Relativistic Toda Hamiltonians associated with a family of cluster algebras

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<u>Combinatorial problem:</u> $\{V_i\}_i$ = finite-dimensional affine algebra modules

Feigin-Loktev: grading on $\bigotimes_{i=1}^{\ell} V_i \simeq \bigoplus_{m\geq 0} W[m]$

graded components

Compute character $\chi_{\{V_i\}}(\underline{x};q) = \sum_{m\geq 0}' q^m \, ch_{\underline{x}} W[m] = \sum_{\lambda}' K_{\{V_i\},\lambda}(q)\, s_\lambda(x)$

If all $\{V_i\} \simeq$ "fundamental" (e.g. $\simeq V(\omega_j)$ for $\mathfrak{g} = \hat{sl}_N$)

$\chi_{\{V_i\}}(\underline{x};q)$ = q-Whittaker function = $\lim_{t\to\infty}$ Macdonald polynomial $P_\lambda(x;q,t)$

$x = q^{\mu+\rho}$; $y = q^{\lambda}$ $(\lambda = \sum n_i \omega_i) \Rightarrow \chi_{\{V_i\}} = \chi_\mu(\{q^{\lambda_i}\}, q^{-1})$

<u>Problem:</u> Explain this (for all $\mathfrak{g}$)

Type $A^{(1)}_{N-1}$ Spherical DAHA

generators: $\left\{ \begin{array}{l} e_a(X_1,\dots,X_N)\,;\, e_a(Y_1,\dots,Y_N) \\ e_a(X_1^{-1},\dots,X_N^{-1})\,;\, e_a(Y_1^{-1},\dots,Y_N^{-1}) \end{array} \right\}_{a=1}^{N}$

Functional Representation on $\mathbb{C}_{q,t}[\{x_i^{\pm 1}\}]^{S_N}$

- $e_a(\{X_i\}) \longmapsto e_a(\{x_i\})$ (acts by multiplication)

- $e_a(Y_i) \longmapsto \sum_{\substack{I \subset [1,N] \\ |I|=a}} A_I(\underline{x}) \prod_{i\in I} T_{x_i,q} = \mathcal{D}_a(\underline{x})$ = Macdonald Operators

q-difference in $\{x_i\}$

$T_{x_i,q}: \; x_j \mapsto q^{\delta_{i,j}} x_j$

$$A_I = \prod_{\substack{i\in I \\ j\notin I}} \frac{t x_i - x_j}{x_i - x_j}$$

- rational in $\{x_i/x_j = x^{-\beta} \,/\, \beta \in R_+\}$

- polynomial in t

Macdonald - Fourier Transform and Duality

Eigenfunctions: $\mathcal{D}_a(\underline{x})P_\lambda(\underline{x}) = t^{-\binom{a}{2}} e_a(\underline{s})P_\lambda(\underline{x})$ $\quad (s_i = t^{N-i}q^{\lambda_i})$

- Solution $P_\lambda(x)$ unique up to normalization
- When $\lambda =$ integer partition $P_\lambda(x) \to$ Macdonald polynomials

(1) If $\mathcal{D}(x) =$ q-difference operator in $\{x_i\}$ & $\mathcal{D}(\underline{x})P_\lambda(\underline{x}) = \bar{\mathcal{D}}(\underline{s})P_\lambda(\underline{x})$

$\bar{\mathcal{D}}(\underline{s}) =$ "Macdonald-Fourier transform" of $\mathcal{D}(\underline{x})$: q-difference operator in $\underline{q}^{\lambda}$

(2) Duality: Define $P(\underline{x}|\underline{s}) = t^{-\sum \lambda_i(N-i)} P_\lambda(x) = t^{-\rho\cdot\lambda}P_\lambda(x)$

Thm: $$\boxed{\frac{P(\underline{x}|\underline{s})}{\Delta(\underline{x})} = \frac{P(\underline{s}|\underline{x})}{\Delta(\underline{s})}}$$

$$\Delta(x) = \prod_{i>j} \frac{(q\frac{x_i}{x_j};q)_\infty}{(qt^{-1}\frac{x_i}{x_j};q)_\infty}, \quad (x;q)_\infty := \prod_{n\geq 0}(1-q^n x)$$

Pieri Rule from Duality

$$\boxed{D_1(x)\,P(x|s) = e_1(s)\,P(x|s)}$$

$$\Rightarrow \quad D_1(x)\,\frac{\Delta(x)}{\cancel{\Delta(s)}}\,P(s|x) = e_1(s)\,\frac{\Delta(x)}{\cancel{\Delta(s)}}\,P(s|x) \quad \leftarrow \text{Duality} \quad \frac{P(x|s)}{\Delta(x)} = \frac{P(s|x)}{\Delta(s)}$$

$$\Rightarrow \quad \left(\Delta^{-1}(x)\,D_1(x)\,\Delta(x)\right) P(s|x) = e_1(s)\,P(s|x)$$

$$\Rightarrow \quad \left(\Delta^{-1}(s)\,D_1(s)\,\Delta(s)\right) P(x|s) = e_1(x)\,P(x|s) \quad \leftarrow \text{Rename } s \leftrightarrow x$$

$$\Rightarrow \quad \boxed{e_1(x)\,P_\lambda(x) = \mathcal{H}_1(s)\,P_\lambda(x)} \qquad \Rightarrow \quad \overline{e_a(x)} = \mathcal{H}_a(s)$$

$$\{\mathcal{H}_a(s)\}_{a=1}^{N} = q\text{-diff ops in } s = \text{“commuting Hamiltonians”}$$

Ex: $$\mathcal{H}_1(s) = \sum_{i=1}^{N} \prod_{j<i} \left(\frac{1-t^{j-i+1}q^{\lambda_i-\lambda_j}}{1-t^{j-i}q^{\lambda_i-\lambda_j}}\right)\left(\frac{1-t^{j-i-1}q^{\lambda_i-\lambda_j+1}}{1-t^{j-i}q^{\lambda_i-\lambda_j+1}}\right) T_{q^{\lambda_i},q}\,, \qquad T_{q^{\lambda_i},q}\,q^{\lambda_j} = q^{\delta_{ij}}\,q^{\lambda_j}\,T_{q^{\lambda_i},q}$$

q-Whittaker limit : $(t \to \infty)$

- $t^{-a(N-a)} \mathcal{D}_a(x) \longrightarrow D_a(x) = \sum'_{|I|=a} \Big[\prod_{\substack{j \notin I \\ i \in I}} \frac{x_i}{x_i - x_j} \Big] \prod_{i \in I} T_{x_i,q}$

- eigenvalues $t^{-a(N-a)-\binom{a}{2}} e_a(s) \xrightarrow[t\to\infty]{} q^{\lambda_1 + \cdots + \lambda_a} = q^{\lambda \cdot \omega_a}$

- $\mathcal{H}_1(S) \longrightarrow H_1(\lambda) = \sum'^{N}_{i=2} (1 - q^{\lambda_{i+1} - \lambda_i}) T_{q^{\lambda_i},q} + T_{q^{\lambda_1},q}$

  quantum relativistic gl_N Toda acting on q^λ

- $e_a(x) \to e_a(x)$

$$\boxed{\begin{aligned} D_a(x)\,\Pi_\lambda(x) &= q^{\omega_a \cdot \lambda}\,\Pi_\lambda(x) \;; \\ e_a(x)\,\Pi_\lambda(x) &= H_a(\lambda)\,\Pi_\lambda(x) \end{aligned}}$$

q-Whittaker function $\Pi_\lambda(x)$:

(function of q^λ

$x = q^{\rho+\mu}$)

Discrete evolution commuting with $\{H_a(\lambda)\}$?

$SL_2(\mathbb{Z})$ acts on DAHA.

Thm: $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ acts on functional rep by $\mathrm{Ad}_{\gamma^{-1}(x)}$

where $$\boxed{\gamma(x) = \exp \sum_{i=1}^{N} \frac{(\log x_i)^2}{2\log q}}$$

- Action on $\mathcal{D}_a(x)$: Use $\mathrm{Ad}_{\gamma^{-1}} T_{x,q} = q^{\frac{1}{2}} x\, T_{x,q}$, $\mathcal{D}_a \xrightarrow{\mathrm{Ad}_{\gamma}^{-k}} \mathcal{D}_{a,k}(x)$

- $[\gamma(x), e_a(x)] = 0 \Rightarrow$ Fourier transform of $\gamma(x)$ commutes with $H_a(s)$

- How to compute $\overline{\gamma}(\lambda)$?

Answer below in q-Whittaker limit

Theorem: $D_{a,k}(x) = q^{-\frac{ak}{2}} \gamma(x)^{-k} D_a(x) \gamma(x)^{k}$ satisfy qQ-system evolution...

Proof:

Define $g(\lambda)$ by $\bar{D}_{a,k} = q^{\frac{ak}{2}} g^{+k} \bar{D}_a g^{-k}$ where $\bar{D}_{a,k}$ satisfy the recursion

$$\boxed{\begin{array}{ll} (1) & q^{a}\, \bar{D}_{a,k-1} \bar{D}_{a,k+1} = \bar{D}_{a,k}^{2} - \bar{D}_{a-1,k} \bar{D}_{a+1,k} \\ (2) & \bar{D}_{a,k+1} \bar{D}_{b,k} = q^{\min(a,b)} \bar{D}_{b,k} \bar{D}_{a,k+1} \quad (q\text{-comm}) \end{array}}$$

(exchange)

$(a = 1, \dots, N, \quad k \in \mathbb{Z})$

Initial Data: $\bar{D}_{a,0} = q^{\lambda \circ \omega_a}$ (eigenvals of $D_a(x)$), $\bar{D}_{a,1} = q^{\lambda \circ \omega_a} T_{q;q}^{-\omega_a}$

Then $g(\lambda) \overset{!}{=} \gamma(T_{q;q}) \prod_{i=1}^{N-1} (q^{\lambda_{i+1} - \lambda_i}; q)_\infty^{-1}$ commutes with $H_a(\lambda) \Rightarrow g(\lambda) = \bar{\gamma}(\lambda)$

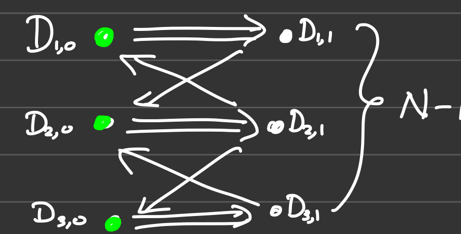
[from $g(\lambda)\Pi_\lambda(x) = \gamma(x)\Pi_\lambda(x)$ and uniqueness of qWhittaker function]

$\Rightarrow D_{a,k}(x)$ satisfy opposite qQ-system

Cluster algebra

The q-Whittaker limit of sDAHA has CA structure

$$\boxed{\begin{aligned} &(1) \quad q^{a}\, D_{a,k+1} D_{a,k-1} = D_{a,k}^{2} - D_{a-1,k} D_{a+1,k} \\ &(2) \quad D_{a,k} D_{b,k+1} = q^{\min(a,b)} D_{b,k+1} D_{a,k} \end{aligned}}$$

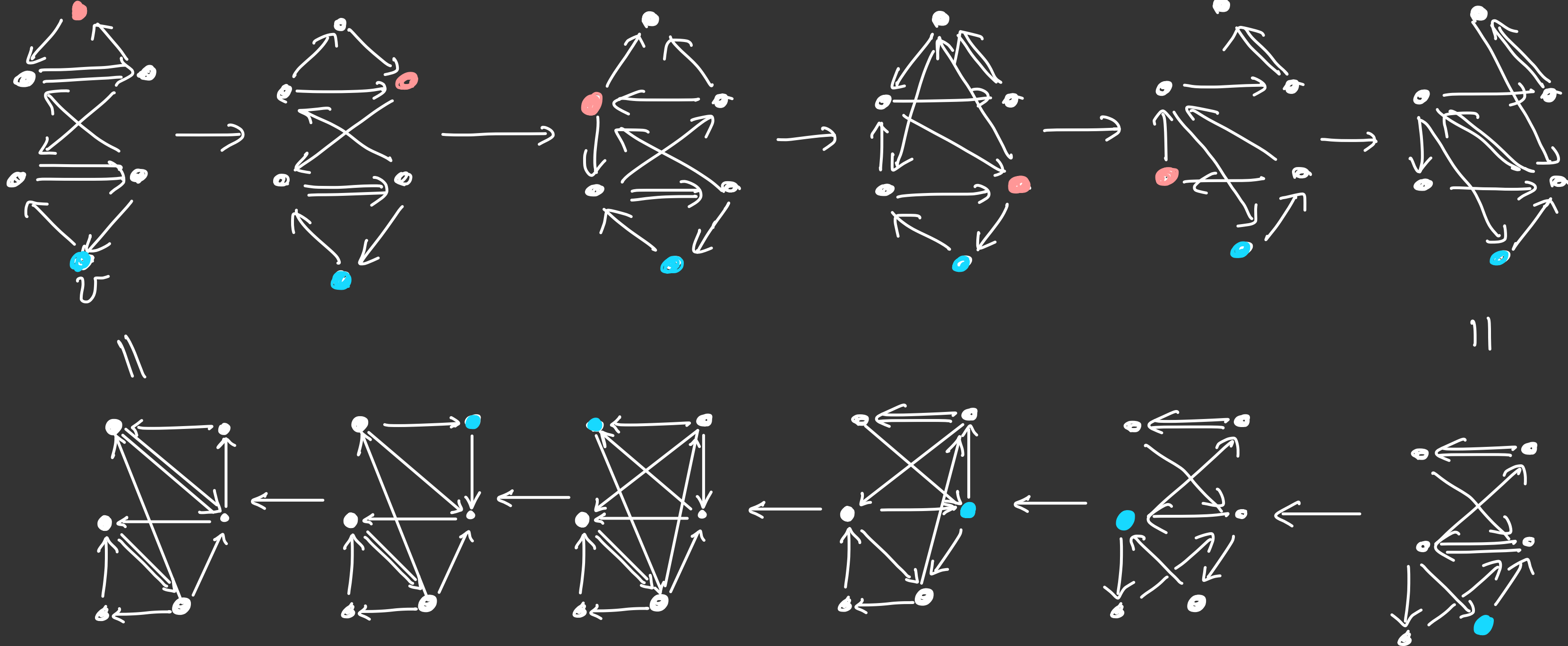


- $\{D_{a,k}\}$ are $\mathcal{A}$-type cluster variables

- The evolution Ad_g is a sequence of mutations ("Dehn Twist")
  at (•)

- <u>Theorem:</u> $g(\lambda)$ = Evaluation of a "Baxter operator" $Q(u,v)$ at
  $u = v = 1$

Schrader - Shapiro Baxter operator



$Q(u,v)$ = sequence of mutations = adjoint action of $\prod$ Dilogs(X_i)

<u>Theorem</u>: $\mathrm{Ad}_{Q(1,1)} = \mathrm{Ad}_{g(\lambda)^{-1}}$ (using identities on dilogs). [DFK]

"Type BC" sDAHA $= \langle \hat{e}_a(\{X_i\}), \hat{e}_a(\{Y_i\}) \rangle$

$\hat{e}_1 = X_1 + \cdots + X_N + X_N^{-1} + \cdots + X_1^{-1}$

W^C- elementary symmetric functions

Functional representation:

- $\hat{e}_1(\underline{Y}) \longmapsto$ Koornwinder operator $\mathcal{D}_1(X) = \sum_{i=1}^{N} \Phi_{i,\varepsilon}(\underline{X}) T_{x_i, q^\varepsilon} + \varphi(x)$

$$\boxed{\Phi_{i,\varepsilon}(\underline{X}) = \prod_{u \in \{a,b,c,d\}} (1 - u X_i^{\varepsilon})(1 - X_i^{2\varepsilon})^{-1}(1 - q X_i^{2\varepsilon})^{-1} \prod_{\substack{j \neq i \\ \eta = \pm 1}} \frac{t X_i^{\varepsilon} X_j^{\eta} - 1}{X_i^{\varepsilon} X_j^{\eta} - 1}}$$

$(a, b, c, d; q, t)$ = parameters

- $\mathcal{D}_1(x)$ has eigenvalue $\sigma t^{N-1} \hat{e}_1(\underline{s})$, $s_i = \sigma t^{N-i} q^{\lambda_i}$, $\sigma = \sqrt{\frac{abcd}{q}}$

eigenfunctions = Koornwinder functions

Koornwinder Duality

Theorem: $\exists$ normalization of Koornwinder eigenfunctions $P(x|s)$ s.t.

$$\boxed{\frac{P(x|s)}{\Delta(x)} = \frac{P^*(s|x)}{\Delta^*(s)}}$$

$s_i = \sigma t^{N-i} q^{\lambda_i}$

$\sigma = \sqrt{\frac{abcd}{q}}$

Involution $*: (a,b,c,d) \longrightarrow (a^*, b^*, c^*, d^*) = \left(\sqrt{\frac{abcd}{q}}, -\sqrt{q\frac{ab}{cd}}, \sqrt{q\frac{ac}{bd}}, -\sqrt{q\frac{ad}{bc}}\right)$

$$\Delta(x) = \prod_{i=1}^{N} \frac{(q x_i^{-2};q)_\infty}{\prod_{u=a,b,c,d} (q/ux_i;q)_\infty} \prod_{\substack{i<j \\ \varepsilon=\pm 1}} \frac{(q x_i^{-1} x_j^{\varepsilon};q)_\infty}{(q/t \, x_i^{-1} x_j^{\varepsilon};q)_\infty}$$

Pieri operators $\quad \mathcal{H}_a(s) P(x|s) = \hat{e}_a(x) P(x|s)$

where $\quad \mathcal{H}_a(s) = \Delta^*(s)^{-1} \vec{D}_a(s) \Delta^*(s)$ $\quad$ Notice $*$

<u>q-Whittaker limit</u> Depends on how (a,b,c,d) scale with t:

q-Whittaker limit $t\to\infty$ of $\mathcal{H}_a(s)$ different for each specialization

$$\lim_{t\to\infty} \mathcal{H}_a^{(\mathfrak{g})}(s) = H_a^{(\mathfrak{g})}(\lambda)$$

Affine algebra	R	R^*	a	b	c	d	σ
$D_N^{(1)}$	D_N	D_N	1	-1	$\sqrt{q}$	$-\sqrt{q}$	1
$B_N^{(1)}$	B_N	C_N	t	-1	$\sqrt{q}$	$-\sqrt{q}$	$t^{1/2}$
$C_N^{(1)}$	C_N	B_N	$\sqrt{t}$	$-\sqrt{t}$	$\sqrt{tq}$	$-\sqrt{tq}$	t
$A_{2N-1}^{(2)}$	C_N	C_N	$\sqrt{t}$	$-\sqrt{t}$	$\sqrt{q}$	$-\sqrt{q}$	$t^{1/2}$
$D_{N+1}^{(2)}$	B_N	B_N	t	-1	$t\sqrt{q}$	$-\sqrt{q}$	t
$A_{2N}^{(2)}$	BC_N	BC_N	t	-1	$\sqrt{tq}$	$-\sqrt{tq}$	t

<u>Example:</u> $$H_a^{(\mathfrak{g})}(\lambda) = T_1 + \sum_{i=1}^{N-1}(1-\Lambda^{-\alpha_i^*})(T_{i+1}+T_i^{-1}) + (1-\Lambda^{-\alpha_N^*})T_N^{-1}$$

when $\mathfrak{g} = C_N^{(1)}, A_{2N-1}^{(2)}$, $\alpha_i^* =$ simple roots of R^*

$(= \alpha_i$ if $i<N)$

Discrete time evolution

(⚠ if α_1 = short root)

$[H_a^{(\mathfrak{g})}(\lambda), g^{(\mathfrak{g})}(\lambda)] = 0$ for some "time translation" operator $g^{(\mathfrak{g})}(\lambda)$

Theorem: For each $\mathfrak{g} = X_N^{(r)}$ $(X = A, B, C, D\ ;\ r = 1, 2)$

the time-translation operator $g^{(\mathfrak{g})}(\lambda)$ is the evolution of the $\mathfrak{g}$-type quantum Q-system

- $q = 1$: Recursion relation for characters of finite-dim $U_q(\mathfrak{g})$-reps
- $q \neq 1$: quantization of cluster algebra(*) structure

(*except for $A_{2N}^{(2)}$)

Q-system evolutions

$$\boxed{Q_{a,k+1} = Q_{a,k-1}^{-1}\left(Q_{a,k}^{2} - \prod_{b\sim a} Q_{b,k}^{[-C_{ba}]_+}\right)}$$

(simply-laced or $A_N^{(2)}$, $D_N^{(2)}$)

TBA 80's
Kirillov-Reshetikhin late 80's
Kuniba et al 90's

Discrete, integrable evolution in k

C = Cartan matrix of type R

If $\mathfrak{g} \neq A_{2N}^{(2)}$, this is a cluster algebra with exchange matrix

$$\left[\begin{array}{c|c} C^T - C^{*T} & -C^T \\ \hline C^{*T} & 0 \end{array}\right]$$

$\rightsquigarrow$ q-deformation is quantum cluster algebra.

Quantum Q-system

Recursion for non-commuting variables $\{\hat{Q}_{a,k}\}_{a\in[1,N],\, k\in\mathbb{Z}}$

$$\begin{cases} (1) \quad \hat{Q}_{a,0}\,\hat{Q}_{b,1} = q^{(w_a^*, w_b)}\,\hat{Q}_{b,1}\,\hat{Q}_{a,0} & \text{q-commutations} \\ (2) \quad q^{(w_a^*, w_a)}\,\hat{Q}_{a,k+1}\,\hat{Q}_{a,k-1} = \hat{Q}_{a,k}^2 - \mathcal{M}_{a,k}^{(g)} & \text{"mutations"} \end{cases}$$

monomial in $\hat{Q}_{b,k'}$, $b\sim a$

Discrete integrable evolution in k

Thm: $H_a^{(g)}(\lambda)$ (computed from Koornwinder specialization) commute with qQ-system time-translation $g(\lambda)^{(g)}$ $: \{\hat{Q}_{a,k}\} \to \{\hat{Q}_{a,k+t_a}\}$

- $g(\lambda)$ is q-Whittaker transform of $\gamma^{-1}(x)$ (or $\gamma^{-2}(x)$)
- $H_a^{(g)}$ are elements in "upper cluster algebra", Laurent in $(Q_{a,k})$

How to prove claim [DFK21]:

1. Compute $g^{(\mathfrak{g})}(\lambda)$ from qQ-system equations
2. Show $[H_a^{(\mathfrak{g})}(\lambda), g(\lambda)] = 0$ (by computation)
3. $\Rightarrow g(\lambda)\,\Pi_\lambda(x)$ = eigenfunction of $H_a^{\mathfrak{g}}(\lambda) \Rightarrow \propto \Pi_\lambda(x)$
4. Compute proportionality constant is $= \gamma(x)$ (or $\gamma^2(x)$)

$$\Downarrow$$

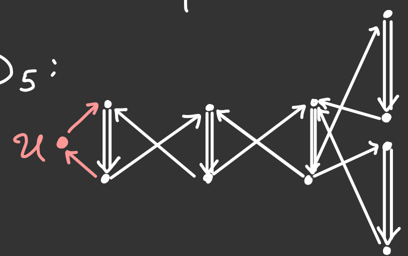
- $D^{(\mathfrak{g})}_{a,k}(x)$ satisfy qQ-system of type $\mathfrak{g}$
- q-Whittaker functions $\Pi_\lambda(x) \propto \prod_{a=1}^{N} D_{a,1}^{n_a}(x)\cdot 1$, $\lambda = \sum n_a \omega_a$
- $D_{a,k}(x)\cdot 1$ = graded character of KR-module of $U_q(\mathfrak{g})$
- $\overrightarrow{\prod} D_{a,k}^{n_{a,k}}(x)\cdot 1$ = characters of FL graded product of KR-modules

Remarks

- Baxter operators as sequences of mutations for BC types:

  Baxter operator = q-exponential generating fn of $H_a^{(g)}(\lambda)$

  ex D_5:



- When $g = B_N^{(1)}, C_N^{(1)}, A_{2N}^{(2)}$ Hamiltonians are new:

  (B_N, C_N Toda come from $A_{2N-1}^{(2)}, D_{N+1}^{(2)}$)

- Finite t time translation: $A_N^{(1)}$ ✓ Koornwinder?