe that the HL polynomials appear in the integrable system theory, but for a special value of the parameter $t = -1$. Namely, they come to the definition of the so-called Q-Schur functions, which are useful for defining the BKP hierarchy. However, for constructing this hierarchy one uses the usual partial derivatives.

Whereas, for a generic t (in our notation for a generic q) the notion of the partial derivatives in the generators l_i^j of the RE algebras must be crucially modified.

I do not here describe a way of introducing such "quantum partial derivatives". I want only to observe that they are introduced via the so-called Quantum Doubles, which generalize the notion of the Heisenberg algebras.

Observe that the polynomials $p_k(L)$ are super-symmetric in $q^{-1}\,\mu_i$ and $q\,\nu_j$.

Recall that by definition a polynomial in two sets of indeterminates μ_i and ν_j is called super-symmetric if it is symmetric in μ_i and ν_j separately and the polynomial in which one puts $\mu_1=\nu_1=s$ does not depend on s.

Consider the even case in more details. It is known that the full symmetric functions and the power sums are related by the Cauchy formula

$$exp(\sum_{k=1}\frac{p_k}{k}z^k)=\sum_{k=0}h_k z^k.$$

As for other Schur functions, they can be expressed via the full symmetric elements h_k by means of the Jacobi-Trudi formulae

$$s_\lambda=det(h_{\lambda_i}-i+j).$$

In our "braided setting" the above Jacobi-Trudi formulae are still valid. In general, the classical Littlewood-Richardson rule with usual coefficients is still valid. However, the Cauchy formula is not valid any more.
Instead of the Cauchy formula the following ones are valid

$$P(-z)\,E(q\,z) = E(q^{-1}\,z), \;\; P(z)\,H(q^{-1}\,z) = H(q\,z),$$

where

$$P(z) = 1 + (q - q^{-1})\sum_{k=1} p_k(L)\,z^k,$$

$$E(z) = \sum_{k=0} e_k(L)\,z^k, \;\; H(z) = \sum_{k=0} h_k(L)\,z^k.$$

Besides, the relations between the quantum Schur functions and the quantum power sums can be found via the determinant formulae presented above and the q-Frobenius formula.

Also, I'll introduce an analog of the so-called cut-and-join operators related to the RE algebras. In the classical setting such operators play a very important role in constructing τ-functions of different models. Thus, the τ-functions can be constructed from the simplest τ-function via applying the cut-and-join operators. Observe that in the classical setting there exist different ways to introduce these operators. One of them can be generalised to the braided case.

As we noticed above, there exists a way to introduce analogs of the partial derivatives $\partial_i^j = \partial_{l_j^i}$ in the generators l_i^j of a given RE algebra $\mathcal{L}(R)$. We constitute the matrix D with entries ∂_i^j.
Note that in the classical setting the matrix

$$\hat{L} = L\,D$$

generates the enveloping algebra $U(gl(N))$. Moreover, its generators are represented by the Euler vector fields:

$$\hat{l}_i^j = l_i^k\,\partial_k^j.$$

By passing to the braided case we have the following

Theorem.

Let $L = \|l_i^j\|_{1\le i,j\le N}$ be the generating matrix of an algebra $\mathcal{L}(R)$ and $D = \|\partial_i^j\|_{1\le i,j\le N}$ be the matrix composed from the partial derivatives. Then the matrix

$$\hat{L} = L\, D$$

generates the modified RE algebra.

We consider two classes of quantum invariant differential operators: Laplace operators $Tr_R L^k$ and Casimir operators $Tr_R \hat{L}^k$. Besides, it is possible to define the normal ordering of the latter operators (and their products), for instance

$$W^{k_1 \dots k_p} =: Tr_R \hat{L}^{k_1} \dots Tr_R \hat{L}^{k_p} : .$$

This normal product is defined via the initial "permutation relations" between the generators ∂_i^j and l_i^j with omitting the "constant terms". In the classical case the initial "permutation relations" are $\partial_i^j \, l_k^m = l_k^m \, \partial_i^j + \delta_i^m \, \delta_k^j$ whereas the normal ordering is defined via relations $\partial_i^j \, l_k^m = l_k^m \, \partial_i^j$, i.e. without terms $\delta_i^m \, \delta_k^j$.
Observe that in the classical setting namely the operators $W^{k_1 \dots k_p}$ properly normalized are called cut-and-join ones.
We call them (normally) ordered Casimir operators.

Observe that in the braided case all quantum invariant differential operators (Laplaces, Casimirs, ordered Casimirs) possess the natural property: while acting on the algebra $\mathcal{L}(R)$, they preserve its characteristic subalgebra (hopefully, coinciding with the center $Z(\mathcal{L}(R))$).

By studding the ordered Casimirs we obtained (as a byproduct) the following quantum matrix Capelli identity

$$(L_1 - \mathcal{P}_1)(L_{\overline{2}} - \mathcal{P}_2)...(L_{\overline{k}} - \mathcal{P}_k) = M_1...M_{\overline{k}}\, D_{\overline{k}}...D_1 \prod_{s=1}^{k} J_s^{-1},$$

where

$$\mathcal{P}_1 = I,\ \mathcal{P}_{k+1} = \frac{I - J_{k+1}^{-1}}{q - q^{-1}},\ k \geq 1$$

and J_s are the Jucys-Murphy elements in the R-matrix representation (they are distinguished elements in the Hecke algebra, generating a commutative subalgebra).

This formula is a quantum analog of an Okounkov's result.

Now, we exhibit an application more of the RE algebras. First, recall a result by Perelomov-Popov. In the enveloping algebra $U(gl(m))$ we consider the power sums TrL^k. By representing them in any finite dimensional space, we get the Casimir operators denoted now Cas_k.

It is known that all irreducible finite dimensional reps of $U(gl(m))$ are labeled by the partitions $\lambda = (\lambda_i, ..., \lambda_m)$. The Casimir operator represented in such a module is a scalar operator $\chi_\lambda(Cas_k)\, I$.

For any central element $G \in U(gl(m))$ the quantity $\chi_\lambda(G)$ (which is defined in the same manner) is called the character of G and is denoted $\chi_\lambda(G)$.
As shown by Perelomov-Popov the eigenvalue of the operator Cas_k in the irreducible module, labelled by $\lambda = (\lambda_i, ..., \lambda_m)$, equals

$$Cas_k = \sum_i^k \hat{\mu}_i^k\, d_i, \quad d_i = \prod_{p=1, p\neq i}^m \frac{\hat{\mu}_i - \hat{\mu}_p - 1}{\hat{\mu}_i - \hat{\mu}_p},$$

where $\hat{\mu}_i = \lambda_i + m - i$.

In fact, this result can be formulated in two steps. First, consider the CH identity for the generating matrix $\hat{L}$ of the enveloping algebra $U(gl(m))$

$$\hat{L}^m - a_{m-1}\,\hat{L}^{m-1} + ... + (-1)^m\,a_0\,I = 0.$$

The coefficients are central elements of $U(gl(m))$. Introduce eigenvalues $\hat{\mu}_i$ of L as explained above. They are assumed to be central in the extended algebra $U(gl(m))[\hat{\mu}_1, ..., \hat{\mu}_m]$.

We assign the character χ_λ to these eigenvalues by setting

$$\chi_\lambda(\hat{\mu}_i) = \lambda_i + m - i.$$

Thus, the PP formula can be expressed as follows

$$\chi_\lambda(Cas_k) = \sum_i^k \chi_\lambda(\hat{\mu}_i)^k \prod_{p=1, p\neq i}^m \frac{\chi_\lambda(\hat{\mu}_i) - \chi_\lambda(\hat{\mu}_p) - 1}{\chi_\lambda(\hat{\mu}_i) - \chi_\lambda(\hat{\mu}_p)}.$$

Our next aim is to get similar characters for eigenvalues μ_i of the generating matrix L of the RE algebra by assuming the initial Hecke symmetry R to be even.

In this case the Casimir elements are expressed via the eigenvalues as follows

$$Cas_k = p_k(L) = q^{-1} \sum_i^m \mu_i^k \prod_{p=1, p\neq i}^m \frac{\mu_i - q^{-2}\mu_p}{\mu_i - \mu_p}.$$

Proposition.

By assuming that the characters of the eigenvalues μ_i in the representations labeled by λ are

$$\chi_\lambda(\mu_i) = q^{-(\lambda_i + m - i)},$$

we get the characters of the Casimir elements Cas_k.

Now, by using the relation $L = I - (q - q^{-1})\,\hat{L}$, we passe to the modified RE algebra. The Casimir elements in this algebra are expressed via the eigenvalues $\hat{\mu}_i$ as follows

$$Cas_k = p_k(\hat{L}) = q^{-1} \sum_{i}^{m} \hat{\mu}_i^k \prod_{p=1,p\neq i}^{m} \frac{\hat{\mu}_i - q^{-2}\hat{\mu}_p - q^{-1}}{\hat{\mu}_i - \hat{\mu}_p}.$$

By setting

$$\chi_\lambda(\hat{\mu}_i) = q^{-(\lambda_i+m-i)}(\lambda_i + m - i)_q,$$

we get the characters of the Casimir elements Cas_k, arising from the action of the modified RE algebra.
Now, by passing to the limit $q = 1$, we get the PP formula.

What next? We plan

– to express the ordered Casimir operators via the Casimir ones,

–to find their eigenvalues in irreducible modules,

–by finding their good normalization, to define braided cut-and-join operators,

–to understand the role of these operators in constructing τ-functions of "braided matrix models".