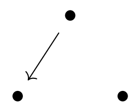
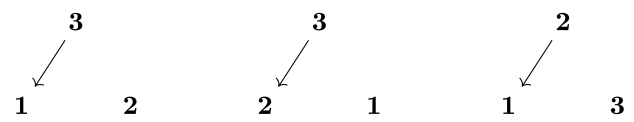
Miracle of integer eigenvalues

Example: start with a finite poset $(P, \preccurlyeq)$ with $n \geqslant 1$ elements, e.g. in the case $n = 3$:



Here is the list $L = L_P$ of *linear extensions* of P=*total orderings* of P, in other words, monotonic maps $\phi : (P, \preccurlyeq) \to (\{1, \ldots, n\}, \leqslant)$ which are *bijections* of sets:

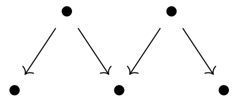


Construct a square matrix filled by permutations $\in Sym_n$, the entry at $(\phi, \psi) \in L_P \times L_P$ is $\phi \circ \psi^{-1} \in Sym_n$:

	3 1 2	3 2 1	2 1 3
3 1 2	**123**	**213**	**132**
3 2 1	**213**	**123**	**231**
2 1 3	**132**	**312**	**123**

$$\rightsquigarrow \begin{pmatrix} \nearrow\nearrow & \searrow\nearrow & \nearrow\searrow \\ \searrow\nearrow & \nearrow\nearrow & \nearrow\searrow \\ \nearrow\searrow & \searrow\nearrow & \nearrow\nearrow \end{pmatrix} \rightsquigarrow M_P = \begin{pmatrix} t_{00} & t_{10} & t_{01} \\ t_{10} & t_{00} & t_{01} \\ t_{01} & t_{10} & t_{00} \end{pmatrix}$$

$$\text{Eigenvalues}(M_P) = (t_{00} + t_{01} + t_{10}, \quad t_{00} - t_{01}, \quad t_{00} - t_{10})$$



$$M_P = \begin{pmatrix}
a_1 & a_5 & a_7 & a_3 & a_9 & a_2 & a_6 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\
a_5 & a_1 & a_7 & a_3 & a_9 & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_2 & a_8 & a_4 & a_{10} & a_6 \\
a_5 & a_7 & a_1 & a_9 & a_3 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_8 & a_2 & a_{10} & a_4 & a_6 \\
a_5 & a_3 & a_9 & a_1 & a_7 & a_6 & a_2 & a_8 & a_4 & a_{10} & a_6 & a_2 & a_8 & a_3 & a_9 & a_5 \\
a_5 & a_9 & a_3 & a_7 & a_1 & a_6 & a_8 & a_2 & a_{10} & a_4 & a_6 & a_8 & a_2 & a_9 & a_3 & a_5 \\
a_2 & a_6 & a_8 & a_3 & a_9 & a_1 & a_5 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\
a_6 & a_2 & a_8 & a_3 & a_9 & a_5 & a_1 & a_7 & a_3 & a_9 & a_5 & a_4 & a_{10} & a_2 & a_8 & a_6 \\
a_6 & a_8 & a_2 & a_9 & a_3 & a_5 & a_7 & a_1 & a_9 & a_3 & a_5 & a_{10} & a_4 & a_8 & a_2 & a_6 \\
a_6 & a_2 & a_8 & a_4 & a_{10} & a_5 & a_3 & a_9 & a_1 & a_7 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 \\
a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_1 & a_5 & a_9 & a_3 & a_8 & a_2 & a_6 \\
a_6 & a_8 & a_2 & a_{10} & a_4 & a_5 & a_9 & a_3 & a_7 & a_5 & a_1 & a_9 & a_3 & a_8 & a_6 & a_2 \\
a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_4 & a_{10} & a_2 & a_8 & a_6 & a_1 & a_7 & a_3 & a_9 & a_5 \\
a_5 & a_9 & a_3 & a_8 & a_2 & a_6 & a_{10} & a_4 & a_8 & a_2 & a_6 & a_7 & a_1 & a_9 & a_3 & a_5 \\
a_6 & a_4 & a_{10} & a_2 & a_8 & a_5 & a_3 & a_9 & a_2 & a_8 & a_6 & a_3 & a_9 & a_1 & a_7 & a_5 \\
a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_9 & a_2 & a_6 & a_9 & a_3 & a_7 & a_1 & a_5 \\
a_6 & a_{10} & a_4 & a_8 & a_2 & a_5 & a_9 & a_3 & a_8 & a_6 & a_2 & a_9 & a_3 & a_7 & a_5 & a_1
\end{pmatrix}$$

Eigenvalues of $M_P = (a_1 - a_4 - a_7 + a_{10})_3\ ,\ a_1 - a_4 + a_7 - a_{10}\ ,\ (a_1 + a_2 - a_5 - a_6)_2\ ,\ (a_1 - a_2 - a_5 + a_6)_2\ ,$

$$(a_1 - a_2 - a_3 + a_4 + a_7 - a_8 - a_9 + a_{10})_2\ ,\ (a_1 - a_2 - a_3 + a_4 - a_7 + a_8 + a_9 - a_{10})_2\ ,$$

$$(a_1 - a_4 + a_5 - a_6 + a_7 - a_{10})_2\ ,\ a_1 + 2a_2 + 2a_3 + a_4 - a_7 - 2a_8 - 2a_9 - a_{10}\ ,$$

$$a_1 + 2a_2 + 2a_3 + 2a_5 + 2a_6 + a_7 + 2a_8 + 2a_9 + a_{10}$$

Theorem: for any finite poset $(P, \preccurlyeq)$ all eigenvalues of M_P are **integer** linear combinations of variables t_ϵ, where $\epsilon \in \{0,1\}^n, \qquad n = \#P.$

In general, matrix M_P can be written as the linear combination of matrices $M_{P,\epsilon}$ with the entries equal to 0 or 1:

$$M_P = \sum_{\epsilon\in\{0,1\}^n} t_\epsilon \cdot M_{P,\epsilon}$$

Individual matrices $M_{P,\epsilon}$ *do not commute* with each other, so the fact that the eigenvalues are *linear functions* (instead of *algebraic functions*) in parameters (t_ϵ) is surprising.

The matrix M_P (as well as each of summands $M_{P,\epsilon}$ is *stochastic*, i.e. sums of all rows are the same $\iff$ the *column* vector $(1, 1, \ldots, 1)$ is its eigenvector.

The explanation of the miracle of integer eigenvalues comes from the existence of certain filtration

$$A^{\leqslant 0} \subseteq A^{\leqslant 1} \subseteq \cdots \subseteq A^{\leq \frac{n(n-1)}{2}} = A$$

on the algebra A of functions on L_P *preserved* by all operators $M_{P,\epsilon}$ and such that the induced operators on associated graded spaces $A^{\leqslant i}/A^{\leqslant (i-1)}$ **commute** with each other.

The space $A^{\leqslant 0}$ is 1-dimensional and spanned by the constant function 1 on L_P (hence the stochasticity of M_P).

The next space $A^{\leqslant 1}$ is spanned by $A^{\leqslant 0}$ and all functions $f_{a,b}$ on L_P of the form

$$f_{a,b} : \phi \in L_P \mapsto \mathrm{sign}(\phi(a) - \phi(b)) \in \{-1, +1\} \subset \mathbb{R}, \qquad \forall\, a \neq b \in P$$

Finally, the filtration is *strictly multiplicative*, i.e.

$$A^{\leq i} = \underbrace{A^{\leqslant 1} \cdot A^{\leqslant 1} \cdot \quad \dots \quad \cdot A^{\leqslant 1}}_{i \text{ times}}$$

Proof:

First, we will construct certain "universal" family of matrices acting on functions on a finite set, such they are preserving certain strictly multiplicative filtration and commute on the associated graded spaces.

For a given $N \geqslant 1$ consider the set V_N of vertices of an N-dimensional cube, it has 2^N elements. For any *face* of the cube we have a natural idempotent map (retraction) from V_N to itself, contracting V_N to the subset of vertices lying in the given face:

$$F_N \to \mathrm{Maps}(V_N, V_N), \qquad F_N := \{\text{faces of } V_N\}, \quad \#F_N = 3^N$$

(in fact F_N is a monoid acting on V_N).

Lemma: operators on the space $A = \mathbb{R}^{V_N}$ of functions on V_N induced by elements of F_N preserve the filtration given by the restriction of *polynomials* of degree $\leqslant 0, \leqslant 1, \leqslant 2, \ldots$ on $\mathbb{R}^N \supset V_N$, and the corresponding operators on associated graded spaces *commute* with each other.

Proof: Enough to check the case $N = 1$: we get 3 operators acting on $\mathbb{R}^2 = \mathbb{R}^{V_1}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

preserving the 1-dimensional subspace spanned by the constant function 1. ■

Now consider the case $N = \binom{n}{2} = \frac{n(n-1)}{2}$, and an abstract n-element set S.
The set of vertices V_N will be the set of *tournaments* on S, i.e. all possible orientations of edges on the complete graph with the vertices equal to S.

Definition: a *filtration* on finite set S is a surjective map $\mathsf{f} : S \twoheadrightarrow \{1, \ldots, l\}$ for some $1 \leqslant l \leqslant \#S$.

Each filtration gives a face of the cube of tournaments, the corresponding retraction forces all edges for pairs $(a, b) \in S^2$, $a \neq b$ such that $\mathsf{f}(a) > \mathsf{f}(b)$ to be directed from a to b, i.e. $a \to b$.

Among all $2^{\frac{n(n-1)}{2}}$ tournaments we have a class of $n!$ special ones corresponding to the *total orderings* of S.
Fact 1 : retractions on tournaments corresponding to filtrations *preserve* the class of total orderings.

Hence, we get a smaller class of matrices (of smaller size!) with the integral eigenvalue property.

Finally, if S is endowed with a partial order $\preccurlyeq$ (i.e. it is a poset), which we now denote by $P := S$, we have even smaller class of

- *filtrations compatible with* $\preccurlyeq$, ($\iff$ *monotonic* surjections $P \twoheadrightarrow \{1, \ldots, l\}$), as well as
- *total orderings compatible with* $\preccurlyeq$, or equivalently, linear extensions L_P of $\preccurlyeq$.

Fact 2: retractions corresponding to filtrations compatible with $\preccurlyeq$, preserve the class of total orderings compatible with $\preccurlyeq$.

Hence, we get operators T_{f} with the integer eigenvalue property corresponding to the filtrations f of the poset P.

For any ordered partition $n = n_1 + \cdots + n_l, \quad n_i \geqslant 1$, $l \geqslant 1$ define

$$T_{n_1,\ldots,n_l} := \sum_{\substack{\mathsf{f}:P \twoheadrightarrow \{1,\ldots,l\} \text{ monotonic} \\ \#\mathsf{f}^{-1}(i)=n_i \ \forall i=1,\ldots,l}} T_{\mathsf{f}}$$

In this way we get 2^{n-1} operators acting on functions on L_P. These operators are *not* exactly our operators M_ϵ which we previously constructed, but closely related. Namely, both types of indices $(n_1, \ldots, n_l)$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_{n-1}) \in \{0,1\}^{n-1}$ we can naturally identify with the *subsets* $J \subseteq \{1, \ldots, n-1\}$:

$$(n_1, \ldots, n_l) \mapsto J = \{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{l-1}\} \subseteq \{1, \ldots, n-1\}$$

$$(\epsilon_1, \ldots, \epsilon_{n-1}) \mapsto J = \left\{i \in \{1, \ldots, n-1\} \middle|\, \epsilon_i = 1\right\}$$

Then one has

$$T_J = \sum_{J' \subseteq J} M_{J'} \quad \Longrightarrow \quad M_J = \sum_{J' \supseteq J} (-1)^{\#J' - \#J} \cdot T_{J'}$$

This proves the Theorem. ■

Special case: the *trivial order* on a finite set S, $\#S = n$.

Then $\#L_S = n!$, and the group $\mathrm{Aut}(S) \simeq Sym_n$ commutes with M_S. Hence for each isomorphism class of irreducible representations of Sym_n (i.e. a partition λ of n) we get a non-empty class of eigenvalues $\subset \oplus_\epsilon \mathbb{Z} \cdot t_\epsilon$ depending on λ.

Surprisingly, there are a lot of coincidences, and we get all together again *only* $p(n)$ different eigenvalues!
This means that we have a decomposition of the regular $n!$-dimensional representation of Sym_n into the sum of (highly reducible) subrepresentations R_λ labeled by partitions $\lambda \vdash n$.
It turns out that each R_λ has dimension equal to the cardinality of the conjugacy class $C_\lambda \subset S_n$ corresponding to λ.
Consider the centralizer of the element $(1)^{a_1}(2)^{a_2}\cdots \in Sym_n$ in conjugacy class C_λ, where $\sum_k a_k \cdot k = n$:

$$Cent_{a_1,a_2,\dots} = \prod_{k\geqslant 1} Sym_{a_k} \ltimes (\mathbb{Z}/k\mathbb{Z})^{a_k}, \qquad C_\lambda = Sym_n / Cent_{a_1,a_2,\dots}$$

There is a canonical 1-dimensional representation (character)

$$\chi = \chi_{a_1,a_2,\ldots} : Cent_{a_1,a_2,\ldots} \to \mathbb{C}^\times :$$

$$\chi_{|Sym_{a_k}} = 1, \quad \chi_{|(\mathbb{Z}/k\mathbb{Z})^{a_k}} : (u_1, \ldots, u_{a_k}) \mapsto e^{\frac{2\pi \mathrm{i}}{k}(u_1+\cdots+u_{a_k})}$$

Then

$$R_\lambda = \mathrm{Ind}_{Cent_{a_1,a_2,\ldots}}^{Sym_n} \chi_{a_1,a_2,\ldots}$$