Demi-shuffle duals of Magnus polynomials in a free associative algebra

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Plan
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$\boxed{2}$ Application
$\boxed{3}$ Prototype duality
$\boxed{4}$ Reutenauer's tool.

$\boxed{1}$ Intro.

k = fd of char. 0

$k\langle X, Y\rangle := k\langle W\rangle$ W = free unitary monoid $\Downarrow$ $\{X,Y\}^*$

w: monomial (word) in X, Y

has k-linear basis $\{w \mid w \in W\}$

∘ Magnus polynomial assoc. to $w = X^{k_1} Y X^{k_2} Y \cdots X^{k_d} Y X^{k_\infty}$

is defined by $(k_1, \ldots, k_d ; k_\infty) \in \mathbb{N}_0^d \times \mathbb{N}_0$

tailed tuple depth d

$$M(w) := ad_X^{k_1}(Y)\cdot ad_X^{k_2}(Y)\cdots ad_X^{k_d}(Y)\cdot X^{k_\infty}$$

where $ad_X(\#) = [X,\#] = X\cdot\# - \#\cdot X$.

<u>Ex.</u> $M(XYYX^2) = ad_X(Y)\cdot YX^2 = [X,Y]\cdot YX^2$

$$= XYYX^2 - YXYX^2$$

Th (Magnus 1937)

$\{M(w)\}_{w\in W}$ forms a k-linear basis of $k\langle X,Y\rangle$.

<u>Aim</u> Construct the dual basis $\{S(w)\mid w\in W\}$ with respect to the standard pairing

$$(w|w') = \delta_{w,w'} = \begin{cases} 0 & w\neq w' \\ 1 & w = w' \end{cases}$$

<u>Define</u> <u>demi-Shuffle poly $S(w)$</u> assoc. to $w\in W$

by $S(w) = (\cdots(X^{k_1}Y)\,\text{Ш}\,X^{k_2})Y)\cdots Y)\,\text{Ш}\,X^{k_d}).Y)\,\text{Ш}\,X^{k_\infty}$

Alternating operation $\left\{ \begin{array}{l} \text{Ш follows } X^{k_i} \\ \bullet \text{ follows } Y \end{array} \right\}$

Theorem (N)

$\{M(w)\}_{w \in W}$ and $\{S(w)\}_{w \in W}$ are mutually dual basis to each other. w.r.t the (monomial) pairing $(* \mid *)$.

i.e. $$\big(M(w) \mid S(w')\big) = \delta_{w,w'}$$

for all $w, w' \in W$.

NB. M.P. Schützenberger (1957), J.-L. Loday (1995) introduced half-shuffle (Zinbiel) operation $\prec$

$$(x_0 x_1 \cdots x_p) \prec (x_{p+1} \cdots x_{p+q}) := x_0 \big((x_1 \cdots x_p) ш (x_{p+1} \cdots x_{p+q})\big)$$

and its axiomatic formalizm has been studied by several authors

2 Application

Let Δ be the standard comultiplication of $k\langle\langle X,Y\rangle\rangle$ formal series

Def. $J \in k\langle\langle X,Y\rangle\rangle$ is called "group-like" series if

$$\begin{cases} \text{const term of } J = 1 \\ \Delta(J) = J \otimes J \end{cases}$$

Write $J = \sum_{w \in W} \underbrace{\mathrm{coeff}_w(J)}_{= (w|J)} . w$

Terminology $w \in W$ is called regular if

$$w = X^{k_1} Y X^{k_2} Y \cdots X^{k_d} Y \leftrightarrow (k_1, \cdots k_d ; 0)$$

Le-Murakami, Furusho formula (extended to arbitrary gp-like series)

All coeff of group-like series J can be written explicitly by regular coefficients:

Let $c_X := (J|X)$, $w = X^{k_1} Y X^{k_2} Y \cdots X^{k_d} Y X^{k_0}$

Then

$$\mathrm{Coeff}_w(J) = \sum_{s=0}^{k_0} \frac{c_X^s}{s!} \sum \pm (\text{prod. of bino. coeff}) \begin{pmatrix} \text{regular} \\ \text{coeff. of } J \end{pmatrix}$$

Background Original Le-Murakami, Furuslo formula

$k=\mathbb{C}$ KZ-associator along path $\vec{01} \rightsquigarrow z \in \mathbb{C}\setminus\{0,1\}$

def. by $$J(z) := 1 + \sum_{d=0}^{\infty} \mathrm{It}\int_{\vec{01}}^{z} \left(\frac{dz}{z}X + \frac{dz}{z-1}Y\right)^{d+1}$$

regular coef = convergent iterated integral
(multiple polylog)

irregular coef need "regularization"

The appearance of $S(w)$ playing certain roles for KZ-associator had been noted by Vincel Hoang Ngo Minh (~ 2019)

Analog: p-adic case (Furusho, Besser, ...)
ℓ-adic Galois case (Wojtkowich)

3 Proto-type duality (Schützenberger 1958)

$\forall\, w \in W$ has the standard Lyndon factorization

$$w = l_1^{k_1} l_2^{k_2} \cdots l_r^{k_r} \quad \left(\begin{array}{l} k_i \in \mathbb{N}_0,\ l_i : \text{“Lyndon word”} \\ l_1 > l_2 > \cdots \ \text{decreasing lex. order} \end{array} \right)$$

$\rightarrow$ associ. Poincaré-Birkhoff-Witt polynomial

$$P_w = P_{l_1}^{k_1} \cdots P_{l_r}^{k_r}$$

where for $l \in$ Lyndon

$P_X = X$, $P_Y = Y$, $P_l = [P_{l_1}, P_{l_2}]$

$l = l_1 \cdot l_2$ (st. factorization in Lyndon so that l_2: longest)

$\Rightarrow P_l \in \mathrm{Lie}(X,Y) \subset k\langle X,Y\rangle$

Ex. $l = XYXYY \in$ Lyndon

$$P_{XYXYY} = [[X,Y],[[X,Y],Y]]$$

$\rightarrow$ associ. shuffle polynomial

$$S_w := \frac{1}{k_1! \cdots k_r!}\, S_{l_1}^{\sqcup\!\sqcup k_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup S_{l_r}^{\sqcup\!\sqcup k_r}$$

with $S_1 = 1$

$S_l = a\, S_{l'}$ if $l = a \cdot l'$ for any $\forall\, l \in$ Lyndon

(while $l' \in W$)

Fact $\quad \mathrm{Lie}(X,Y) \subset k\langle X,Y\rangle$

$\uparrow$ k-linear basis $\{P_l\}_{l:\,\text{Lyndon}}$

$\uparrow$ k-linear dual bases $\{S_w\}_{w\in W} \leftrightarrow \{P_w\}_{w \in W}$

$\boxed{4}$ Reutenauer's tool: $\underbrace{\mathbb{Q}_k\langle X,Y\rangle}_{\sqcup\!\sqcup} \hat{\otimes} \underbrace{\mathbb{Q}_k\langle X,Y\rangle}_{\bullet}$

$\rightsquigarrow$ J = group like, $C_X = \mathrm{coeff}_X(J) = (J|X)$

$$J = \underbrace{\prod_{l>X} \exp\big((S_l|J)\cdot P_l\big)}_{\text{regular}} \cdot \underbrace{\left(\sum_{i=0}^{\infty} \frac{1}{i!}(C_X\cdot X)^i\right)}_{\text{irregular Tail}}$$

decreasing product

See that all coeff can be available from $\{$reg. coeff, & $C_X\}$

but nested def of S_l makes expression complicated.

Using our Magnus–DemiShuttle duality, the same

$(\{M(w)\} \leftrightarrow \{S(w)\})$

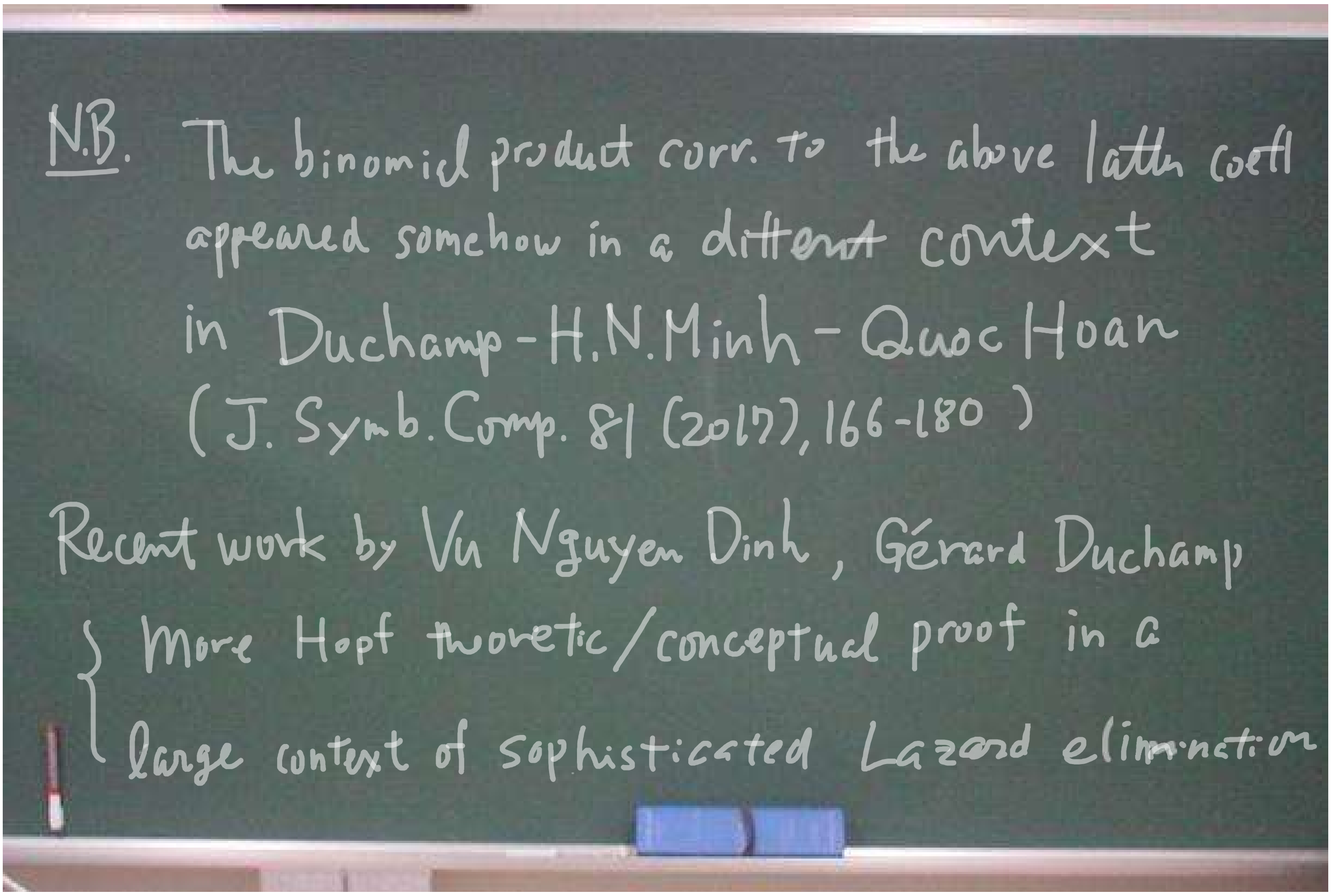
tool derives

$$J = \underbrace{\left(\sum_{w:\,\text{regular}} (S(w)|J)\,M(w)\right)}_{\text{can expressible by products of binomial coeff.}} \cdot \left(\sum_{i=0}^{\infty} \frac{C_X^i}{i!} X^i\right)$$

Reason: Explicit expansion

$$\begin{cases} M(w) = \sum (\quad)\cdot w \\ S(w) = \sum (\quad)\cdot w \end{cases}$$

elementary counting of words




```
> ### Standard Pairing ###
> Pairing(XYXYX, XYXYX)
                                1
> Pairing(XYXYY, XYXYX)
                                0
> Pairing(2·XYXYX + XYXYY, XYXYX)
                                2
> MagnusPoly(XYXYX)
                XYXYX − XYYXX − YXXYX + YXYXX
>
> demiShuffle(XYXYX)
        XYXYX + 2 XYXXY + 4 XXYXY + 2 XXYYX + 6 XXXYY
>
> Pairing(demiShuffle(XYXYX), MagnusPoly(XYXYX))
                                1
> Pairing(demiShuffle(XYXYY), MagnusPoly(XYXYX))
                                0
```

```
> ### ALL WORDS of two X's and three Y's ####
> wlist := WordList_SizeDepth(2, 3)
```

$$wlist := [YYYXX, YYXYX, YYXXY, YXYYX, YXYXY, YXXYY, XYYYX, XYYXY, XYXYY, XXYYY]$$

```
> N := nops(wlist)
```

$$N := 10$$

```
> ### checking standard pairing ###
> Pairing_WordList(wlist, wlist)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
>
```

```
> wlist := WordList_SizeDepth(2, 3); N := nops(wlist)
```

$$wlist := [YYYXX, YYXYX, YYXXY, YXYYX, YXYXY, YXXYY, XYYYX, XYYXY, XYXYY, XXYYY]$$

$$N := 10$$

```
> list1 := [seq(demiShuffle(wlist[i]), i = 1 ..N)]
```

$$list1 := [YYYXX + YYXYX + YYXXY + YXYYX + YXYXY + YXXYY + XYYYX + XYYXY + XYXYY + XXYYY, YYXYX + 2\,YYXXY + 2\,YXYXY + 2\,XYYXY + YXYYX + 2\,YXXYY + 2\,XYXYY + XYYYX + 2\,XXYYY, YYXXY + YXYXY + YXXYY + XYYXY + XYXYY + XXYYY, YXYYX + YXYXY + 2\,YXXYY + 2\,XYXYY + XYYYX + XYYXY + 2\,XXYYY, YXYXY + 2\,YXXYY + 2\,XYXYY + XYYXY + 2\,XXYYY, YXXYY + XYXYY + XXYYY, XYYYX + XYYXY + XYXYY + 2\,XXYYY, XYYXY + XYXYY + 2\,XXYYY, XYXYY + 2\,XXYYY, XXYYY]$$

```
> list2 := [seq(MagnusPoly(wlist[i]), i = 1 ..N)]
```

$$list2 := [YYYXX, YYXYX - YYYXX, YYXXY - 2\,YYXYX + YYYXX, YXYYX - YYXYX, YXYXY - YXYYX - YYXXY + YYXYX, YXXYY - 2\,YXYXY + YYXXY, XYYYX - YXYYX, XYYXY - XYYYX - YXYXY + YXYYX, XYXYY - XYYXY - YXXYY + YXYXY, XXYYY - 2\,XYXYY + YXXYY]$$

```
> ### checking demiShuffle-Magnus Duality ####
> Pairing_WordList(list1, list2)
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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>
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