Solving combinatorial equations via computer algebra

Combinatorics and Arithmetic for Physics, 15-17 November 2023

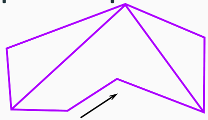
Hadrien Notarantonio (Inria Saclay – Sorbonne Université)

Based on joint works with:

Alin Bostan (Inria Saclay) *Mohab Safey El Din (Sorbonne Université)* *Sergey Yurkevich (University of Vienna)*

Which type of equations are we looking at?

rooted planar maps



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fixed-point in F $\rightsquigarrow$ **unique** solution in $\mathbb{Q}[u][[t]]$

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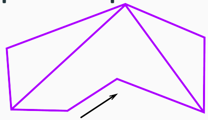

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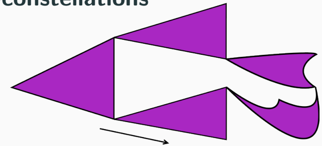

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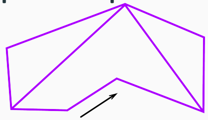

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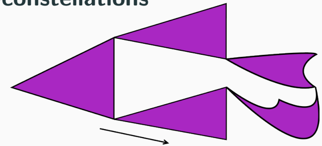

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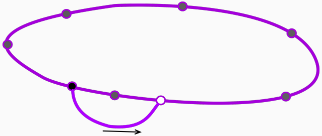
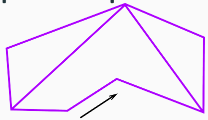
hard particles on planar maps



$$\begin{cases} F(t,u) = x - y + G(t,u) + tu\left(uF(t,u)^2 + \frac{uF(t,u)-F(t,1)}{u-1}\right) \\ G(t,u) = y + tsu\left(F(t,u)G(t,u) + \frac{G(t,u)-G(t,1)}{u-1}\right) \end{cases}$$

How to relate these combinatorial objects to such equations?

rooted planar maps



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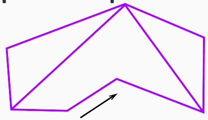

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$a_{n,d} := \#\{\text{planar maps with } n \text{ edges, } d \text{ of them on the external face}\}$

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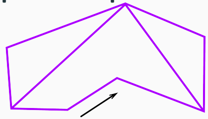

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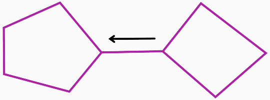

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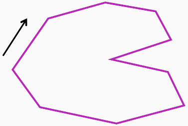

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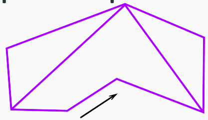

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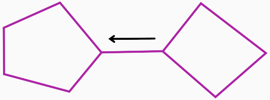

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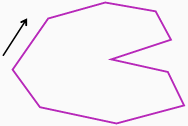

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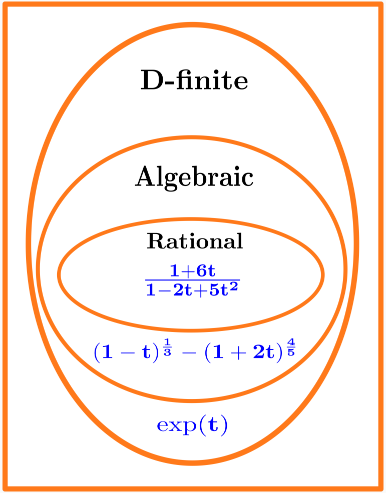
$$F(t,1) = \sum_{n=0}^{\infty} a_n t^n$$

Solving functional equations



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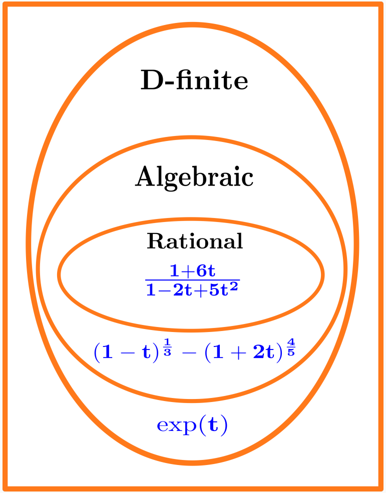

In this talk

Solving = Classifying the initial series $F(t,1)$

+ Computing a witness of this classification

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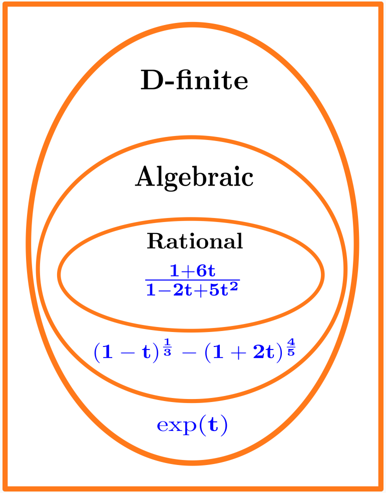

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Going back to our planar maps...

$F(t,1) = 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \cdots \in \mathbb{Q}[[t]]$

annihilated by $R = 27t^2z^2 + (1-18t)z + 16t - 1 \in \mathbb{Q}[z,t]$

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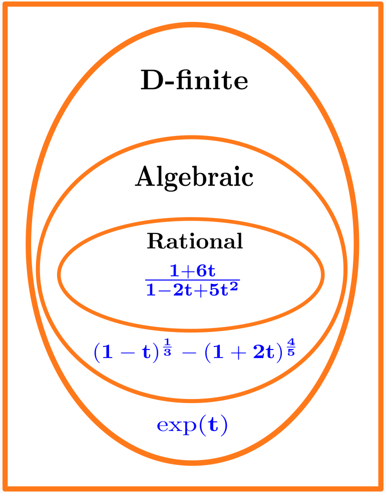

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From R:

- **(Recurrence)** $a_0 = 1$ and $(n+3)a_{n+1} - 6(2n+1)a_n = 0$,
- **(Closed-form)** $a_n = 2\frac{3^n(2n)!}{n(n+2)!}$,
- **(Asymptotics)** $a_n \sim 2\frac{12^n}{\sqrt{\pi n^5}}$, when $n \to +\infty$.

Content of the talk

Objectives

- **Introduce** so-called Discrete Differential Equations (DDEs),
- **Determine the nature** of the solutions of DDEs,
- Provide an **efficient algorithm** for computing a witness,
- Implementation in action $\rightsquigarrow$ Solving a problem **previously out of reach**.

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Plan

I Perform the above points for DDEs [Bostan, N., Safey El Din '23]

II Perform the above points for *systems* of DDEs [N., Yurkevich '23]

Objects of interest: Discrete Differential Equations

Definition

Given $f \in \mathbb{Q}[u]$, $k \geq 1$, and $Q \in \mathbb{Q}[y_0, \ldots, y_k, t, u]$,

$$F = f + t \cdot Q(F, \Delta F, \ldots, \Delta^k F, t, u) \qquad \textbf{(DDE)}$$

is a *Discrete Differential Equation*, where $\Delta : F \in \mathbb{Q}[u][[t]] \mapsto \frac{F(t,u) - F(t,1)}{u-1} \in \mathbb{Q}[u][[t]]$, and where for $\ell \geq 1$ we define $\Delta^{\ell+1} = \Delta^{\ell} \circ \Delta$.

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Theorem
[Bousquet-Mélou, Jehanne '06]

The unique solution in $\mathbb{Q}[u][[t]]$ *of* **(DDE)** *is* **algebraic** *over* $\mathbb{Q}(t,u)$.

$\rightsquigarrow$ **Constructive proof** $\Longrightarrow$ **algorithm**

Bousquet-Mélou and Jehanne's algorithm

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- Eliminate all series but $F(t,1)$

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$\rightarrow$ **Resultants**

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$$\partial_u F(t,u) \cdot \partial_x P(F(t,u), u, F(t,1), \partial_u F(t,1)) + \partial_u P(F(t,u), u, F(t,1), \partial_u F(t,1)) = 0,$$

- **Show** that there exist distinct $U_1, U_2 \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]]$ s.t. $\partial_x P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0,$
- **Set up**

For $1 \leq i \leq 2$, $\begin{cases} P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \\ \partial_x P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \\ \partial_u P(F(t,U_i), U_i, F(t,1), \partial_u F(t,1)) = 0, \end{cases}$

$$m \cdot (U_1 - U_2) - 1 = 0.$$

Elimination theory

- Eliminate all series but $F(t,1)$

$\rightarrow$ **Resultants**

$\rightarrow$ **Gröbner bases**

$$\mathcal{S}: \qquad \text{For } 1 \leq i \leq 2, \begin{cases} P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \\ \partial_x P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \\ \partial_u P(F(t,U_i), F(t,1), \partial_u F(t,1), t, U_i) = 0, \end{cases} \qquad U_1 - U_2 \neq 0.$$

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- U_1, U_2 are **distinct series**,
- $\mathcal{S}$ has **finitely many solutions** in $\overline{\mathbb{Q}(t)}^6$,
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Quantitative estimates with A. Bostan and M. Safey El Din

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[Bostan, N., Safey El Din '23]

Under the above assumptions: $\delta := \deg(P)$

- There exists some nonzero polynomial $R \in \mathbb{Q}[z_0, t]$ whose partial degrees are upper bounded by $\delta^2(\delta-1)^4/2$, such that $R(F(t,1), t) = 0$.
- There exists an algorithm computing this R in $O_{\log}(\delta^{17})$ ops. in $\mathbb{Q}$.

(We proved a **general version** of this result)

Some preliminaries on Gröbner bases

$\mathcal{A} := \mathbb{Q}[x, \boldsymbol{y}]$ polynomial ring, where $\boldsymbol{y} = y_1, \ldots, y_s$.

Monomial orders

- $x^4 y_1^3 y_2^2 \succ_{lex} x^3 y_1^4 y_2^2$ for a **lexicographic order**,
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Properties

- Such bases always **exist** and **generate** $\mathcal{I}$,
- Computing Gröbner bases is **NP-hard**,
- Gröbner bases are a **powerful tool** in elimination theory.

New geometric modelling of the problem with A. Bostan and M. Safey El Din

There exist 2 solutions $(x, \mathbf{u}) \in \overline{\mathbb{Q}(t)}^2$ with **distinct** $\mathbf{u}$-coordinates to

$$\begin{cases} \mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}, \\ \partial_{\mathbf{x}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{0}, \\ \partial_{\mathbf{u}}\mathbf{P}(\mathbf{x}, \mathbf{u}, \mathbf{F}(\mathbf{t}, \mathbf{0}), \partial_{\mathbf{u}}\mathbf{F}(\mathbf{t}, \mathbf{0})) = \mathbf{0}. \end{cases}$$

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Characterize with polynomial constraints

$\mathcal{F}_2 := \{\alpha_{\underline{z}} \in \overline{\mathbb{Q}(t)}^2 \mid \# \, \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} \geq 2\}$

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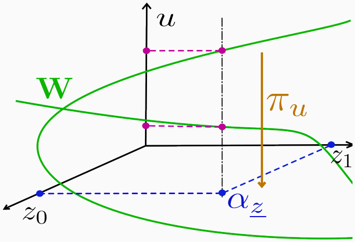

$\# \, \pi_u^{-1}(\alpha_{\underline{z}}) \cap \mathbf{W} = 2$

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Solving a toy example... with A. Bostan and M. Safey El Din

Input: $F(t,u) = 1 + t\left(uF(t,u) + \frac{F(t,u) - F(t,0) - u\partial_u F(t,0)}{u^2}\right)$, **k = 2**

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$\mathbf{B_0}$: γ_0

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[Extension theorem]

$\alpha \in \pi_u(V(G_u)) \implies \mathrm{LeadingCoeff}_u(\mathbf{g_2}) \neq 0$

Distinct solutions in $u \implies \mathrm{disc}_u(\mathbf{g_2}) \neq 0$ (**inequations**)

... yields an algorithm based on elimination theory (ISSAC'23)

Projecting	$\Longrightarrow$	Elimination theorem
Lifting points of the projections	$\Longrightarrow$	Extension theorem

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Projecting $\Longrightarrow$ Elimination theorem

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[Proposition] Let $g \in (\mathbb{Q}(t)[z_0, z_1])[u]$. Then g has **at least i distinct solutions** at $\alpha \in \overline{\mathbb{Q}(t)}^2$ **if and only if** the $(i \times i)$-minors of the **Hermite quadratic form** associated with g **do not vanish simultaneously** at α.

$\rightsquigarrow$ Reduces to studying the **multiplication maps** $(M_{u^\ell} : q \mapsto q \cdot u^\ell)_{\ell \geq 1}$ in $(\mathbb{Q}[t, z_0, z_1])[u]/\langle g \rangle$

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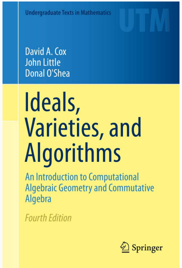

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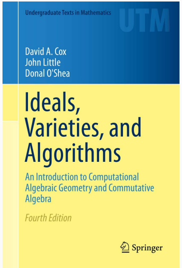

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[Bostan, N., Safey El Din '23]

Disjunction of conjunctions of polynomial equations and inequations whose zero set is $\mathcal{F}_2$

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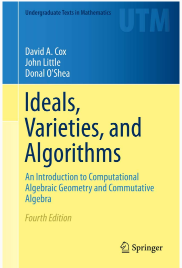

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[5—constellations $k = 4$]

Strategy	**Timing**	(d_{z_0}, d_t)
Duplication	> 5d	?
Elimination	**2d21h**	$(9, 3)$

What could be extended to systems?

Systems of Discrete Differential Equations

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Modelling **special Eulerian planar orientations**:

$$\begin{cases} F(t,u) = 1 + t \cdot \left(u + 2uF(t,u)^2 + 2uG(t,1) + u\frac{F(t,u) - uF(t,1)}{u-1} \right), \\ G(t,u) = t \cdot \left(2uF(t,u)G(t,u) + uF(t,u) + uG(t,1) + u\frac{G(t,u) - uG(t,1)}{u-1} \right). \end{cases}$$

[Bonichon, Bousquet-Mélou, Dorbec, Pennarun '17]

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[Bonichon, Bousquet-Mélou, Dorbec, Pennarun '17]

Modelling **hard particles on planar maps**:

$$\begin{cases} F(t,u) = x - y + G(t,u) + tu\left(uF(t,u)^2 + \frac{uF(t,u) - F(t,1)}{u-1} \right) \\ G(t,u) = y + tsu\left(F(t,u)G(t,u) + \frac{G(t,u) - G(t,1)}{u-1} \right) \end{cases}$$

[Bousquet-Mélou, Jehanne '06]

Popescu's theorem yielding algebraicity of the solutions

[Popescu '86, Swan '98]

(1.4) THEOREM. *Let k be a field, $k\langle X\rangle$ the algebraic power series ring in $X=(X_1,\cdots,X_r)$ over k, f a finite system of polynomial equations over $k\langle X\rangle$ and $\hat{y}=(\hat{y}_1,\cdots,\hat{y}_n)\in k[\![X]\!]^n$ a formal solution of f such that $\hat{y}_i\in k[\![X_1,\cdots,X_{s_i}]\!]$, $1\leqslant i\leqslant n$ for some positive integers $s_i\leqslant r$. Then there exists a solution $y=(y_1,\cdots,y_n)$ of f in $k\langle X\rangle$ such that $y_i\in k\langle X_1,\cdots,X_{s_i}\rangle$, $1\leqslant i\leqslant n$.*

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- Solutions of systems of DDEs are **unique** with components in $\mathbb{Q}[u][[t]]$ $\Longrightarrow$ they are **algebraic**!

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[planar maps] $$H(t,u) = 1 + t\left(u^2 H(t,u)^2 + u\frac{uH(t,u) - G(t,u)}{u-1}\right)$$

- There exists a solution $(H, G) = (F, F(t,1))$, where $F \in \mathbb{Q}[u][[t]]$,
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The proof is **highly not constructive**... How to compute witnesses?

Constructive algebraicity theorem for solutions of systems of DDEs (FPSAC′23)

[N., Yurkevich ′23]

Let $n, k \geq 1$ be integers and $f_1, \dots, f_n \in \mathbb{Q}[u]$, $Q_1, \dots, Q_n \in \mathbb{Q}[y_1, \dots, y_{n(k+1)}, t, u]$ be polynomials. Denote $\nabla^k F := F, \Delta F, \dots, \Delta^k F$. Then the **system** of DDEs

$$\begin{cases} (\mathbf{E_{F_1}}): \quad F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ (\mathbf{E_{F_n}}): \quad F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u). \end{cases} \qquad \textbf{(SDDEs)}$$

admits a **unique** vector of solutions $(F_1, \dots, F_n) \in \mathbb{Q}[u][[t]]^n$, and all its components are **algebraic** over $\mathbb{Q}(t, u)$.

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[Proof sketch]

- There exists a **polynomial system** $\mathcal{S}$ defined over $\mathbb{Q}(t)$ in $\boldsymbol{nk(n+2)}$ equations and unknowns, that admits a solution $\mathcal{P}$ with $F_1(t, 1)$ as one of its coordinates,
- **The Jacobian of $\mathcal{S}$ is invertible at $\mathcal{P}$** $\implies$ $F_1(t, 1)$ is **algebraic** over $\mathbb{Q}(t)$.

Identifying more polynomial equations

Consider $\rightsquigarrow F_1, F_2 \equiv F_1(t,u), F_2(t,u) \in \mathbb{Q}[u][[t]]$

$$\begin{cases} 0 = (1-F_1)\cdot(\mathbf{u}-1) + t\mathbf{u}\cdot(2\mathbf{u}F_1^2 - \mathbf{u}F_1(t,1) + 2\mathbf{u}F_2(t,1) - 2F_1^2 + \mathbf{u} + F_1 - 2F_2(t,1) - 1), \\ 0 = F_2\cdot(1-\mathbf{u}) + t\mathbf{u}\cdot(2\mathbf{u}F_1F_2 + \mathbf{u}F_1 - 2F_1F_2 - F_1 + F_2 - F_2(t,1)). \end{cases}$$

Denote by $E_1, E_2 \in \mathbb{Q}(t)[x_1, x_2, \mathbf{u}, z_0, z_1]$ polynomials such that

$$\text{for } i \in \{1,2\},\quad E_i(F_1(t,\mathbf{u}), F_2(t,\mathbf{u}), \mathbf{u}, F_1(t,1), F_2(t,1)) = 0. \qquad (\equiv E_i(\mathbf{u}))$$

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$$\begin{pmatrix} (\partial_{x_1}E_1)(\mathbf{u}) & (\partial_{x_2}E_1)(\mathbf{u}) \\ (\partial_{x_1}E_2)(\mathbf{u}) & (\partial_{x_2}E_2)(\mathbf{u}) \end{pmatrix} \cdot \begin{pmatrix} \partial_{\mathbf{u}}F_1 \\ \partial_{\mathbf{u}}F_2 \end{pmatrix} + \begin{pmatrix} (\partial_{\mathbf{u}}E_1)(\mathbf{u}) \\ (\partial_{\mathbf{u}}E_2)(\mathbf{u}) \end{pmatrix} = 0.$$

$$\textbf{For } \mathbf{U(t)} \in \bigcup_{d\geq 1} \overline{\mathbb{Q}}[[t^{\frac{1}{d}}]], \begin{cases} \textbf{if} & (\partial_{x_1}E_1\cdot\partial_{x_2}E_2 - \partial_{x_1}E_2\cdot\partial_{x_2}E_1)(\mathbf{U(t)}) = 0, \\ \textbf{then} & (\partial_{x_1}E_1\cdot\partial_{\mathbf{u}}E_2 - \partial_{x_1}E_2\cdot\partial_{\mathbf{u}}E_1)(\mathbf{U(t)}) = 0. \end{cases}$$

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Does this yield an **elimination procedure**?

A polynomial system for systems of 2 DDEs of order 1

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(F_1, \Delta F_1, F_2, \Delta F_2, t, u), \\ F_2 = f_2(u) + t \cdot Q_2(F_1, \Delta F_1, F_2, \Delta F_2, t, u). \end{cases} \qquad \textbf{(SDDEs)}$$

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Define the "numerators" E_1, E_2 and the polynomials

$$\text{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \partial_{x_2} E_1 \\ \partial_{x_1} E_2 & \partial_{x_2} E_2 \end{pmatrix} \quad \text{and} \quad P := \det \begin{pmatrix} \partial_{x_1} E_1 & \partial_u E_1 \\ \partial_{x_1} E_2 & \partial_u E_2 \end{pmatrix}.$$

Set up the duplicated polynomial system $\mathcal{S}$, consisting in the **2** duplications of the polynomials $(E_1, E_2, \text{Det}, P)$: it has **8** equations and unknowns.
Moreover, one of its solutions in $\overline{\mathbb{Q}(t)}^8$ is

$$\mathcal{P} := (F_1(t, U_1), F_2(t, U_1), F_1(t, U_2), F_2(t, U_2), U_1, U_2, F_1(t, 1), F_2(t, 1)).$$

Compute an element of $\langle \mathcal{S}, m \cdot (U_1 - U_2) - 1 \rangle \cap \mathbb{Q}[z_0, t]$.

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The geometric strategy previously described for solving one DDE can be extended here!

Conclusion and perspectives

- **Decidability**: **geometry-driven algorithm** computing $R \in \mathbb{Q}[z,t] \setminus \{0\}$ s.t. $R(F_1(t,1),t) = 0$,
- **Resolution** of the DDE of 5-constellations in an **automatic fashion**,
- **Constructive proof** of **algebraicity** of solutions of SDDEs.

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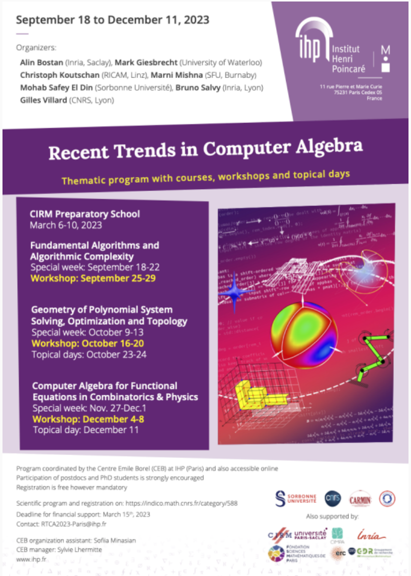
- **Implementing** the algorithm in a *Maple* package?
  (Work in progress)
- **Expanded algorithmic comparison** in the system case?
  (Work in progress with S. Yurkevich)
- More **nested** catalytic variables in the direction of Popescu's theorem?
  (Work in progress with M. Bousquet-Mélou)

RTCA: Computer Algebra for Functional Equations in Combinatorics & Physics



November 27th → December 11th

- **Special week:**
  November 27th → December 1st,
- **Workshop:**
  December 4th → December 8th,
- **Topical day:**
  December 11th.

Thank you for your attention!

Solving 5-constellations using a Hybrid Guess-and-Prove strategy

Input: (The rather big DDE associated with the enumeration of 5-constellations)
Output: $15625t^2F(t,1)^5 - 31250t^2F(t,1)^4 + (25000t^2 - 1000t)F(t,1)^3 - (10000t^2 - 8700t)F(t,1)^2 + (2000t^2 - 15855t + 16)F(t,1) - 160t^2 + 8139t - 16 = 0$

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- **Draw at random** a prime number p and some $c \in \mathbb{F}_p$,
- **Compute** upper bounds $(9,3)$ on the bidegree of $M \in \mathbb{F}_p[z,t]$ annihilating $F(t,1)$ modulo p,
- **Expand** the truncated series $F(t,1)$ mod t^{55}, $\quad 55 = 2 \cdot 9 \cdot 3 + 1$
- **Guess** $R \in \mathbb{Q}[z,t]$ such that $R(F(t,1),t) = O(t^{(9+1)\cdot(3+1)-1})$,
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[Bostan, N., Safey El Din '23]

$\rightsquigarrow$ **elimination strategy**,

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Strategy	**Timing**	(d_z, d_t)
Elimination	2d21h	$\mathbf{(9,3)}$
Hybrid G-P	**2h40min**	$\mathbf{(5,2)}$

A polynomial system for systems of DDEs

Consider

$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u). \end{cases} \qquad \textbf{(SDDEs)}$$

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Perturbe **(SDDEs)** and **define** the “numerators” $E_1, \dots, E_n$ and the polynomials

$$\mathrm{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \quad \text{and} \quad P := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_{n-1}} E_1 & \partial_u E_1 \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_{n-1} & \dots & \partial_{x_{n-1}} E_{n-1} & \partial_u E_{n-1} \\ \partial_{x_1} E_n & \dots & \partial_{x_{n-1}} E_n & \partial_u E_n \end{pmatrix},$$

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Set up the duplicated polynomial system $(\mathcal{S}_{\mathrm{dup}})$, consisting in the nk duplications of the polynomials $E_1, \dots, E_n, \mathrm{Det}, P$. It has $nk(n+2)$ variables and equations.

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$$\begin{cases} F_1 = f_1(u) + t \cdot Q_1(\nabla^k F_1, \dots, \nabla^k F_n, t, u), \\ \quad \vdots \qquad\qquad\qquad \vdots \\ F_n = f_n(u) + t \cdot Q_n(\nabla^k F_1, \dots, \nabla^k F_n, t, u). \end{cases} \qquad \textbf{(SDDEs)}$$

Perturbe **(SDDEs)** and **define** the "numerators" $E_1, \dots, E_n$ and the polynomials

$$\mathrm{Det} := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_n} E_1 \\ \vdots & \ddots & \vdots \\ \partial_{x_1} E_n & \dots & \partial_{x_n} E_n \end{pmatrix} \quad \text{and} \quad P := \det \begin{pmatrix} \partial_{x_1} E_1 & \dots & \partial_{x_{n-1}} E_1 & \partial_u E_1 \\ \vdots & \ddots & \vdots & \vdots \\ \partial_{x_1} E_{n-1} & \dots & \partial_{x_{n-1}} E_{n-1} & \partial_u E_{n-1} \\ \partial_{x_1} E_n & \dots & \partial_{x_{n-1}} E_n & \partial_u E_n \end{pmatrix},$$

Set up the duplicated polynomial system $(\mathcal{S}_{\text{dup}})$, consisting in the nk duplications of the polynomials $E_1, \dots, E_n, \mathrm{Det}, P$. It has $nk(n+2)$ variables and equations.

Compute a non-trivial element of $(\langle \mathcal{S}_{\text{dup}} \rangle : \det(\mathrm{Jac}_{\mathcal{S}_{\text{dup}}})^\infty) \cap \mathbb{K}[t, z_0, \epsilon]$, then set ϵ to 0.