Categorified Crystal Bases on Localized Quantum Coordinate Rings and Cellular Crystals

10th Combinatorics and Arithmetic for Physics

Institut des Hautes Études Scientifiques

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References

[KKOP] Kashiwara M., Kim M., Oh S-J. and Park E., Localization for quiver Hecke algebras, Pure Appl.Math. Q. 17(2021), no.4, 1465–1548.

[KKOP2] —–, Localization for quiver Hecke algebras II, arXiv:2208.01255.

[Kanakubo-N] Kanakubo Y. and Nakashima T., Half potential on geometric crystals and connectedness of cellular crystals, Transformation Groups **28** (2023), no.1, 327–373.

[Lauda-Vazirani] Lauda A.D. and Vazirani M., Crystals from categorified quantum groups, Adv.Math., **228**, (2011), 803–861.

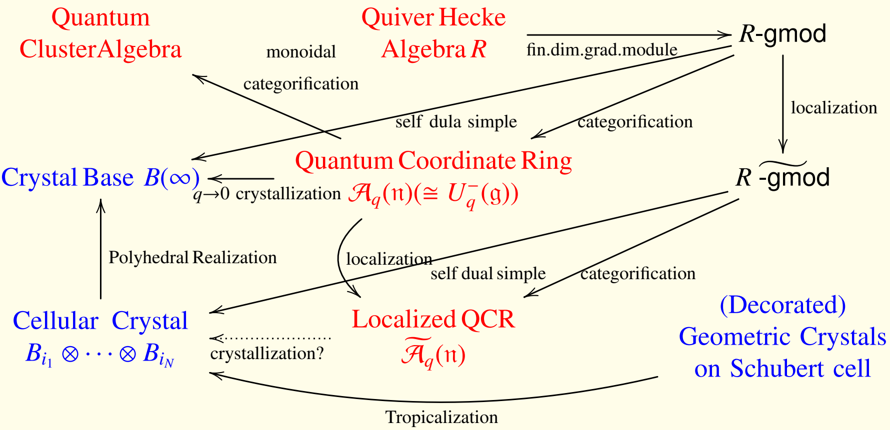
[N1] Nakashima T., Polyhedral Realizations of Crystal Bases and Braid -type Isomorphisms, Contemporary Mathematics **248**,(1999), 419–435.

[N2] Nakashima T., Polytopes for Crystallized Demazure Modules and Extremal Vectors, Commun. in Algebra, **30** (3), (2002), 1349–1367.

[N3] Nakashima T.,Categorified Crystal Structure on Localized Quantum Coordinate Ring, arXiv:2208.08396

[N-Zelevinsky] Nakashima T. and Zelevinsky A., Polyhedral Realizations of Crystal Bases for Quantized Kac-Moody Algebras, Adv. Math, **131**, No.1, (1997), 253–278.

Introduction



Preliminaries

- $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_- = \langle e_i, h_i, f_i \rangle$: Simple Lie algebra,
$A = (a_{ij})_{i,j \in I := \{1,2,\cdots,n\}}$: Cartan matrix for $\mathfrak{g}$
- $\{\alpha_i : i \in I\}$: set of simple roots, $\{h_i : i \in I\}$: set of simple coroots such that $a_{ij} = \alpha_j(h_i)$. Define the root lattice $Q := \bigoplus_i \mathbb{Z}\alpha_i \supset Q_+ := \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$. For $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k} \in Q_+$, define the height of β by $|\beta| = k$.
- (,) : symm.bilinear form on $\mathfrak{t}^*$ s.t. $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ and $\lambda(h_i) = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $\lambda \in \mathfrak{t}^*$. (We shall use the notation $\langle h_i, \lambda \rangle$ for $\lambda(h_i)$.)
- $P := \{\lambda \in \mathfrak{t}^* \,|\, \langle h_i, \lambda \rangle \in \mathbb{Z} (\forall i \in I)\}$: weight lattice$\supset P_+$: dominant weights
- $P^* := \{h \in \mathfrak{t} \,|\, \langle h, P \rangle \subset \mathbb{Z}\}$: dual weight lattice
- $W = \langle s_i \mid i \in I \rangle$: Weyl group ass. P.
- $U_q(\mathfrak{g}) := \langle e_i,\ f_i,\ q^h \rangle_{i \in I, h \in P^*}$: quantum algebra/$\mathbb{Q}(q)$
- $U_q^-(\mathfrak{g}) := \langle f_i \rangle_{i \in I}$, $U_q^+(\mathfrak{g}) := \langle e_i \rangle_{i \in I}$: nilpotent subalgebras

Quantum coordinate ring I

(Unipotent) quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ is defined as a restricted dual of $U_q^+(\mathfrak{g})$:

$$\mathcal{A}_q(\mathfrak{n}) = \bigoplus_{\beta \in Q_-} \mathcal{A}_q(\mathfrak{n})_\beta \qquad \mathcal{A}_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q))$$

$\exists$Embedding of algebra $\mathcal{A}_q(\mathfrak{n}) = \langle F_i^* \mid i \in I \rangle \hookrightarrow \mathcal{F}^*$(=quantum shuffle algebra).
Note that we get the isomorphism of $\mathbb{Q}(q)$-algebras

$$U_q^-(\mathfrak{g}) \quad \stackrel{\sim}{\longrightarrow} \quad \mathcal{A}_q(\mathfrak{n}) \qquad (f_i \longmapsto F_i^*).$$

The $\mathbb{Z}$-form $\mathcal{A}(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$ is defined by:

$$\mathcal{A}(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]} := \{a \in \mathcal{A}(\mathfrak{n}) \mid \langle a, U^+_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})\rangle \subset \mathbb{Z}[q, q^{-1}]\}.$$

Crystal Base of $U_q^-(\mathfrak{g}) \cong \mathcal{A}_q(\mathfrak{n})$

- Define the **Kashiwara operators** $\tilde{e}_i, \tilde{f}_i \in \mathrm{End}_{\mathbb{Q}(q)}(U_q^-(\mathfrak{g}))$ by

$$\tilde{e}_i\left(\sum_{k=0}^{l} f_i^{(k)} u_k\right) = \sum_{k=1}^{l} f_i^{(k-1)} u_k, \quad \tilde{f}_i\left(\sum_{k=0}^{l} f_i^{(k)} u_k\right) = \sum_{k=0}^{l} f_i^{(k+1)} u_k, \quad (u_k \in \mathrm{Ker}(e_i') \cap U_q^-(\mathfrak{g}))$$

Theorem (Kashiwara)

The **crystal base** $(L(\infty), B(\infty))$ *of* $U_q^-(\mathfrak{g})$ *is defined by*

$$L(\infty) := \sum_{k\geq 0, i_1,\cdots,i_k \in I} \mathbb{A}\tilde{f}_{i_1}\cdots\tilde{f}_{i_k}u_\infty \quad (u_\infty := 1 \in U_q^-(\mathfrak{g})),$$

($\mathbb{A} \subset \mathbb{Q}(q)$ is the local subring at $q = 0$)

$$B(\infty) = \{\tilde{f}_{i_1}\cdots\tilde{f}_{i_k}u_\infty \bmod qL(\infty) \mid k \geq 0, i_1, \cdots, i_k \in I\} \setminus \{0\},$$

$$\varepsilon_i(b) = \max\{k : \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle,$$

which satisfies: $U_q^-(\mathfrak{g}) \cong \mathbb{Q}(q) \otimes_{\mathbb{A}} L(\infty)$, *$B(\infty)$ is a $\mathbb{Q}$-basis of $L(\infty)/qL(\infty)$,* $\tilde{e}_i B(\infty) \subset B(\infty) \sqcup \{0\}$, $\tilde{f}_i B(\infty) \subset B(\infty)$ *and* $\tilde{e}_i u = v \Leftrightarrow \tilde{f}_i v = u$ *for* $\forall u, v \in B(\infty)$, $\forall i \in I$.

Crystals

A crystal is a combinatorial object obtained by abstracting the properties of crystal bases.

Definition (Crystal)

A 6-tuple $(B, \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\tilde{e}_i\}, \{\tilde{f}_i\})_{i\in I}$ is a crystal if B is a set and $\exists 0 \notin B$ and maps:

$$\mathrm{wt}: B \to P, \quad \varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\} \quad (i \in I)$$
$$\tilde{e}_i : B \sqcup \{0\} \to B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\} \ (i \in I),$$

satisfying :

1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$.
2. If $b, \tilde{e}_i b \in B$, then $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.
3. If $b, \tilde{f}_i b \in B$, then $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.
4. For $b, b' \in B$ and $i \in I$, one has $\tilde{f}_i b = b' \iff b = \tilde{e}_i b'$.
5. If $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = \tilde{f}_i b = 0$ and $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$.

Definition

For a crystal B, its crystal graph is an oriented I-colored graph defined by

$$b \xrightarrow{i} b' \Leftrightarrow \tilde{f}_i b = b'$$

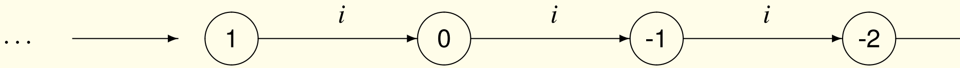
Examples of Crystals

Example

For $i \in I$, set $B_i := \{(n)_i \mid n \in \mathbb{Z}\}$ and its crystal structure is given by

$$
\begin{aligned}
&\mathrm{wt}((n)_i) = n\alpha_i,\ \ \varepsilon_i((n)_i) = -n,\ \ \varphi_i((n)_i) = n,\\
&\varepsilon_j((n)_i) = \varphi_j((n)_i) = -\infty \ \ (i \neq j),\\
&\tilde{e}_i((n)_i) = (n+1)_i, \quad \tilde{f}_i((n)_i) = (n-1)_i,\\
&\tilde{e}_j((n)_i) = \tilde{f}_j((n)_i) = 0 \ \ (i \neq j)
\end{aligned}
$$

Crystal graph of B_i:



Explicit Crystal Structure of $B_{i_1} \otimes \cdots \otimes B_{i_m}$ I

It is well-known that crystals hold a natural tensor product structure. Then, for crystals $B_{i_1}, \cdots, B_{i_m}$, their tensor product $B_{i_1} \otimes \cdots \otimes B_{i_m}$ is defined.

Remark. As a set $B_{i_1} \otimes \cdots \otimes B_{i_m} = B_{i_1} \times \cdots \times B_{i_m}$.

Fix a sequence of indices $\mathbf{i} = (i_1, \cdots, i_m) \in I^m$ and write

$$(x_1, \cdots, x_m) := \tilde{f}_{i_1}^{x_1}(0)_{i_1} \otimes \cdots \otimes \tilde{f}_{i_m}^{x_m}(0)_{i_m} = (-x_1)_{i_1} \otimes \cdots \otimes (-x_m)_{i_m},$$

where if $n < 0$, then $\tilde{f}_i^n(0)_i$ means $\tilde{e}_i^{-n}(0)_i$.

The crystal structure on $B_{i_1} \otimes \cdots \otimes B_{i_m}$ is given by: for $x = (x_1, \cdots, x_m)$, define

$$\sigma_k(x) := x_k + \sum_{j<k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j \quad (k \in [1, m]),$$

$$\widetilde{\sigma}^{(i)}(x) := \max\{\sigma_k(x) \mid 1 \le k \le m \,\text{and}\, i_k = i\}, \quad (i \in I),$$

$$\widetilde{M}^{(i)} = \widetilde{M}^{(i)}(x) := \{k \mid 1 \le k \le m,\ i_k = i,\ \sigma_k(x) = \widetilde{\sigma}^{(i)}(x)\} \quad (i \in I),$$

$$\widetilde{m}_f^{(i)} = \widetilde{m}_f^{(i)}(x) := \max \widetilde{M}^{(i)}(x), \quad \widetilde{m}_e^{(i)} = \widetilde{m}_e^{(i)}(x) := \min \widetilde{M}^{(i)}(x) \quad (i \in I).$$

Explicit Crystal Structure of $B_{i_1} \otimes \cdots \otimes B_{i_m}$ II

Now, we have the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ and the functions ε_i, φ_i and wt as

$$\tilde{f}_i(x)_k := x_k + \delta_{k,\widetilde{m}_f^{(i)}}, \qquad \tilde{e}_i(x)_k := x_k - \delta_{k,\widetilde{m}_e^{(i)}},$$

$$\mathrm{wt}(x) := -\sum_{k=1}^{m} x_k \alpha_{i_k}, \quad \varepsilon_i(x) := \widetilde{\sigma}^{(i)}(x), \quad \varphi_i(x) := \langle h_i, \mathrm{wt}(x) \rangle + \varepsilon_i(x).$$

Function β_k

For $\mathbf{i} = i_1, \cdots, i_m \in I^m$ and $k \in [1, m]$, define
$k^+ := \min(\{l \in [1, m] \mid i_k = i_l, k < l\} \sqcup \{m+1\})$. For $x = (x_1, \cdots, x_m)$, define

$$\beta_k(x) := \sigma_{k^+}(x) - \sigma_k(x) = x_k + \sum_{k<j<k^+} \langle h_i, \alpha_{i_j} \rangle x_j + x_{k^+} \quad (k^+ \le m)$$

Then, $\widetilde{m}_f^{(i)}$ and $\widetilde{m}_e^{(i)}$ are determined by $\{\beta_k(x) \mid i_k = i\}$.

Kashiwara embedding and Polyhedral Realization

Theorem (Kashiwara embedding)

For the crystal $B(\infty)$ of $U_q^-(\mathfrak{g})$ and any $i \in I$, $\exists$strict embedding of crystals:

$$\Psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i, \qquad u_\infty \mapsto u_\infty \otimes (0)_i.$$

Iterating these according to $i_1, i_2, \ldots, i_l$ $(i_j \in I)$, we obtain the embedding:

$$\Psi_{i_1,\ldots,i_l} := \Psi_{i_1} \circ \cdots \circ \Psi_{i_l} : B(\infty) \hookrightarrow B(\infty) \otimes B_{i_1} \otimes \cdots \otimes B_{i_l}.$$

In (semi-)simple setting, for any reduced longest word $\mathbf{i} = i_1 \cdots i_N$, we obtain ([N2])

$$\Psi_{\mathbf{i}} : B(\infty) \hookrightarrow u_\infty \otimes B_{i_1} \otimes \cdots \otimes B_{i_N} \cong \mathbb{Z}^N \quad (N := \mathrm{length}(w_0))$$

Polyhedral Realization [N-Zelevinsky]

The image $\mathrm{Im}(\Psi_{\mathbf{i}}) \cong B(\infty)$ is described by "polyhedral realization" method.

Example

A_2-case: for $\mathbf{i} = 121$, $B(\infty) \cong \mathrm{Im}(\Psi_{121}) = \{(x, y, z) \in \mathbb{Z}^3 \mid 0 \le x \le y,\ z \ge 0\}$.

Braid-type isomorphisms

Proposition (Braid-type isomorphisms [N1])

There exist the following isomorphisms of crystals $\phi_{ij}^{(k)}$ $(k = 0, 1, 2, 3)$:

$$\begin{aligned}
&\phi_{ij}^{(0)} : B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i, && a_{ij}a_{ji} = 0,\\
&\phi_{ij}^{(1)} : B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_j \otimes B_i \otimes B_j, && a_{ij}a_{ji} = 1,\\
&\phi_{ij}^{(2)} : B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i, && a_{ij}a_{ji} = 2,\\
&\phi_{ij}^{(3)} : (B_i \otimes B_j)^{\otimes 3} \xrightarrow{\sim} (B_j \otimes B_i)^{\otimes 3}, && a_{ij}a_{ji} = 3.
\end{aligned}$$

They also satisfy $\phi_{ij}^{(k)} \circ \phi_{ji}^{(k)} = \mathrm{id}$.

For any $w \in W$ and its reduced words $i_1 \cdots i_l$ and $j_1 \cdots j_l$,

$$B_{i_1} \otimes \cdots \otimes B_{i_l} \cong B_{j_1} \otimes \cdots \otimes B_{j_l}$$

Cellular Crystal $\mathbb{B}_{\mathbf{i}} = \mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes \cdots \otimes B_{i_k}$

For a reduced word $\mathbf{i} = i_1 i_2 \cdots i_k$, we call the crystal $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_k}$ a **cellular crystal** associated with $\mathbf{i}$. Indeed, it is obtained by the tropicalization from the positive geometric crystal on the Langlands dual Schubert cell $X_w^\vee$ $(w = s_{i_1} \cdots s_{i_k})$.

Theorem ([Kanakubo-N])

For any simple Lie algebra $\mathfrak{g}$ and any reduced word $i_1 i_2 \cdots i_k$,
the cellular crystal $\mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_k}$ is connected (as a crystal graph).

$N = l(w_0)$: the length of the longest element. For $\forall k \leq N$,

$$\mathbb{B}_{i_1 i_2 \cdots i_N} \text{ is connected} \Longrightarrow \mathbb{B}_{i_1 i_2 \cdots i_k} \text{ is connected}$$

since $B_1 \otimes B_2$ is connected $\Longrightarrow$ both B_1 and B_2 are connected.

Cellular Crystal $\mathbb{B}_{\mathbf{i}}$ – Subspace $\mathcal{H}_{\mathbf{i}}$

Fix a longest reduced word $\mathbf{i} = i_1 \cdots i_N$ and take the function β_k as above:

$$\beta_k(x) = x_k + \sum_{k<j<k^+} \langle h_{i_k} \alpha_{i_j} \rangle x_j + x_{k^+} \quad (1 \le k \le N)$$

Now, we define $\mathcal{H}_{\mathbf{i}} \subset \mathbb{Z}^N$, which is the key object for this talk.

$$\mathcal{H}_{\mathbf{i}} := \{x \in \mathbb{Z}^N (= \mathbb{B}_{\mathbf{i}}) \,|\, \beta_k(x) = 0 (\forall k \text{ s.t. } k^+ \le N)\} \subset \mathbb{B}_{\mathbf{i}}$$

Proposition (Kanakubo-N)

For $\mathbf{i} = i_1 i_2 \cdots i_N$, $k = 1, 2, \cdots, N$ *and a fundamental weight* Λ_i, *we define*

$$h_i^{(k)} := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda_i \rangle, \quad \mathbf{h}_i := (h_i^{(1)}, h_i^{(2)}, \ldots, h_i^{(N)}) \in \mathbb{B}_{\mathbf{i}}$$

$$\Longrightarrow \mathcal{H}_{\mathbf{i}} = \mathbb{Z}\mathbf{h}_1 \oplus \mathbb{Z}\mathbf{h}_2 \oplus \cdots \oplus \mathbb{Z}\mathbf{h}_n$$

By the fact that $B(\infty)$ is connected and $B(\infty) \subset \mathbb{B}_{\mathbf{i}} = \mathbb{Z}^N$ ($\forall \mathbf{i}$), we obtain

Lemma (Kanakubo-N, N3)

For $h \in \mathcal{H}_{\mathbf{i}}$, define

$$B^h(\infty) := \{x + h \in \mathbb{Z}^N = \mathbb{B}_{\mathbf{i}} \mid x \in B(\infty)\}.$$

1. *For any $x + h \in B^h(\infty)$ and $i \in I$, we obtain*

$$\tilde{e}_i(x+h) = \tilde{e}_i(x) + h, \qquad \tilde{f}_i(x+h) = \tilde{f}_i(x) + h,$$

   and then $B^h(\infty)$ is connected.

2. *For any $h \in \mathcal{H}_{\mathbf{i}}$, we have $B(\infty) \cap B^h(\infty) \neq \emptyset$.*
3. $$\mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^h(\infty)$$

By this lemma, we can show that $\mathbb{B}_{\mathbf{i}}$ is connected.

For a finite index set I and a field $\mathbf{k}$, let $(Q_{i,j}(u,v))_{i,j\in I} \subset \mathbf{k}[u,v]$ be polynomials satisfying: $Q_{i,j}(u,v) = Q_{j,i}(v,u)$, $Q_{i,i}(u,v) = 0$ for any $i, j \in I$ and some other conditions. For $\beta = \sum_i m_i\alpha_i \in Q_+$ with $|\beta| := \sum_i m_i = m$.

Definition

For $\beta \in Q_+$, the quiver Hecke algebra $R(\beta)$ associated with a Cartan matrix $A = (a_{ij})_{i,j=1,2,\ldots,n}$ and polynomials $(Q_{ij}(u,v))_{i,j\in I}$ is the algebra generated by

$$\{e(\nu) | \nu \in I^\beta := \{((\nu_1, \cdots, \nu_m) \mid \sum_{k=1}^{m} \alpha_{\nu_k} = \beta\}\}, \quad \{x_k | 1 \le k \le n\}, \quad \{\tau_i | 1 \le i \le n-1\}$$

with the relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu), \quad \sum_{\nu\in I^\beta} e(\nu) = 1, \quad e(\nu)x_k = x_k e(\nu), \quad x_k x_l = x_l x_k,$$

$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k\tau_l = \tau_l\tau_k \text{ if } |k-l| > 1,$$

Quiver Hecke Algebra II

$$\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k,\ \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1,\ \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \overline{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\overline{Q}_{i,j}(u,v,w) = \frac{Q_{i,j}(u,v)-Q_{i,j}(w,v)}{u-w} \in \mathbf{k}[u,v,w]$ and set $R := \bigoplus_{\beta\in Q_+} R(\beta)$.

Grading

The relations above are homogeneous if we define

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

Thus, $R(\beta)$ becomes a $\mathbb{Z}$-graded algebra. Here we define the *weight* of $R(\beta)$-module M as $\mathrm{wt}(M) = -\beta$.

R-modules I

R-modules

1. Define a *grading shift functor* q on a $\mathbb{Z}$-graded $R(\beta)$-module $M = \bigoplus_{k\in\mathbb{Z}} M_k$ by:

$$qM := \bigoplus_{k\in\mathbb{Z}} (qM)_k, \quad \text{where } (qM)_k = M_{k-1}.$$

2. For $M \in R(\beta)$-Mod and $N \in R(\beta')$-Mod, define the convolution product by

$$M \circ N := R(\beta+\beta')e(\beta,\beta') \otimes_{R(\beta)\otimes R(\beta')} (M \otimes N) \quad (e(\beta,\beta') := \sum_{\nu\in I^{\beta},\, \nu'\in I^{\beta'}} e(\nu,\nu'))$$

3. $M \nabla N := \mathrm{hd}(M \circ N)$ (head), $M \Delta N := \mathrm{soc}(M \circ N)$ (socle), where the head of a module is the quotient by its radical and the socle of a module is the summation of all its simple submodules.
4. A simple R-module M is *real* $\Longleftrightarrow M \circ M$ is simple.
5. If $M \cong M^*$, we say M is *self-dual*.

Categorification of $U_q^-(\mathfrak{g})$ and $\mathcal{A}_q(\mathfrak{n})$

$R(\beta)\text{-gmod}$: Category of finite-dimensional graded $R(\beta)$-modules
$R(\beta)\text{-gproj}$: Category of finitely generated graded projective $R(\beta)$-modules
$*$ N.B. These categories are monoidal categories by the convolution product.

Theorem ([Khovanov-Lauda, Rouquier])

Let $\mathcal{K}(R\text{-gmod})$ (resp. $\mathcal{K}(R\text{-gproj})$) be the Grothendieck ring of the monoidal category $R\text{-gmod}$ (resp. $R\text{-gproj}$). Then we obtain

$$\mathcal{K}(R\text{-gproj}) \cong U_q^-(\mathfrak{g})_{\mathbb{Z}}, \qquad \mathcal{K}(R\text{-gmod}) \cong \mathcal{A}_q(\mathfrak{n})_{\mathbb{Z}}$$

Define the functors

$$E_i \quad : \quad R(\beta)\text{-gmod} \to R(\beta-\alpha_i)\text{-gmod by} \quad E_i(M) := e(\alpha_i, \beta-\alpha_i)M$$
$$F_i \quad : \quad R(\beta)\text{-gmod} \to R(\beta+\alpha_i)\text{-gmod by} \quad F_i(M) = L(i) \circ M,$$

where $e(\alpha_i, \beta-\alpha_i) := \sum_{\nu \in I^\beta, \nu_1 = i} e(\nu)$ and $L(i)$ is a 1-dim. simple $R(\alpha_i)$-module. They satisfy e.g., $E_iF_i = q_i^{-2}F_iE_i + \mathrm{id}$ (q-boson relation) and q-Serre relations.

Categorfication of $B(\infty)$ by Lauda and Vazirani

For a simple module $M \in R(\beta)$-gmod, define

$$\begin{aligned}
&\mathrm{wt}(M) = -\beta, \\
&\varepsilon_i(M) = \max\{n \in \mathbb{Z} \,|\, E_i^n M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \mathrm{wt}(M)\rangle, \\
&\widetilde{E}_i M := q_i^{1-\varepsilon_i(M)} \mathrm{soc}(E_i M) \cong q_i^{\varepsilon_i(M)-1} \mathrm{hd}(E_i M) \quad (q_i := q^{\frac{(\alpha_i,\alpha_i)}{2}}), \\
&\widetilde{F}_i M := q_i^{\varepsilon_i(M)} \mathrm{hd}(F_i M).
\end{aligned}$$

Set $\mathbb{B}(R\text{-gmod}) := \{S \,|\, S$ is a self-dual simple module in R-gmod$\}$

Theorem (Lauda-Vazirani)

The 6-tuple, $(\mathbb{B}(R\text{-gmod}), \{\widetilde{E}_i\}, \{\widetilde{F}_i\}, \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\})$ holds a crystal structure and there exists the following isomorphism of crystals:

$$\Psi : \mathbb{B}(R\text{-}gmod) \xrightarrow{\sim} B(\infty)$$

Braiders and Real Commuting Family I

Let Λ be $\mathbb{Z}$-lattice and $\mathcal{T} = \oplus_{\lambda\in\Lambda}\mathcal{T}_\lambda$ be a $\mathbf{k}$-linear Λ-graded monoidal category with $1 \in \mathcal{T}_0$ and the bifunctor $\circ : \mathcal{T}_\lambda \times \mathcal{T}_\mu \to \mathcal{T}_{\lambda+\mu}$. (Later Λ will be the root lattice Q)

Definition ([KKOP])

q: grading shift functor on $\mathcal{T}$. A **graded braider** *is a triple (C, R_C, ϕ), where $C \in \mathcal{T}$, $\mathbb{Z}$-linear map $\phi : \Lambda \to \mathbb{Z}$ and a morphism:*

$$R_C : C \circ X \to q^{\phi(\lambda)} X \circ C \quad (X \in \mathcal{T}_\lambda),$$

which is functorial in X and satisfying the commutative diagram

$$\begin{array}{ccc} C \circ X \circ Y & \xrightarrow{R_C(X)\circ Y} & q^{\phi(\lambda)}\, X \circ C \circ Y \\ & \searrow^{R_C(X\circ Y)} & \downarrow{\scriptstyle X\circ R_C(Y)} \\ & & q^{\phi(\lambda+\mu)}(X \circ Y) \circ C \end{array} \qquad (X \in \mathcal{T}_\lambda,\ Y \in \mathcal{T}_\mu)$$

Braiders and Real Commuting Family II

Let I be an index set and $\Gamma := \oplus_{i\in I}\mathbb{Z}e_i$ and $\Gamma_+ := \oplus_{i\in I}\mathbb{Z}_{\geq 0}e_i$. (Later Γ will be the weight lattice P and Γ_+ be the set of dominant weights P_+.)

Definition ([KKOP])

We say $(C_i, R_{C_i}, \phi_i)_{i\in I}$ *a* **real commuting family of graded braiders** *in* $\mathcal{T}$ *if*

1. $C_i \in \mathcal{T}_{\lambda_i}$ *for some* $\lambda_i \in \Lambda$, *and* $\phi_i(\lambda_i) = 0$, $\phi_i(\lambda_j) + \phi_j(\lambda_i) = 0$ $(i, j \in I)$.
2. $R_{C_i}(C_i) \in \mathbf{k}^\times \mathrm{id}_{C_i \circ C_i}$ $(i \in I)$, $\quad R_{C_i}(C_j) \circ R_{C_j}(C_i) \in \mathbf{k}^\times \mathrm{id}_{C_i \circ C_j}$ $(i, j \in I)$.
   (Note: $R_{C_i}(C_j)$*'s satisfy the "Yang-Baxter equation" on* $C_i \circ C_j \circ C_k$*.)*

Lemma ([KKOP])

For a real commuting family $(C_i, R_{C_i}, \phi_i)_{i\in I}$, $\exists$*bilinear map* $H : \Gamma \times \Gamma \to \mathbb{Z}$ *such that* $\phi_i(\lambda_j) = H(e_i, e_j) - H(e_j, e_i)$ *and there exist*

1. *an object* C^α *for any* $\alpha \in \Gamma_+$.
2. *an isomorphism* $\xi_{\alpha,\beta} : C^\alpha \circ C^\beta \xrightarrow{\sim} q^{H(\alpha,\beta)} C^{\alpha+\beta}$ *for any* $\alpha, \beta \in \Gamma_+$

such that $C^0 = 1$ *and* $C^{e_i} = C_i$ *and satisfying some commutative diagrams.*

Localization I

Let $(C_i, R_{C_i}, \phi_i)_{i\in I}$ be a real commuting family in $\mathcal{T}$ and $\{C^\alpha\}_{\alpha\in\Gamma_+}$ be objects as in the previous lemma. We define a partial order $\leq$ on Γ by

$$\alpha \leq \beta \Longleftrightarrow \beta - \alpha \in \Gamma_+$$

For $\alpha_1, \alpha_2, \cdots \in \Gamma$, define

$$\mathcal{D}_{\alpha_1,\alpha_2,\cdots} := \{\delta \in \Gamma \,|\, \alpha_i + \delta \in \Gamma_+ \ \forall i = 1, 2, \cdots\}$$

For $(X, \alpha), (Y, \beta) \in \mathrm{Ob}(\mathcal{T}) \times \Gamma$ ($X \in \mathcal{T}_\lambda,\ Y \in \mathcal{T}_\mu,\ \alpha, \beta \in \Gamma$) and $\delta \in \mathcal{D}_{\alpha,\beta}$, set

$$H_\delta((X, \alpha), (Y, \beta)) := \mathrm{Hom}_{\mathcal{T}}(C^{\delta+\alpha} \circ X, q^{H(\delta,\beta-\alpha)+\phi(\delta+\beta,\mu)} Y \circ C^{\beta+\delta}),$$

where $\phi : \Gamma \times \Lambda \to \mathbb{Z}$ $(e_i, \lambda) \mapsto \phi_i(\lambda)$. For $\delta \leq \delta'$ define the map

$$\zeta_{\delta,\delta'} : H_\delta((X, \alpha), (Y, \beta)) \to H_{\delta'}((X, \alpha), (Y, \beta))$$

by some proper way.

Localization II

Lemma ([KKOP])

In the setting above, we obtain

$$\zeta_{\delta,\delta'} \circ \zeta_{\delta',\delta''} = \zeta_{\delta,\delta''} \text{ for } \delta \le \delta' \le \delta''$$

and then we find that $\{H_\delta((X,\alpha),(Y,\beta))\}_{\delta\in\mathcal{D}_{\alpha,\beta}}$ *becomes an inductive system.*

Definition (Localization [KKOP])

We define the **localization** *of monoid.cat.* $\mathcal{T}$ *denoted by* $\widetilde{\mathcal{T}}$ *or* $\mathcal{T}[C_i^{\circ -1} \mid i \in I]$:

$$\mathrm{Ob}(\widetilde{\mathcal{T}}) := \mathrm{Ob}(\mathcal{T}) \times \Gamma,$$

$$\mathrm{Hom}_{\widetilde{\mathcal{T}}}((X,\alpha),(Y,\beta)) := \varinjlim_{\delta\in\mathcal{D}(\alpha,\beta),\ \lambda+L(\alpha)=\mu+L(\beta)} H_\delta((X,\alpha),(Y,\beta)),$$

$$(X,\alpha) \circ (Y,\beta) := (q^{-\phi(\beta,\lambda)+H(\alpha,\beta)}(X \circ Y), \alpha+\beta),$$

where $X \in \mathcal{T}_\lambda$, $Y \in \mathcal{T}_\mu$ *and* $L : \Gamma \to \Lambda$ $(e_i \mapsto \lambda_i)$

Localization III

Theorem ([KKOP])

$\widetilde{\mathcal{T}}$ *becomes a monoidal category. Moreover, there exists an* **exact monoidal functor** $\Upsilon : \mathcal{T} \to \widetilde{\mathcal{T}}$ *with certain universality s.t.*

1. $\Upsilon(C_i)$ *is* **invertible** *(*$\Leftrightarrow X \mapsto X \circ \Upsilon(C_i)$ *and* $X \mapsto \Upsilon(C_i) \circ X$ *are equiv. of cat.)*
2. $\forall i \in I$ *and* $\forall X \in \mathcal{T}$, $\Upsilon(R_{C_i}(X)) : \Upsilon(C_i \circ X) \to \Upsilon(X \circ C_i)$ *is an isomorphism.*

Proposition ([KKOP])

Under the setting above, we obtain

1. $(X, \alpha + \beta) \cong q^{-H(\beta,\alpha)}(C^\alpha \circ X, \beta)$, $(1, \beta) \circ (1, -\beta) \cong q^{-H(\beta,\beta)}(1, 0)$ *for* $\alpha \in \Gamma_+, \beta \in \Gamma$ *and* $X \in \widetilde{\mathcal{T}}$.
2. $\mathcal{T}$ *is an abelian category* $\Longrightarrow$ *so is* $\widetilde{\mathcal{T}}$.
3. *If the functor* $- \circ Y$ *and* $Y \circ -$ *are exact for any* Y *in* $\mathcal{T}$*, then the functors* $\widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ $(X \mapsto X \circ Y$ *(resp.* $X \to Y \circ X$*)) are exact for any* Y *in* $\widetilde{\mathcal{T}}$.

Determinantial Modules

Let us find "**real commuting family of graded braiders**" in R-gmod. Let $L(i^n) := q_i^{\frac{n(n-1)}{2}} L(i)^{\circ n}$ be a simple $R(n\alpha_i)$-module satisfying $\mathrm{qdim}(L(i^n)) = [n]_i!$.

Definition

For $M \in R$-gmod, define

$$\widetilde{F}_i^n(M) := L(i^n) \nabla M.$$

For a Weyl group element w, let $s_{i_1} \cdots s_{i_l}$ be its reduced expression.
For $\Lambda \in P_+$ and w, set

$$m_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_l} \Lambda \rangle \qquad (k = 1, \cdots, l).$$

We define the **determinantial module** *associated with w and Λ by*

$$\mathbf{M}(w\Lambda, \Lambda) := \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_l}^{m_l} \mathbf{1},$$

which does not depend on the choice of $i_1 \dots i_l$.

Definition (KKOP)

M: simple R-module. A graded braider (M, R_M, ϕ) is **non-degenerate** *if $R_M(L(i)) : M \circ L(i) \to q^{\phi(\alpha_i)} L(i) \circ M$ is non-zero.*

Set $C_\Lambda := \mathbf{M}(w_0\Lambda, \Lambda)$. In particular, for $i \in I$ set $C_i = C_{\Lambda_i}$. Then we obtain

Theorem ([KKOP])

Define the function $\phi_{C_i} : Q \to \mathbb{Z}$ by $\phi_{C_i}(\beta) := -(\beta, w_0\Lambda_i + \Lambda_i)$
$\Longrightarrow \exists\{(C_i, R_{C_i}, \phi_{C_i})\}_{i\in I}$ a non-deg. real comm. family of graded braiders in R-gmod.
Take $\Gamma = P$ and $\Gamma_+ = P_+$. Then, we obtain the localization of R-gmod

$$\widetilde{R\text{-gmod}} := R\text{-gmod}[C_i^{\circ -1} \mid i \in I]$$

by $\{(C_i, R_{C_i}, \phi_{C_i})\}_{i\in I}$.

Its Grothendieck ring $\mathcal{K}(\widetilde{R\text{-gmod}})$ defines the **localized quantum coordinate ring** $\widetilde{\mathcal{A}_q(\mathfrak{n})} := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{K}(\widetilde{R\text{-gmod}})$.

Localization $\widetilde{R\text{-gmod}}$ II

Proposition (KKOP)

We get $\mathcal{K}(\widetilde{R\text{-gmod}}) \cong \mathcal{S}^{-1}\mathcal{K}(R\text{-gmod})$= *the left ring of quotients of the ring* $\mathcal{K}(R\text{-gmod})(\cong \mathcal{A}_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]})$ *with respect to the multiplicative set*

$$\mathcal{S} := \{q^k \prod_{i\in I}[C_i]^{a_i} \mid k \in \mathbb{Z},\ (a_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}\}$$

Proposition (KKOP)

Let $\Phi : R\text{-gmod} \to \widetilde{R\text{-gmod}}$ *be the canonical functor. Then,*

1. $\widetilde{R\text{-gmod}}$ *is an abelian category and the functor* Φ *is exact.*
2. $\widetilde{C}_i := \Phi(C_i)$ $(i \in I)$ *is invertible central graded braider in* $\widetilde{R\text{-gmod}}$.
3. $S \in R\text{-gmod}$ *is simple* $\Longrightarrow \Phi(S)$ *is simple in* $\widetilde{R\text{-gmod}}$.

   For $\nu \in P$, *define* $\widetilde{C}_\nu$ *by* $\widetilde{C}_{\lambda+\mu} = \widetilde{C}_\lambda \circ \widetilde{C}_\mu$ *(up to grading) and* $\widetilde{C}_{-\Lambda_i} = C_i^{\circ -1}$
4. *For* $\forall$*simple* $M \in \widetilde{R\text{-gmod}}$, *simple* $\exists S \in R\text{-gmod}$ *and* $\exists \Lambda \in P$ *s.t.* $M \cong \widetilde{C}_\Lambda \circ \Phi(S)$ *(not necessarily unique)* .

Crystal Structure on $R\text{-}\widetilde{\text{gmod}}$ I

Lemma ([KOPP])

For $\forall$simple $M \in R\text{-}\widetilde{\text{gmod}}$, $\exists! n \in \mathbb{Z}$ s.t. $q^n M$ is self-dual simple, denoted by $\delta(M)$.

For a simple object $\widetilde{C}_\Lambda \circ \Phi(S) \in R\text{-}\widetilde{\text{gmod}}$ we write simply $C_\Lambda \circ S$.
Set $\mathbb{B}(R\text{-}\widetilde{\text{gmod}}) := \{S \mid S \text{ is a self-dual simple module in } R\text{-}\widetilde{\text{gmod}}\}$

The actions of the Kashiwara operators [N3]

Define the Kashiwara operators $\widetilde{F}_i$ and $\widetilde{E}_i$ ($i \in I$) on $\mathbb{B}(R\text{-}\widetilde{\text{gmod}})$:

$$\widetilde{F}_i(C_\Lambda \circ S) = q^{\delta(C_\Lambda \circ \widetilde{F}_i S)} C_\Lambda \circ \widetilde{F}_i S,$$

$$\widetilde{E}_i(C_\Lambda \circ S) = \begin{cases} q^{\delta(C_\Lambda \circ \widetilde{E}_i S)} C_\Lambda \circ \widetilde{E}_i S & \text{if } E_i S \neq 0, \\ q^{\delta(C_{\Lambda-\Lambda_{i^*}} \circ (\widetilde{E}_i C_{\Lambda_{i^*}} \circ S))} C_{\Lambda-\Lambda_{i^*}} \circ (\widetilde{E}_i C_{\Lambda_{i^*}} \circ S) & \text{if } E_i S = 0, \end{cases}$$

where δ is given in the above lemma and $i^* \in I$ is the index satisfying $\Lambda_{i^*} = -w_0 \Lambda_i$.

Crystal Structure on $\widetilde{R\text{-gmod}}$ II

Crystal structure: ε_i and wt [N3]

Let $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ (Lauda-Vazirani). For $C_\Lambda \circ S \in \mathbb{B}(\widetilde{R\text{-gmod}})$, define

$$\mathrm{wt}(C_\Lambda \circ S) = \mathrm{wt}(\Psi(S)) + w_0\Lambda - \Lambda,$$
$$\varepsilon_i(C_\Lambda \circ S) = \varepsilon_i(\Psi(S)) - \langle h_i, w_0\Lambda \rangle,$$
$$\varphi_i(C_\Lambda \circ S) = \varepsilon_i(\Psi(C_\Lambda \circ S)) + \langle h_i, \mathrm{wt}(C_\Lambda \circ S) \rangle.$$

Theorem ([N3])

The 6-tuple $(\mathbb{B}(\widetilde{R\text{-gmod}}), \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\widetilde{E}_i\}, \{\widetilde{F}_i\})_{i\in I}$ is a crystal.

Indeed, we should show that well-definedness, i.e.,all data do not depend on the presentation $C_\Lambda \circ S \cong C_{\Lambda'} \circ S'$ and for $b = C_\Lambda \circ S$,

$$\widetilde{E}_i\widetilde{F}_i b = \widetilde{F}_i\widetilde{E}_i b = b,$$
$$\varepsilon_i(\widetilde{F}_i(b)) = \varepsilon_i(b) + 1, \qquad \varepsilon_i(\widetilde{E}_i(b)) = \varepsilon_i(b) - 1,$$
$$\mathrm{wt}(\widetilde{E}_i b) = \mathrm{wt}(b) + \alpha_i, \qquad \mathrm{wt}(\widetilde{F}_i b) = \mathrm{wt}(b) - \alpha_i.$$

As we have seen above that the set $\mathcal{H}_{\mathbf{i}} \subset \mathbb{B}_{\mathbf{i}}$ is presented by

$$\mathcal{H}_{\mathbf{i}} = \bigoplus_{i \in I} \mathbb{Z}\mathbf{h}_i, \quad \mathbf{h}_i = ((h_i^{(k)} := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda_i \rangle)_{k=1,\cdots,N}$$

For $\Lambda \in P_+$ set $m_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N} \Lambda \rangle)_{k=1,\cdots,N}$ we get

Lemma ([N3])

For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$ and $\Lambda \in P_+$, we obtain

$$\begin{aligned} \tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots \tilde{f}_{i_N}^{m_N}((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) &= \tilde{f}_{i_1}^{m_1}(0)_{i_1} \otimes \tilde{f}_{i_2}^{m_2}(0)_{i_2} \otimes \cdots \otimes \tilde{f}_{i_N}^{m_N}(0)_{i_N} \\ &= (m_1, m_2, \cdots, m_N) =: \mathbf{h}_\Lambda \in \mathcal{H}_{\mathbf{i}}, \end{aligned}$$

where note that for $\Lambda = \Lambda_i$, one has $m_k = h_i^{(k)}$. Then in this case we obtain

$$\tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots \tilde{f}_{i_N}^{m_N}((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) = \mathbf{h}_i$$

Cellular Crystal $\mathbb{B}_{\mathbf{i}}$ and $\mathbb{B}(\widetilde{R\text{-gmod}})$ II

Observation: Determinantial modules $\{C_\Lambda = \mathbf{M}(w_0\Lambda, \Lambda)\} \longleftrightarrow \mathcal{H}_{\mathbf{i}}$

$$\{C_\Lambda \mid \Lambda \in P_+\} \subset R\text{-gmod} \quad \longleftrightarrow \quad \mathcal{H}_{\mathbf{i}}$$

$$C_\Lambda = \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_N}^{m_N} \mathbf{1} \quad \longleftrightarrow \quad \mathbf{h}_\Lambda = \tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots \tilde{f}_{i_N}^{m_N}((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N})$$

Theorem ([N3])

For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$, $\exists$isomorphism of crystals:

$$\widetilde{\Psi} : \mathbb{B}(\widetilde{R\text{-gmod}}) \xrightarrow{\sim} \mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^h(\infty)$$

$$C_\Lambda \circ S \longmapsto \mathbf{h}_\Lambda + \Psi(S) \in B^{\mathbf{h}_\Lambda}(\infty),$$

where $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ (Lauda-Vazirani), S is simple in $\mathbb{B}(R\text{-gmod})$ and for $\Lambda = \sum_i a_i \Lambda_i$ we have $\mathbf{h}_\Lambda = \sum_i a_i \mathbf{h}_i$.

Problems I

Definition

Let X, Y be objects in a monoidal category $\mathcal{T}$, and $\varepsilon : X \otimes Y \to 1$ and $\eta : 1 \to Y \otimes X$ morphisms in $\mathcal{T}$. We say that a pair (X, Y) is dual pair or X is a left dual to Y or Y is a right dual to X if the following compositions are identities:

$$X \simeq X \otimes 1 \xrightarrow{\mathrm{id}\otimes\eta} X \otimes Y \otimes X \xrightarrow{\varepsilon\otimes\mathrm{id}} 1 \otimes X \simeq X, \quad Y \simeq 1 \otimes Y \xrightarrow{\eta\otimes\mathrm{id}} Y \otimes X \otimes Y \xrightarrow{\mathrm{id}\otimes\varepsilon} Y \otimes 1 \simeq Y$$

We denote a right dual to X by $\mathcal{D}(X)$ and a left dual to X by $\mathcal{D}^{-1}(X)$.

Theorem ([KKOP,KKOP2])

For a quiver Hecke algebra R associated with an arbitrary symmetrizable Kac-Moody Lie algebra, $\widetilde{R\text{-gmod}}$ is rigid, i.e., $\forall X \in \widetilde{R\text{-gmod}}, \exists \mathcal{D}(X), \mathcal{D}^{-1}(X)$.

Problem 1

Let R be of finite type. For a simple object $C_\Lambda \circ S \in \mathbb{B}(\widetilde{R\text{-gmod}})$, describe the right and left duals explicitly: $\widetilde{\Psi}(\mathcal{D}(C_\Lambda \circ S)), \ \widetilde{\Psi}(\mathcal{D}^{-1}(C_\Lambda \circ S)) \in \mathbb{B}_{\mathbf{i}}$

Problems II

Theorem (Kashiwara)

Define the involution $* : U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g})$ *by* $(q^h)^* = q^{-h}, \quad e_i^* = e_i, \quad f_i^* = f_i$.
Set $L^*(\infty) := \{u^* \mid u \in L(\infty)\}$, $B^*(\infty) := \{b^* \mid b \in B(\infty)\}$. *Then we have*

$$L^*(\infty) = L(\infty), \quad B^*(\infty) = B(\infty).$$

Theorem (N3)

There exists the operation $\tilde{\mathfrak{a}} : \widetilde{R\text{-gmod}} \to \widetilde{R\text{-gmod}}$ *such that* $\tilde{\mathfrak{a}}(C_i) = C_{i^*}$ $(\forall i \in I)$, $\tilde{\mathfrak{a}}^2 \cong \mathrm{id}$ *and* $\tilde{\mathfrak{a}}(X \circ Y) \cong \tilde{\mathfrak{a}}(Y) \circ \tilde{\mathfrak{a}}(X)$, *we obtain*

$$\tilde{\mathfrak{a}}(\mathbb{B}(\widetilde{R\text{-gmod}})) = \mathbb{B}(\widetilde{R\text{-gmod}}).$$

Problem 2

Describe $\tilde{\mathfrak{a}}$ on $\mathbb{B}_{\mathbf{i}} = B_{i_1} \otimes \cdots \otimes B_{i_N}$ explicitly.

Problems III

Problem 3

In an arbitrary symmetrizable Kac-Moody setting, for any Weyl group element $w \in W$, there exists the full subcategory $\mathscr{C}_w$ of R-gmod and it admits a localization

$$\widetilde{\mathscr{C}_w} = \mathscr{C}_w[C_i^{\circ -1} \mid i \in I], \quad (C_i = M(w\Lambda_i, \Lambda_i)).$$

Indeed, for semi-simple $\mathfrak{g}$, $\mathscr{C}_{w_0} = R$-gmod.

Q: Does the localization $\widetilde{\mathscr{C}_w}$ have a crystal $\mathbb{B}(\widetilde{\mathscr{C}_w})$? If so,

$$\mathbb{B}(\widetilde{\mathscr{C}_w}) \xrightarrow{\sim} B_{i_1} \otimes \cdots \otimes B_{i_m} \ ?$$

where $i_1 \cdots i_m$ is a reduced word of w. (Almost done by collaboration with M.Kashiwara)

Merci Beaucoup