Inequalities defining polyhedral realizations of affine types and extended Young diagrams

Combinatorics and Arithmetic for Physics, IHES

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17.11.2023

Polyhedral realization : a combinatorial description of crystal bases

Crystal base : a powerful tool to study representations of quantum groups

Goal

We give explicit forms of polyhedral realizations for crystal bases of affine quantum groups in terms of extended Young diagrams.

Plan

1. (Affine) quantum groups
2. Extended Young diagrams
3. Crystal bases and polyhedral realizations
4. Main results

1. (Affine) quantum groups

Lie algebras and their representations (History)

Lie algebra $\mathfrak{g}$: Vector space/$\mathbb{C}$ with Lie bracket product [,].

For $x, y, z \in \mathfrak{g}$,

- $[x, x] = 0$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Lie algebras and their representations (History)

Lie algebra $\mathfrak{g}$: Vector space/$\mathbb{C}$ with Lie bracket product $[\ ,\ ]$.

For $x, y, z \in \mathfrak{g}$,

- $[x, x] = 0$,
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

(I) Finite dimensional simple Lie algebras

(II) Affine Lie algebras

(III) Quantum groups

(IV) Affine quantum groups

(I) Finite dimensional **simple** Lie algebra $\mathfrak{g}$

simple means $\mathfrak{g}$ has no ideal other than $\{0\}$ and $\mathfrak{g}$.

Example) $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{C}, a + d = 0 \right\}$.

$$[x, y] = xy - yx, \qquad x, y \in \mathfrak{g} \Rightarrow [x, y] \in \mathfrak{g}.$$

Putting

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

these are generators of $\mathfrak{g}$. It holds

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Theorem (Serre(1966))

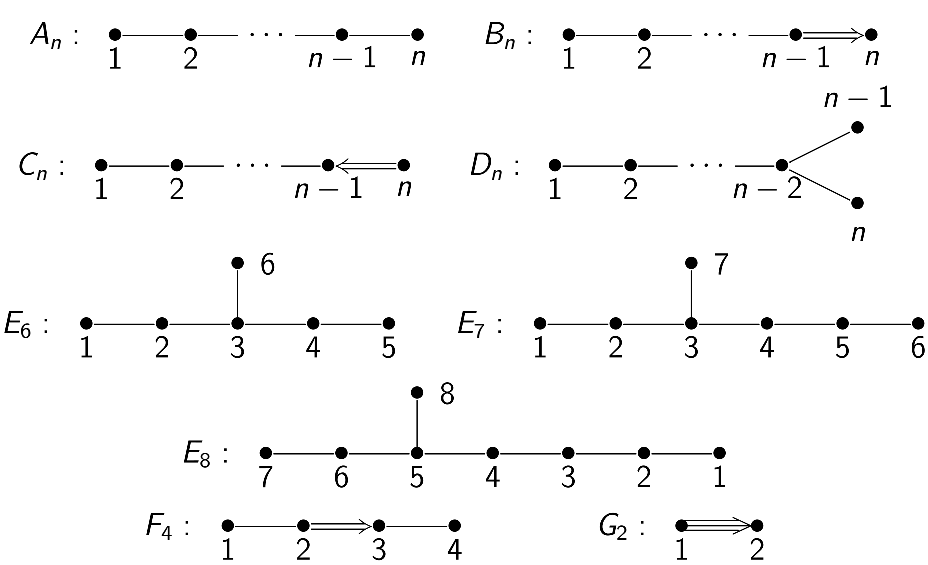
$\mathfrak{g}$: finite dimensional simple Lie algebra. Then $\mathfrak{g}$ has generators e_i, f_i and h_i ($i \in I = \{1, 2, \cdots, n\}$) s.t.

1. $[h_i, h_j] = 0$,
2. $[e_i, f_j] = \delta_{i,j} h_i$,
3. $[h_i, e_j] = a_{ij} e_j$,
4. $[h_i, f_j] = -a_{ij} f_j$,
5. $(\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0$ for $i \neq j$,
6. $(\mathrm{ad}\ f_i)^{1-a_{ij}} f_j = 0$ for $i \neq j$.

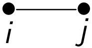
Here, $(\mathrm{ad}\ x)y := [x, y]$. $A = (a_{ij})_{i,j \in I}$ is called a **Cartan matrix** and

- $a_{ii} = 2$ ($i \in I$), $a_{ij} \leq 0$ if $i \neq j$,
- if $a_{ij} = 0$ then $a_{ji} = 0$,
- (a_{ij}) is symmetrizable (i.e., $\exists D$: diagonal matrix s.t. DA is symmetric) and DA is positive definite.

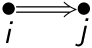
(a_{ij}) is classified by Dynkin diagrams:



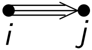
Here,



implies $a_{ij} = a_{ji} = -1$,



implies $a_{ij} = -1$, $a_{ji} = -2$,



implies $a_{ij} = -1$, $a_{ji} = -3$. If i and j are not connected then $a_{ij} = a_{ji} = 0$.

The **structure** and **classification** of cpx. finite dimensional simple Lie algebras are well known.

$\rightsquigarrow$The theory of fin. dim. simple Lie algebras are **quite successful**.

Considering natural **variations** of fin.dim.simple Lie algebras, it can be expected to obtain some new interesting theories.

Variations

- Change $(a_{i,j})$ from positive definite to nonnegative-definite $\to$ **affine Lie algebra**.

  Recall : When $\mathfrak{g}$ is fin. dim. simple,

  - $a_{ii} = 2$ $(i \in I)$, $a_{ij} \leq 0$ if $i \neq j$,
  - if $a_{ij} = 0$ then $a_{ji} = 0$,
  - (a_{ij}) is symmetrizable (i.e., $\exists D$: diagonal matrix s.t. DA is symmetric) and DA is positive definite.

- Quantization $\to$ **quantum group**.
- Both $\to$ **affine quantum group**.

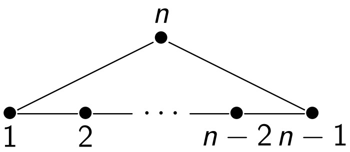
(II) Affine Lie algebras

Let $(a_{ij})_{i,j\in I}$ be a matrix s.t.

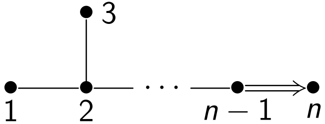
- $a_{ii} = 2$ $(i \in I)$, $a_{ij} \leq 0$ if $i \neq j$,
- if $a_{ij} = 0$ then $a_{ji} = 0$,
- (a_{ij}) is symmetrizable and DA is nonnegative-definite and not positive definite.

(a_{ij}) is classified by affine Dynkin diagrams:

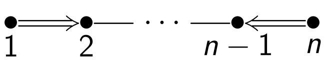
$\mathrm{A}^{(1)}_{n-1}$ $(n \geq 3)$:



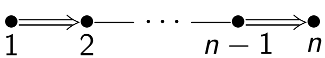
$\mathrm{B}^{(1)}_{n-1}$ $(n \geq 3)$:



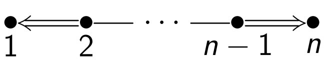
$\mathrm{C}^{(1)}_{n-1}$ $(n \geq 3)$:



$\mathrm{A}^{(2)}_{2n-2}$ $(n \geq 3)$:



$\mathrm{D}^{(2)}_{n}$ $(n \geq 3)$:



and $\mathrm{D}^{(1)}_{n-1}$, $\mathrm{A}^{(2)}_{2n-3}$, $\mathrm{D}^{(3)}_{4}$, $\mathrm{E}^{(1)}_{6}$, $\mathrm{E}^{(1)}_{7}$, $\mathrm{E}^{(1)}_{8}$, $\mathrm{F}^{(1)}_{4}$, $\mathrm{G}^{(1)}_{2}$ and $\mathrm{E}^{(2)}_{6}$.

We need a preparation:

- Let $\mathfrak{h}$ be a $|I|+1$-dimensional vector space/$\mathbb{C}$ and we assume that $\{h_i\}_{i\in I} \cup \{d\}$ is a base of $\mathfrak{h}$.
- We take $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ s.t. $\alpha_j(h_i) = a_{ij}$, $\alpha_j(d) = \delta_{1,j}$.

Definition

Let $\mathfrak{g}$ be a Lie algebra with generators e_i, f_i, h_i $(i \in I)$ and d s.t. for $h, h' \in \mathfrak{h} \subset \mathfrak{g}$,

1. $[h, h'] = 0$,
2. $[e_i, f_j] = \delta_{i,j} h_i$,
3. $[h, e_j] = \alpha_j(h) e_j$,
4. $[h, f_j] = -\alpha_j(h) f_j$,
5. $(\mathrm{ad}\ e_i)^{1-a_{ij}} e_j = 0 \quad$ for $i \neq j$,
6. $(\mathrm{ad}\ f_i)^{1-a_{ij}} f_j = 0 \quad$ for $i \neq j$.

$\mathfrak{g}$ is called an **affine Lie algebra**.

Unlike fin.dim.simple Lie alg,

- Affine Lie algebras are infinite dimensional.
- All non-trivial representations are infinite dimensional.

• Applications for q-series, number theory, physics...

- Jacobi triple product formula,
- Rogers-Ramanujan identities and their variations,
- Solvable lattice models.

(III)Quantum groups (Drinfeld,Jimbo(1985)) (='similar' algebra to $\mathfrak{g}$)

- q : indeterminant,

$[r]_q := \frac{q^r - q^{-r}}{q - q^{-1}}$ for $r \in \mathbb{Z}_{\geq 0}$, $[r]_q! := [r]_q[r-1]_q \cdots [1]_q$,

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q![m-r]_q!}.$$

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra.

Let $U_q(\mathfrak{g})$ be $\mathbb{C}(q)$-algebra with unit 1 with generators e_i, f_i and q^h $(h \in \oplus_{i\in I}\mathbb{Z}h_i)$ satisfying

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$,
4. $e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \ (i \neq j)$,
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \ (i \neq j)$.

Here, $D = \mathrm{diag}(d_1, \cdots, d_n)$. $U_q(\mathfrak{g})$ is called a **quantum group** associated with $\mathfrak{g}$.

When '$q \to 1$' , we get relations of (universal env. alg. of) $\mathfrak{g}$.

(IV) Affine quantum groups

We take $A = (a_{ij})$, vector space $\mathfrak{h}$ and $\alpha_i \in \mathfrak{h}^*$ just as in the definition of affine Lie algebras :

Let $(a_{ij})_{i,j \in I}$ be a matrix s.t.

- $a_{ii} = 2$ $(i \in I)$, $a_{ij} \leq 0$ if $i \neq j$,
- if $a_{ij} = 0$ then $a_{ji} = 0$,
- A is symmetrizable and DA is nonnegative-definite but positive definite.

Let $\mathfrak{h}$ be a $|I| + 1$-dimensional vector space$/\mathbb{C}$ with base $\{h_i\}_{i \in I} \cup \{d\}$.

We take $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ s.t. $\alpha_j(h_i) = a_{ij}$ and $\alpha_j(d) = \delta_{0,j}$.

We can define $U_q(\mathfrak{g})$ by the same way as in the case $\mathfrak{g}$ is fin. dim. simple:

Let $U_q(\mathfrak{g})$ be $\mathbb{C}(q)$-algebra with unit 1 and with generators e_i, f_i and q^h ($h \in \oplus_{i\in I}\mathbb{Z}h_i \oplus \mathbb{Z}d$) satisfying

1. $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$,
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$,
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$,
4. $e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}$,
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{d_i}} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \ (i \neq j)$,
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q^{d_i}} f_i^{1-a_{ij}-k} f_j f_i^k = 0 \ (i \neq j)$.

This $U_q(\mathfrak{g})$ is called **affine quantum group**.

Representations of $U_q(\mathfrak{g})$

$$P := \{\lambda \in \mathfrak{h}^* | \lambda(h_i), \lambda(d) \in \mathbb{Z}\ (i \in I)\} : \text{weight lattice}$$
$$P^+ := \{\lambda \in \mathfrak{h}^* | \lambda(h_i), \lambda(d) \in \mathbb{Z}_{\geq 0}\ (i \in I)\}.$$

For $\lambda \in P^+$, there exists a representation $V(\lambda)$ s.t. $\exists v_\lambda \in V(\lambda)$ and

$$e_i v_\lambda = 0\ (\forall i \in I), \quad q^h v_\lambda = q^{\lambda(h)} v_\lambda\ (\forall h \in \oplus_{i \in I} \mathbb{Z} h_i \oplus \mathbb{Z} d),$$

$$V(\lambda) = \langle f_{j_1} \cdots f_{j_r} v_\lambda | j_1, \cdots, j_r \in I \rangle_{\mathbb{C}(q)-\text{vect.sp.}}.$$

- $V(\lambda)$ is an analog of finite dimensional irreducible representation $L(\lambda)$ of fin.dim.simple Lie algebra.
  c.f.) $\exists \ell_\lambda \in L(\lambda)$ and

$$e_i \ell_\lambda = 0\ (\forall i \in I), \quad h \ell_\lambda = \lambda(h) \ell_\lambda\ (\forall h \in \oplus_{i \in I} \mathbb{Z} h_i),$$

$$L(\lambda) = \langle f_{j_1} \cdots f_{j_r} \ell_\lambda | j_1, \cdots, j_r \in I \rangle_{\mathbb{C}-\text{vect.sp.}}.$$

2. extended Young diagrams

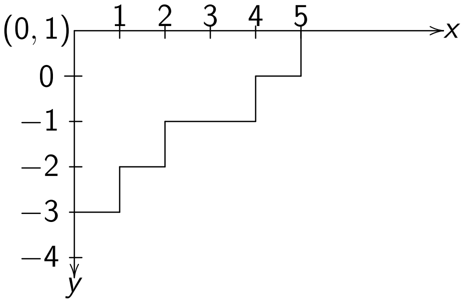
Extended Young diagram

Definition (Hayashi, Misra-Miwa (1990))

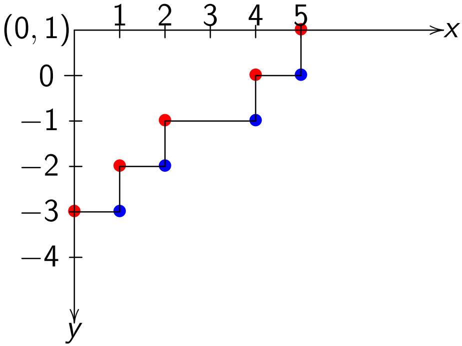
An **extended Young diagram** T is a sequence $(y_k)_{k\in\mathbb{Z}_{\geq 0}}$ s.t.

- $y_k \in \mathbb{Z}$, $y_k \leq y_{k+1}$ for all $k \in \mathbb{Z}_{\geq 0}$,
- $\exists y_\infty \in \mathbb{Z}$ s.t. $y_k = y_\infty$ for $k \gg 0$.

Ex) $y_0 = -3$, $y_1 = -2$, $y_2 = y_3 = -1$, $y_4 = 0$, $y_5 = 1, \cdots, y_\infty = 1$.



Extended Young diagram



- We call red points concave corners and blue points convex corners.

Coloring of corners

Definition

We define a map $\{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\} = I$ as

$$1, 2, \cdots, n \mapsto 1, 2, \cdots, n$$

and extend it to a map $\pi_{A^{(1)}} : \mathbb{Z} \to \{1, 2, \cdots, n\} = I$ by periodicity n.

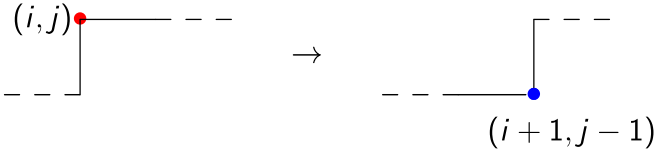
Each integer point (i, j) is colored by $\pi_{A^{(1)}}(i + j) \in I$.

Back ground

Studies in 90's (Hayashi, Jimbo-Misra-Miwa-Okado, Kang-Misra-Miwa)

$\mathcal{F}(\Lambda_k)$: vector space $/\mathbb{Q}(q)$ having all EYDs with $y_\infty = k$ as base.

- If $\mathfrak{g}$ is of type $X = A^{(1)}_{n-1},\ C^{(1)}_{n-1},\ A^{(2)}_{2n-2}$ or $D^{(2)}_n$, the vect. sp. $\mathcal{F}(\Lambda_k)$ is a $U_q(\mathfrak{g})$-module and $U_q(\mathfrak{g})\phi_k \cong V(\Lambda_k)$ ($\phi_k := (k, k, \cdots)$) for almost all $k \in I$. Here, $\Lambda_k(h_j) = \delta_{k,j}$.
- Roughly speaking, acting f_m on EYD T, a concave corner (i, j) s.t. $\pi_X(i + j) = m$ changes to a convex corner.



3. Crystal bases and polyhedral realizations

$\mathfrak{g}$: finite dim. simple or affine

Recall : For $\lambda \in P^+$, there exist representations $V(\lambda)$ of $U_q(\mathfrak{g}) = \langle e_i, f_i, q^h \rangle$ s.t.

$$e_i v_\lambda = 0 \ (i \in I), \quad q^h v_\lambda = q^{\lambda(h)} v_\lambda,$$

$$V(\lambda) = \langle f_{j_1} \cdots f_{j_k} v_\lambda | j_1, \cdots, j_k \in I \rangle_{\mathbb{C}(q)-\text{vect.sp.}}$$

Crystal base $B(\lambda)$ (Properties)

- $B(\lambda)$ is a set with maps

  $$\tilde{e}_i, \tilde{f}_i : B(\lambda) \to B(\lambda) \sqcup \{0\},$$

  which are 'combinatorial analogs' of e_i, $f_i : V(\lambda) \to V(\lambda)$.
- $\exists \ \overline{v}_\lambda \in B(\lambda)$ s.t.

  $$B(\lambda) = \{\tilde{f}_{j_1} \cdots \tilde{f}_{j_k} \overline{v}_\lambda | k \in \mathbb{Z}_{\geq 0}, \ j_1, \cdots, j_k \in I\} \setminus \{0\}.$$

  It holds $\sharp B(\lambda) = \dim V(\lambda)$.
- There is a map $\mathrm{wt} : B(\lambda) \to P$ s.t.

  $\mathrm{wt}(\tilde{f}_{j_1} \cdots \tilde{f}_{j_k} \cdot \overline{v}_\lambda)(h_i) =$ Eigen value of $f_{j_1} \cdots f_{j_k} \cdot v_\lambda \in V(\lambda)$ for q^{h_i}.

By studying $B(\lambda)$, we know eigenvalues of $V(\lambda)$, $\dim V(\lambda)$ and structures of tensor products and so on.

To study $B(\lambda)$, combinatorial descriptions are useful.

Today, we consider the **polyhedral realizations** which describe $B(\lambda)$ in terms of (infinite) integer vectors.

Let $\iota = (\cdots, i_2, i_1)$ be a sequence from $I = \{1, 2, \cdots, n\}$.

$$\mathbb{Z}_\iota^\infty[\lambda] := \{(\cdots, a_2, a_1) | a_j \in \mathbb{Z},\ a_k = 0\ (k \gg 0)\}.$$

One can define maps denoted by $\tilde{f}_i$, $\tilde{e}_i : \mathbb{Z}_\iota^\infty[\lambda] \to \mathbb{Z}_\iota^\infty[\lambda] \sqcup \{0\}$ and $\mathrm{wt} : \mathbb{Z}_\iota^\infty[\lambda] \to P$ as follows:

For $r \in \mathbb{Z}_{\geq 1}$ and $\mathbf{a} = (\cdots, a_2, a_1) \in \mathbb{Z}_\iota^\infty[\lambda]$,

$$\sigma_r(\mathbf{a}) := a_r + \sum_{j>r} a_{i_r, i_j} a_j \quad (r \in \mathbb{Z}_{\geq 1}),$$

$$\sigma_0^{(k)}(\mathbf{a}) := -\lambda(h_k) + \sum_{j \geq 1} a_{i_r, i_j} a_j \quad (k \in I),$$

$$M^{(k)}(\mathbf{a}) := \{r \in \mathbb{Z}_{\geq 1} | i_r = k,\ \sigma_r(\mathbf{a}) = \sigma^{(k)}(\mathbf{a})\}.$$

$\mathrm{wt} : \mathbb{Z}^\infty[\lambda] \to P$ is defined by

$$\mathrm{wt}(\mathbf{a}) := \lambda - \sum_{j=1}^{\infty} a_j \alpha_{i_j}.$$

One can define

$$\tilde{e}_i, \tilde{f}_i : \mathbb{Z}_\iota^\infty[\lambda] \to \mathbb{Z}_\iota^\infty[\lambda] \sqcup \{0\}$$

as

$$(\tilde{f}_k(\mathbf{a}))_r := a_r + \delta_{r,\min M^{(k)}(\mathbf{a})} \text{ if } \sigma^{(k)}(\mathbf{a}) > \sigma_0^{(k)}(\mathbf{a});\ \text{otherwise } \tilde{f}_k(\mathbf{a}) = 0,$$

$$(\tilde{e}_k(\mathbf{a}))_r := a_r - \delta_{r,\max M^{(k)}(\mathbf{a})} \text{ if } \sigma^{(k)}(\mathbf{a}) > 0,\ \sigma^{(k)}(\mathbf{a}) \geq \sigma_0^{(k)}(\mathbf{a})$$
$$;\ \text{o.w. } \tilde{e}_k(\mathbf{a}) = 0$$

for $\mathbf{a} = (\cdots, a_j, \cdots, a_2, a_1) \in \mathbb{Z}_\iota^\infty[\lambda]$.

Essential point : Above $\tilde{e}_i$, $\tilde{f}_i$ and wt are defined by only sum of integers (in particular, **without** representation theory). Everybody can compute them by following the above rule.

Theorem (Nakashima)

There is an injective map

$$\Psi_\iota^{(\lambda)} : B(\lambda) \hookrightarrow \mathbb{Z}_\iota^\infty[\lambda]$$

s.t. $\Psi_\iota^{(\lambda)}(\overline{v}_\lambda) = (\cdots, 0, 0, 0)$ and $\Psi_\iota^{(\lambda)}$ commutes with $\tilde{e}_i$, $\tilde{f}_i$ and preserves wt.

By this theorem, we can reduce calculations of $\underbrace{\tilde{e}_i, \tilde{f}_i, \mathrm{wt}}_{\text{important}}$ in $B(\lambda)$ to

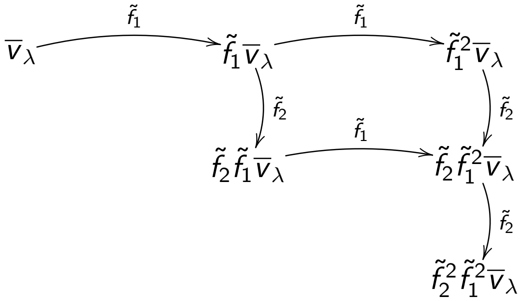
$\underbrace{\tilde{e}_i, \tilde{f}_i, \mathrm{wt}}_{\text{computable}}$ in $\mathbb{Z}_\iota^\infty[\lambda]$.

Definition

$\mathrm{Im}(\Psi_\iota^{(\lambda)})(\cong B(\lambda))$ is called a **polyhedral realization** for $B(\lambda)$.

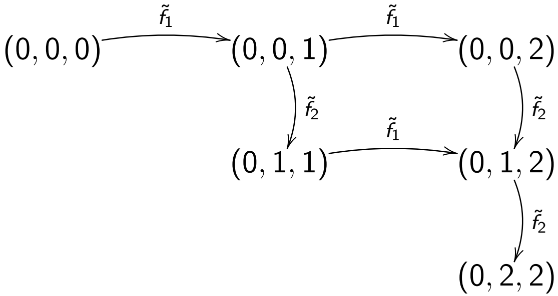
Example) $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \middle| a + e + i = 0 \right\}$,
$\lambda \in P^+$ s.t. $\lambda(h_1) = 2$, $\lambda(h_2) = 0$.

$\underline{B(\lambda) = \{\tilde{f}_{j_1} \cdots \tilde{f}_{j_r} \overline{v}_\lambda\} \setminus \{0\}}$



Thus, $\dim V(\lambda) = 6$.

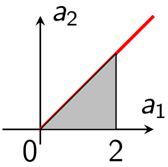
$\underline{\Psi_\iota^{(\lambda)}(B(\lambda))}$



Here, (a_3, a_2, a_1) implies $(\cdots, 0, 0, a_3, a_2, a_1) \in \mathbb{Z}_\iota^\infty[\lambda]$.

integer points (a_3, a_2, a_1) satisfying

$$a_3 = 0,\ 2 \geq a_1 \geq a_2 \geq 0$$



Crystal base $\leftrightarrow$ integer points of a polytope

Question

Find an explicit form of this polytope like as
$a_3 = 0, \; 2 \geq a_1 \geq a_2 \geq 0$.

Case : $\mathfrak{g}$ is fin. dim. simple

Nakashima-Zelevinsky(2002), Hoshino(2006)

- $\iota = (\cdots, n, \cdots, 2, 1, n, \cdots, 2, 1) \Rightarrow$ an explicit form is given.

K-Nakashima (2021)

- If $\mathfrak{g}$ is type A-D, ι is cyclic then the inequalities of explicit forms are described as column tableaux.

Littlemann(1998)

- ι : 'nice decomposition' $\Rightarrow$ an explicit form of **string polytipes** $\mathcal{S}_\iota(\lambda)$ are given. This is a 'dual polytope' in certain sense.

Gleizer-Postnikov(2000), Genz-Koshevoy-Schumann (2021)

- ι : any, $\mathfrak{g}$: type A $\Rightarrow$ explicit forms of our polytopes and $\mathcal{S}_\iota(\lambda)$ are given via **wiring diagrams**, **Reineke crossings**.

Case : $\mathfrak{g}$ is an Affine Lie algebra

Nakashima-Zelevinsky, Hoshino, Hoshino-Nakada

- $\mathfrak{g}$: classical affine type, $\iota = (\cdots, n, \cdots, 1, n, \cdots, 1) \Rightarrow$ an explicit form of the polytope

Goal

In this talk, we consider the case $\mathfrak{g}$ is an affine Lie algebra of type $A_n^{(1)}$, ι is cyclic and describe polytopes in terms of **extended Young diagrams**.

Remark

We obtained a result for types $A_n^{(1)}$, $D_{n+1}^{(2)}$, $C_n^{(1)}$ and $A_{2n}^{(2)}$ (arXiv:2301.05800).

4. Polyhedral realizations of type $\mathrm{A}_{n-1}^{(1)}$

Notation We take $i_1,\cdots,i_n$ s.t. $\{i_1,\cdots,i_n\}=\{1,2,\cdots,n\}=I$ and

$$\iota := (\cdots,i_n,\cdots,i_3,i_2,i_1,i_n,\cdots,i_3,i_2,i_1).$$

- We define $x_k \in \mathrm{Hom}(\mathbb{Z}^\infty,\mathbb{Z})$ as $x_k(\cdots,a_3,a_2,a_1)=a_k$.
- We identify $\mathbb{Z}_{\geq 1}$ with $\mathbb{Z}_{\geq 1}\times I$ as

$$\cdots,2n,\cdots,n+2,n+1,n,,\cdots,2,1$$

$$\leftrightarrow \cdots,(2,i_n),\cdots,(2,i_2),(2,i_1),(1,i_n),\cdots,(1,i_2),(1,i_1).$$

Example) $\iota=(\cdots,2,1,3,2,1,3,2,1,3)$
$(\cdots,x_7,x_6,x_5,x_4,x_3,x_2,x_1)=(\cdots,x_{3,3},x_{2,2},x_{2,1},x_{2,3},x_{1,2},x_{1,1},x_{1,3})$

For $i,j\in I$ $(i\neq j)$,

$$p_{i,j} := \begin{cases} 1 & \text{if } (i_n,\cdots,i_1)=(\cdots,j,\cdots,i,\cdots), \\ 0 & \text{if } (i_n,\cdots,i_1)=(\cdots,i,\cdots,j,\cdots). \end{cases}$$

For $k \in I$, we set $P_k(k) := 0$ and inductively define as

$$P_k(t) := P_k(t-1) + p_{\pi_{A(1)}(t), \pi_{A(1)}(t-1)} \quad \text{if } t > k,$$
$$P_k(t) := P_k(t+1) + p_{\pi_{A(1)}(t), \pi_{A(1)}(t+1)} \quad \text{if } t < k.$$

For an integer point (i, j), $s \in \mathbb{Z}_{\geq 1}$ and $k \in I$, we put

$$L_{s,k}(i,j) := x_{s+P_k(i+j)+\min(k-j,i), \pi_{A(1)}(i+j)} \in \mathrm{Hom}(\mathbb{Z}^\infty, \mathbb{Z}).$$

Key definition

For each extended Young diagram T with $y_\infty = k$, we assign a homomorphism:

$$\begin{aligned} & L_{s,k,\iota}(T) \\ := & \sum_{P:\text{concave corner of } T} L_{s,k}(P) - \sum_{P:\text{convex corner of } T} L_{s,k}(P) \\ \in & \ \mathrm{Hom}(\mathbb{Z}^\infty, \mathbb{Z}). \end{aligned}$$

Let EYD_k be the set of extended Young diagrams with $y_\infty = k$ for $k \in I$ and

$$\mathrm{Comb}_\iota[\infty] := \{L_{s,k,\iota}(T) | s \in \mathbb{Z}_{\geq 1},\ k \in I,\ T \in \mathrm{EYD}_k\}.$$

$$\boxed{r}_k := x_{P_k(r),\pi_{A^{(1)}}(r)} - x_{1+P_k(r-1),\pi_{A^{(1)}}(r-1)} \qquad (r \in \mathbb{Z}_{\geq k+1}),$$

$$\boxed{\tilde{r}}_k := x_{P_k(r-1),\pi_{A^{(1)}}(r-1)} - x_{1+P_k(r),\pi_{A^{(1)}}(r)} \qquad (r \in \mathbb{Z}_{\leq k}).$$

$$\mathrm{Comb}_{k,\iota}[\lambda] :=$$

$$\begin{cases} \{-x_{1,k} + \lambda(h_k)\} & \text{if } \begin{array}{l}(1,k) < (1,\pi_{A^{(1)}}(k+1))\\ (1,k) < (1,\pi_{A^{(1)}}(k-1))\end{array} \\ \{\boxed{\tilde{r}}_k + \lambda(h_k) \mid r \in \mathbb{Z}_{\leq k}\} & \text{if } \begin{array}{l}(1,k) < (1,\pi_{A^{(1)}}(k+1))\\ (1,k) > (1,\pi_{A^{(1)}}(k-1))\end{array} \\ \{\boxed{r}_k + \lambda(h_k) \mid r \in \mathbb{Z}_{\geq k+1}\} & \text{if } \begin{array}{l}(1,k) > (1,\pi_{A^{(1)}}(k+1))\\ (1,k) < (1,\pi_{A^{(1)}}(k-1))\end{array} \\ \{L_{0,k,\iota}(T) + \lambda(h_k) \mid T \in \mathrm{EYD}_k \setminus \{\phi^k\}\} & \text{if } \begin{array}{l}(1,k) > (1,\pi_{A^{(1)}}(k+1))\\ (1,k) > (1,\pi_{A^{(1)}}(k-1))\end{array} \end{cases}$$

Here, $\phi^k \in \mathrm{EYD}_k$ is the extended Young diagram without boxes.

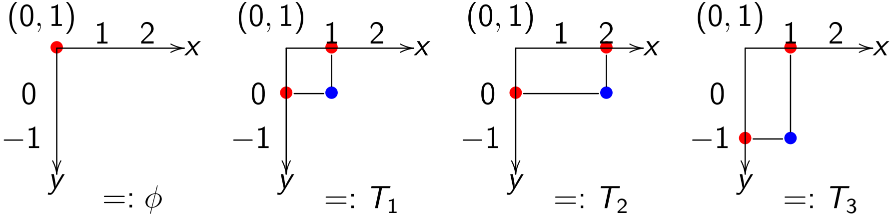
Theorem (K.)

Let $\mathfrak{g}$ be of type $\mathrm{A}_{n-1}^{(1)}$ $(n \geq 2)$. Then

$$\mathrm{Im}(\Psi_{\iota}^{(\lambda)}) = \left\{ \mathbf{a} \in \mathbb{Z}^{\infty} \;\middle|\; \begin{array}{l} \varphi(\mathbf{a}) \geq 0 \\ \forall \varphi \in \mathrm{Comb}_{\iota}[\infty] \cup \bigcup_{k \in I} \mathrm{Comb}_{k,\iota}[\lambda] \end{array} \right\}.$$

Inequalities defining polyhedral realization $\mathrm{Im}(\Psi_{\iota}^{(\lambda)})$ are expressed via combinatorial objects such as extended Young diagrams and boxes.

Example : $\mathfrak{g}$: type $A_2^{(1)}$, $k=1$, $\iota = (\cdots, 3, 1, 2, 3, 1, 2)$.



$$
\begin{aligned}
L_{s,1,\iota}(\phi) &= L_{s,1}(0,1) = x_{s,1}, \\
L_{s,1,\iota}(T_1) &= L_{s,1}(1,1) + L_{s,1}(0,0) - L_{s,1}(1,0) \\
&= x_{s+1,2} + x_{s,3} - x_{s+1,1},
\end{aligned}
$$

$$
\begin{aligned}
L_{s,1,\iota}(T_2) &= L_{s,1}(2,1) + L_{s,1}(0,0) - L_{s,1}(2,0) \\
&= x_{s+1,3} + x_{s,3} - x_{s+2,2},
\end{aligned}
$$

$$L_{s,1,\iota}(T_3) = L_{s,1}(1,1) + L_{s,1}(0,-1) - L_{s,1}(1,-1) = 2x_{s+1,2} - x_{s+1,3}.$$

$$\mathbb{Z}_{\geq 1} \leftrightarrow \mathbb{Z}_{\geq 1} \times I, \quad 1 \leftrightarrow (1,2),\ 2 \leftrightarrow (1,1),\ 3 \leftrightarrow (1,3),$$
$$(1,2) < (1,1) < (1,3).$$

We have $\mathrm{Comb}_{2,\iota}[\lambda] = \{-x_{1,2} + \lambda(h_2)\}$ and

$$\mathrm{Comb}_{1,\iota}[\lambda] = \{\boxed{r}_1 + \lambda(h_1)\ |r \in \mathbb{Z}_{\geq 2}\}.$$

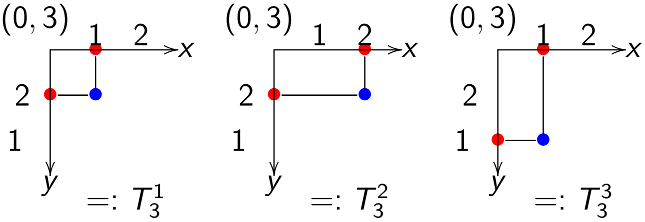
One can compute

$$\boxed{2}_1 = x_{1,2} - x_{1,1},\ \boxed{3}_1 = x_{1,3} - x_{2,2},\ \boxed{4}_1 = x_{2,1} - x_{2,3},$$

$$\boxed{5}_1 = x_{3,2} - x_{3,1},\ \boxed{6}_1 = x_{3,3} - x_{4,2}, \cdots .$$

Next, we consider

$$\mathrm{Comb}_{3,\iota}[\lambda] = \{L_{0,3,\iota}(T) + \lambda(h_3) | T \in \mathrm{EYD}_3 \setminus \{\phi^3\}\}.$$



$$\begin{aligned} L_{0,3,\iota}(T_3^1) &= x_{P_3(4),\pi_{\mathrm{A}(1)}(4)} + x_{P_3(2),\pi_{\mathrm{A}(1)}(2)} - x_{P_3(2),\pi_{\mathrm{A}(1)}(3)} \\ &= x_{1,1} + x_{1,2} - x_{1,3}, \\ L_{0,3,\iota}(T_3^2) &= x_{2,2} + x_{1,2} - x_{2,1}, \\ L_{0,3,\iota}(T_3^3) &= 2x_{1,1} - x_{2,2}. \end{aligned}$$

Therefore, we could partially compute inequalities:

$$\mathrm{Im}(\Psi_{\iota}^{(\lambda)}) = \left\{ \mathbf{a} \in \mathbb{Z}^{\infty} \;\middle|\; \begin{array}{l} s \in \mathbb{Z}_{\geq 1},\ \ a_{s,1} \geq 0, \\ a_{s+1,2} + a_{s,3} - a_{s+1,1} \geq 0, \\ a_{s+1,3} + a_{s,3} - a_{s+2,2} \geq 0, \\ 2a_{s+1,2} - a_{s+1,3} \geq 0, \cdots \\ a_{1,2} \leq \lambda(h_2),\ a_{1,2} - a_{1,1} + \lambda(h_1) \geq 0, \\ a_{1,3} - a_{2,2} + \lambda(h_1) \geq 0, \\ a_{2,1} - a_{2,3} + \lambda(h_1) \geq 0, \\ a_{3,2} - a_{3,1} + \lambda(h_1) \geq 0, \\ a_{3,3} - a_{4,2} + \lambda(h_1) \geq 0, \cdots, \\ a_{1,1} + a_{1,2} - a_{1,3} + \lambda(h_3) \geq 0, \\ a_{2,2} + a_{1,2} - a_{2,1} + \lambda(h_3) \geq 0, \\ 2a_{1,1} - a_{2,2} + \lambda(h_3) \geq 0, \cdots \end{array} \right\}.$$