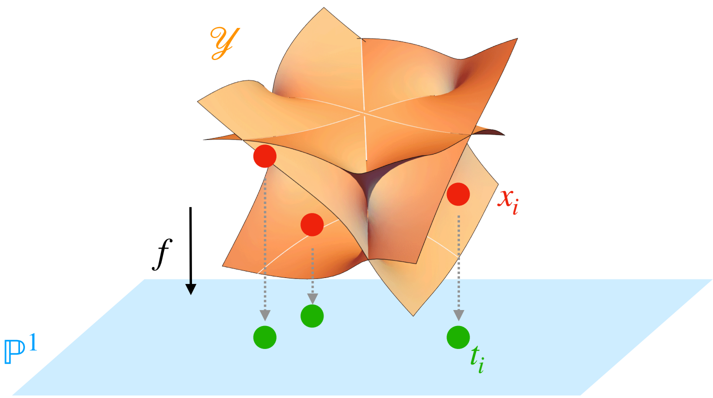
Eric Pichon-Pharabod

Computation of periods of projective hypersurfaces via Picard-Lefschetz theory

Joint work with Pierre Lairez and Pierre Vanhove



Periods as integrals of rational functions

A homogeneous of degree $k \deg P - \deg \Omega$

Ω volume-form of $\mathbb{P}^{n+1}$

$$\int_\gamma \frac{A}{P^k}\Omega$$

A cycle

P defines a smooth hypersurface $\mathcal{X} = V(P) = \{P = 0\}$

The period matrix

Let $\gamma_1, \ldots, \gamma_r \in H_n(\mathcal{X})$ and $\omega_1, \ldots, \omega_r \in H^n_{DR}(\mathcal{X})$ be bases of singular homology and algebraic DeRham cohomology.

The period matrix is

$$\Pi = \left(\int_{\gamma_j} \omega_i \right)_{\substack{1 \le i \le r \\ 1 \le j \le r}}$$

It is an **invertible** matrix that encodes the isomorphism between DeRham cohomology and homology.

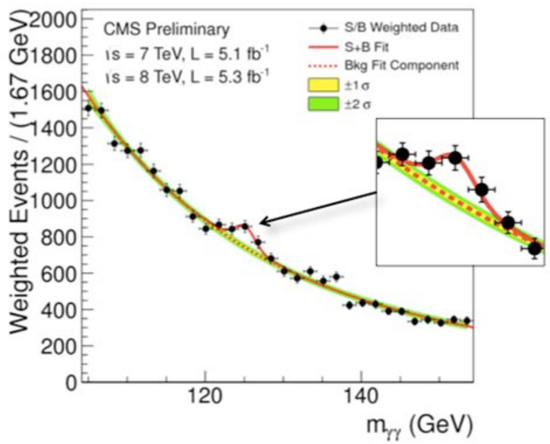
The goal is to compute, given P, the period matrix of $\mathcal{X} = V(P)$.

Why are periods interesting?

The period matrix of $\mathscr{X}$ encodes several **algebraic invariants** of $\mathscr{X}$.

Torelli-type theorems: the period matrix of $\mathscr{X}$ determines its isomorphism class.

Feynman integrals are (relative) periods that arise as scattering amplitudes in quantum field theory.



Previous works

[Deconinck, van Hoeij 2001], [Bruin, Sijsling, Zotine 2018], [Molin, Neurohr 2017]:
Algebraic curves (Riemann surfaces)

[Elsenhans, Jahnel 2018], [Cynk, van Straten 2019]:
Higher dimensional varieties (double covers of $\mathbb{P}^2$ ramified along 6 lines / of $\mathbb{P}^3$ ramified along 8 planes)

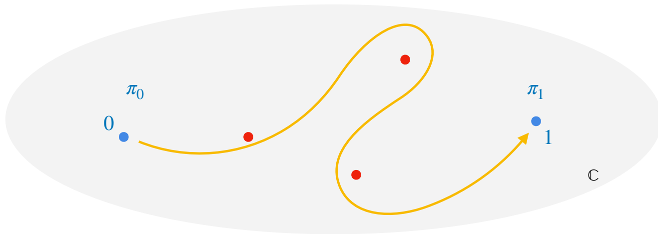
[Sertöz 2019]: compute the period matrix by **deformation.**

Previous works

Sertöz 2019: compute the periods matrix by **deformation :**

We wish to compute $\displaystyle\int_\gamma \frac{\Omega}{X^3+Y^3+Z^3+XYZ}$.

Let us consider instead $\displaystyle\pi_t = \int_{\gamma_t} \frac{\Omega}{X^3+Y^3+Z^3+tXYZ}$,



Exact formulae are known for π_0 **[Pham 65, Sertöz 19]**

Furthermore π_t is a solution to the differential operator
$\mathscr{L} = (t^3+27)\partial_t^2 + 3t^2\partial_t + t$ (Picard-Fuchs equation)

We may numerically compute the analytic continuation of π_0
along a path from 0 to 1 **[Chudnovsky², Van der Hoeven, Mezzarobba]**
This way, we obtain a numerical approximation of π_1.

Previous works

Sertöz 2019: compute the periods matrix by **deformation :**

Two drawbacks :

We rely on the knowledge of the periods of some variety.
[Pham 65, Sertöz 19] provides the periods of the Fermat hypersurfaces $V(X_0^d + \ldots + X_n^d)$.
In more general cases (e.g. complete intersections), we do not have this data.

The differential operators that need to be integrated quickly go beyond what current software can manage:
To compute the periods of a smooth quartic surface in $\mathbb{P}^3$,
one needs to integrate an operator of order 21.

Goal: a more intrinsic description of the integrals should solve both problems.

Contributions

Hundreds of digits

New method for computing periods with high precision:

→ implementation in Sagemath (relying on OreAlgebra) — lefschetz_family

→ sufficiently efficient to compute periods of new varieties (generic quartic surface)

→ homology of complex algebraic varieties

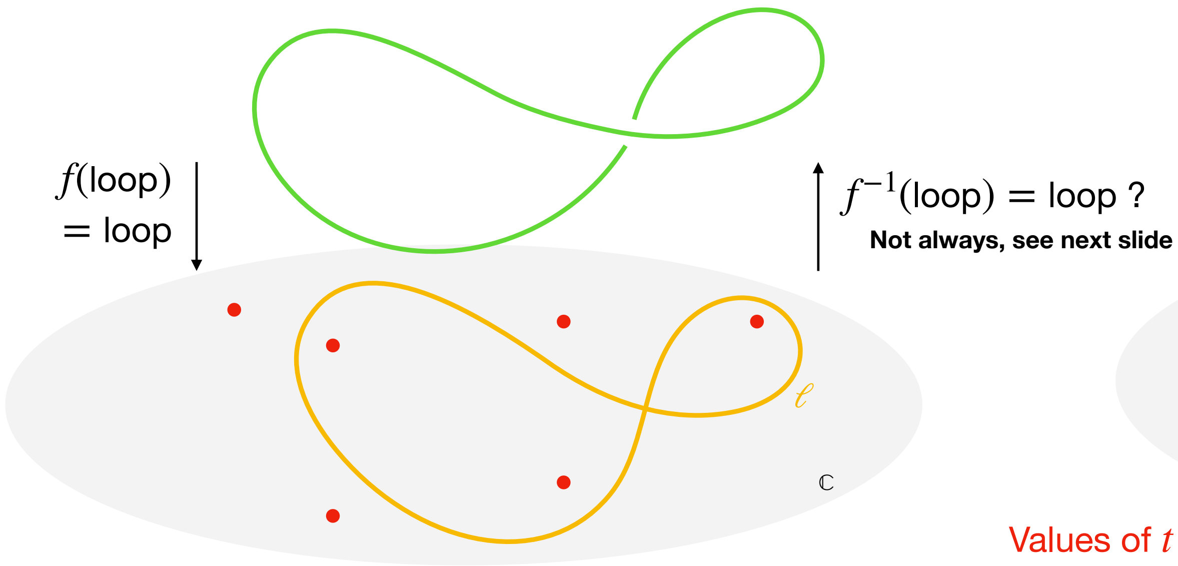
→ generalisable to other types of varieties (e.g. complete intersections, varieties with isolated singularities, etc.)

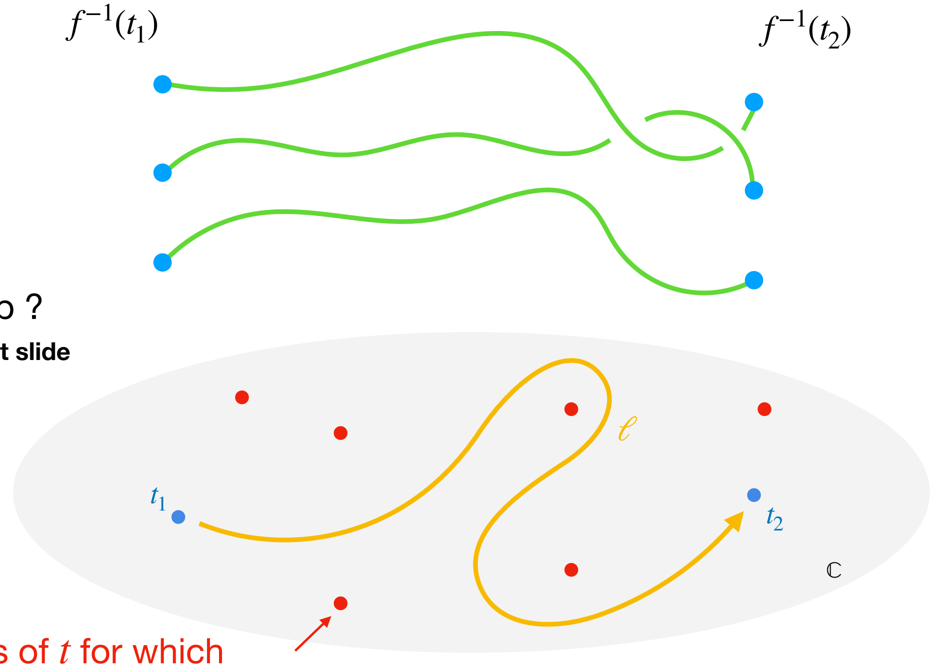
First example: algebraic curves

Let $\mathscr{X}$ be an algebraic curve
with equation $P = y^3 + x^3 + 1 = 0$.
Let $f : (x, y) \mapsto y/(2x + 1)$.

The fibre above $t \in \mathbb{C}$ is $\mathscr{X}_t = f^{-1}(t)$
$= \{(x, t(2x+1)) \mid P\left(x, t(2x+1)\right) = 0\}$.
It deforms continuously with t.

In dimension 1, we are looking for 1-cycles of $\mathscr{X}$
(i.e. closed paths up to deformation).

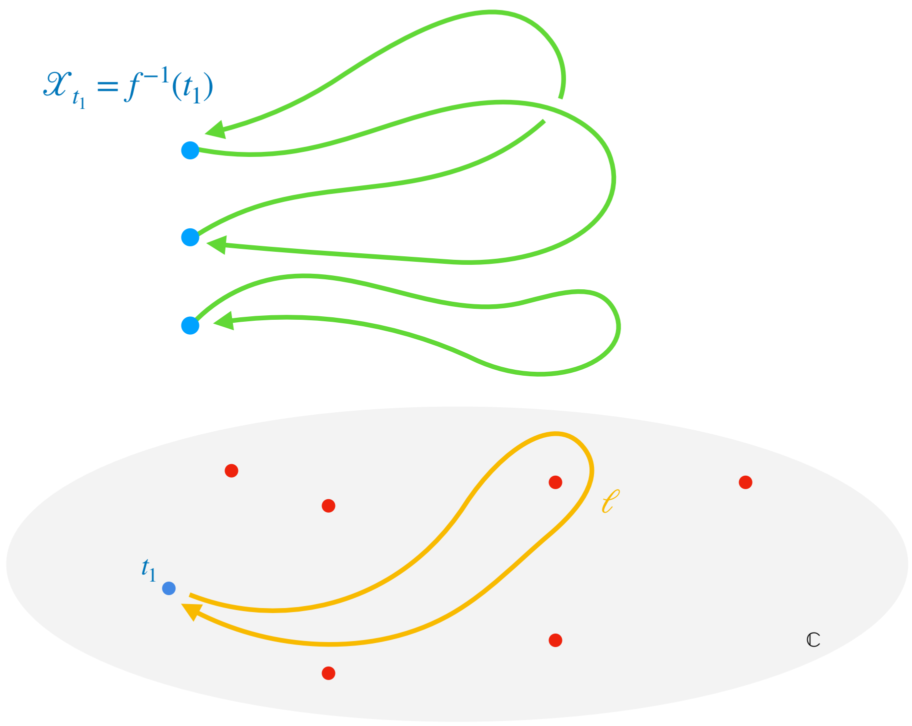


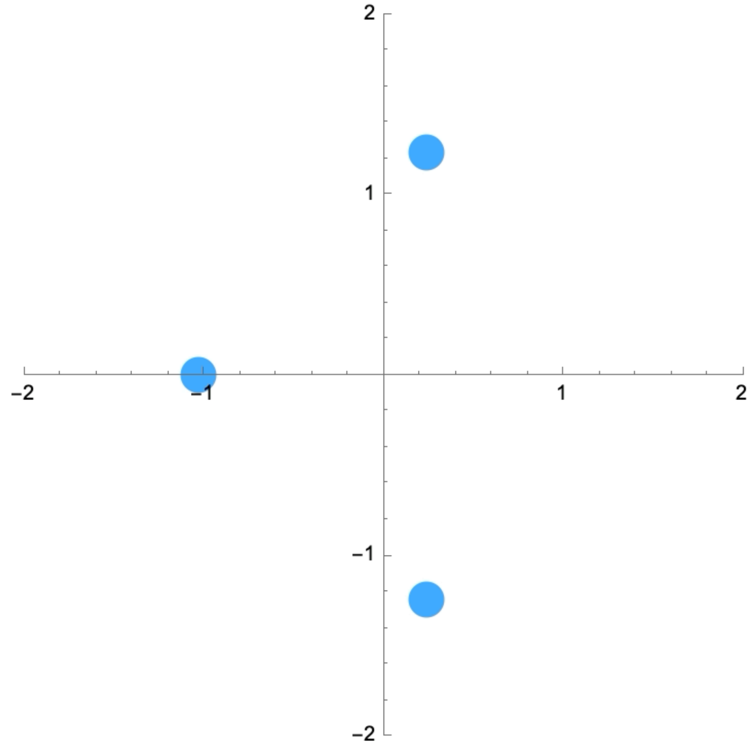


Values of t for which
$P(x, t(2x+1)) = t^3(2x+1)^3 + x^3 + 1$
has double roots (critical values)

What happens when we loop around a critical value?

A loop ℓ in $\mathbb{C}$ based at t_1 induces a permutation of $\mathscr{X}_{t_1}$.



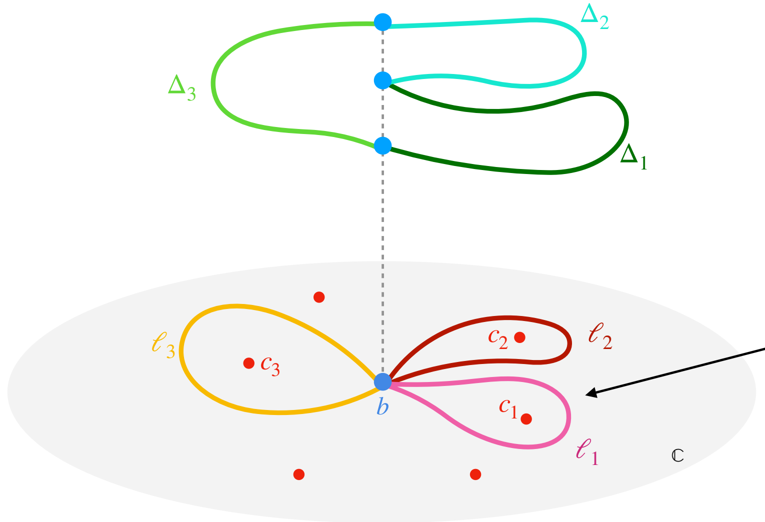


This permutation is called the **action of monodromy along** ℓ on $\mathscr{X}_{t_1}$. It will be denoted ℓ_*

If ℓ is a simple loop around a critical value, ℓ_* is a transposition.

Periods of algebraic curves

The lift of a simple loop ℓ around a critical value c that has a non-trivial boundary in $\mathscr{X}_b$ is called the **thimble** of c. It is an element of $H_1(\mathscr{X}, \mathscr{X}_b)$.



Paths in $\mathscr{X}$ with boundary in $\mathscr{X}_b$

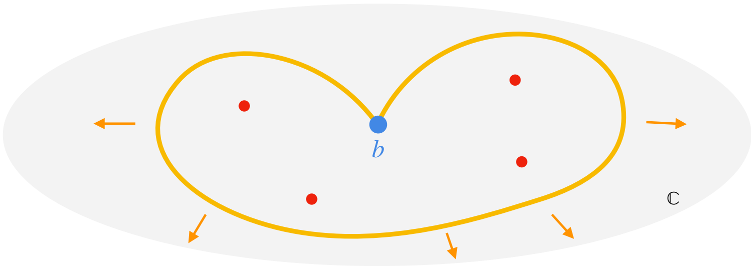
Simple loop around c_1

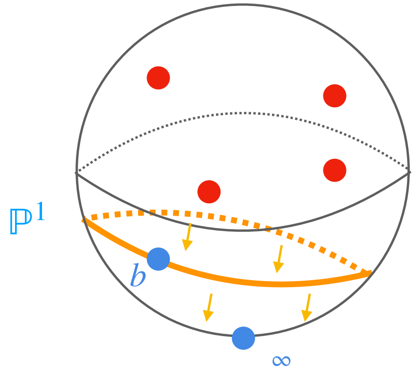
Thimbles serve as "building blocks" to recover $H_1(\mathscr{X})$. Indeed, to find a loop that lifts to a 1-cycle in $\mathscr{X}$, it is sufficient to glue thimbles together in a way such that their boundaries cancels.

Concretely, we take the kernel of the boundary map $\delta : H_1(\mathscr{X}, \mathscr{X}_b) \to H_0(\mathscr{X}_b)$

Fact: all of $H_1(\mathscr{X})$ can be recovered this way.

Certain combinations of thimbles are trivial



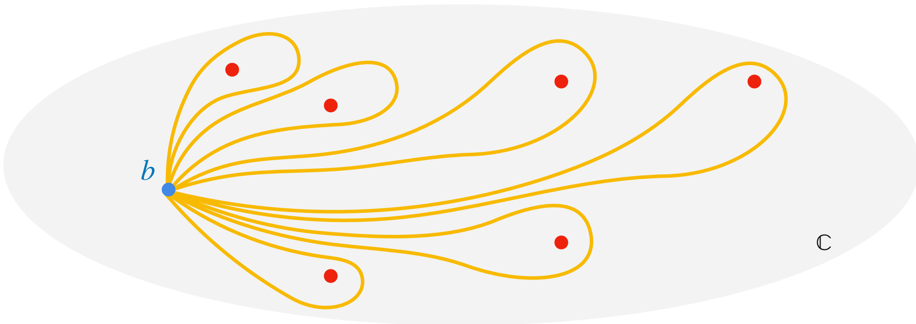


Extensions along contractible paths in $\mathbb{P}^1 \setminus \{\text{crit. val.}\}$
have a trivial homology class in $H_1(\mathscr{X})$.

Fact: these are the only ones — the kernel of the map $\mathbb{Z}^r \mapsto H(\mathscr{X}, \mathscr{X}_b)$,
$k_1, \ldots, k_r \mapsto \sum_i k_i \Delta_i$ is generated by these extensions "around infinity".

Computing periods of algebraic curves

1. Compute simple loops $\ell_1, \ldots, \ell_{\#\text{crit.}}$ around the critical values — basis of $\pi_1(\mathbb{C}\setminus\{\text{crit. val.}\})$



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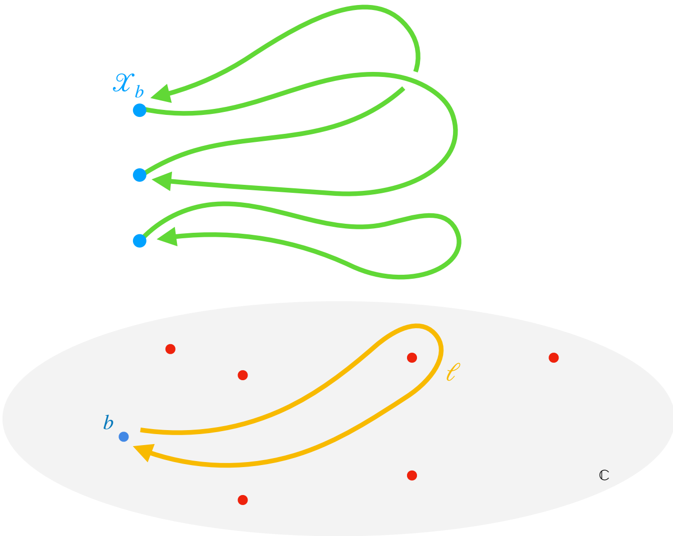

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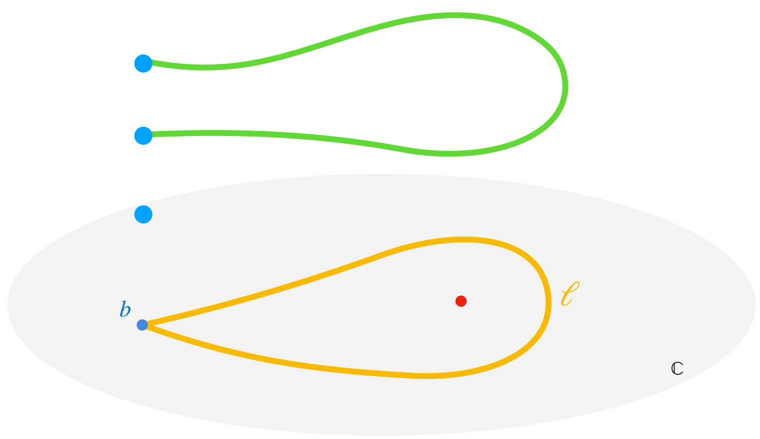
3. This provides the thimble Δ_i. Its boundary is the difference of the two points of $\mathscr{X}_b$ that are permuted.



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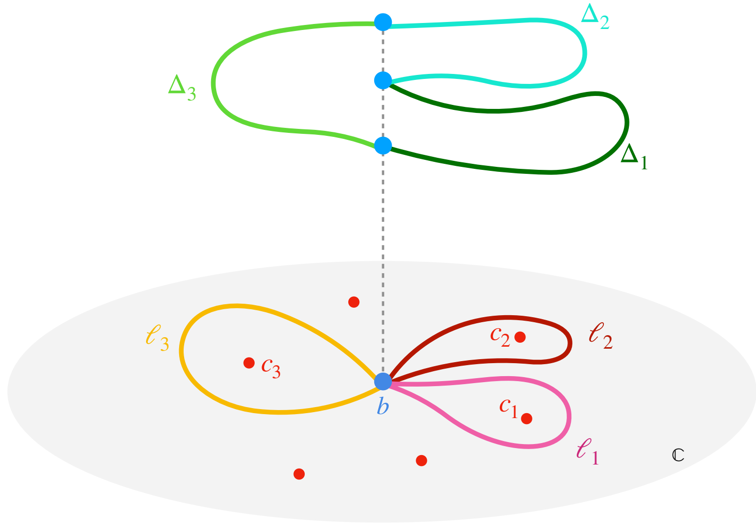
4. Compute sums of thimbles without boundary → basis of $H_1(\mathscr{X})$



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4. Compute sums of thimbles without boundary → basis of $H_1(\mathscr{X})$

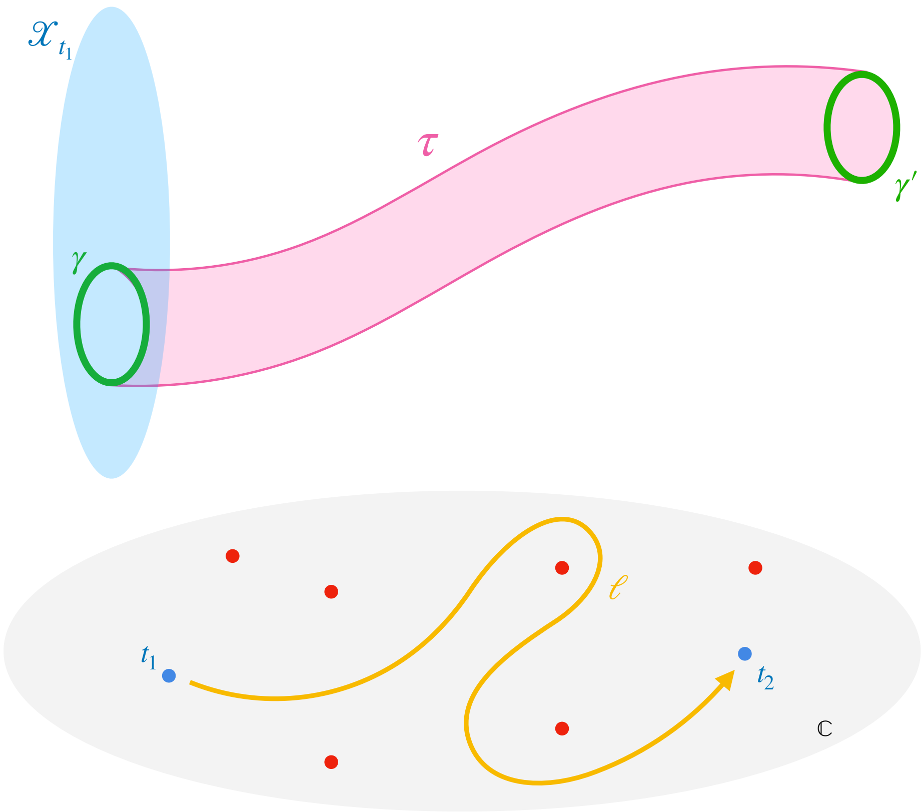
5. Periods are integrals along these loops
→ we have an explicit parametrisation of these paths → numerical integration.

$$\int_\gamma \omega = \int_\ell \omega_t$$

DEMO

Higher dimensions: surfaces

The fibre $\mathscr{X}_t$ is an algebraic curve.
It deforms continuously with respect to t.

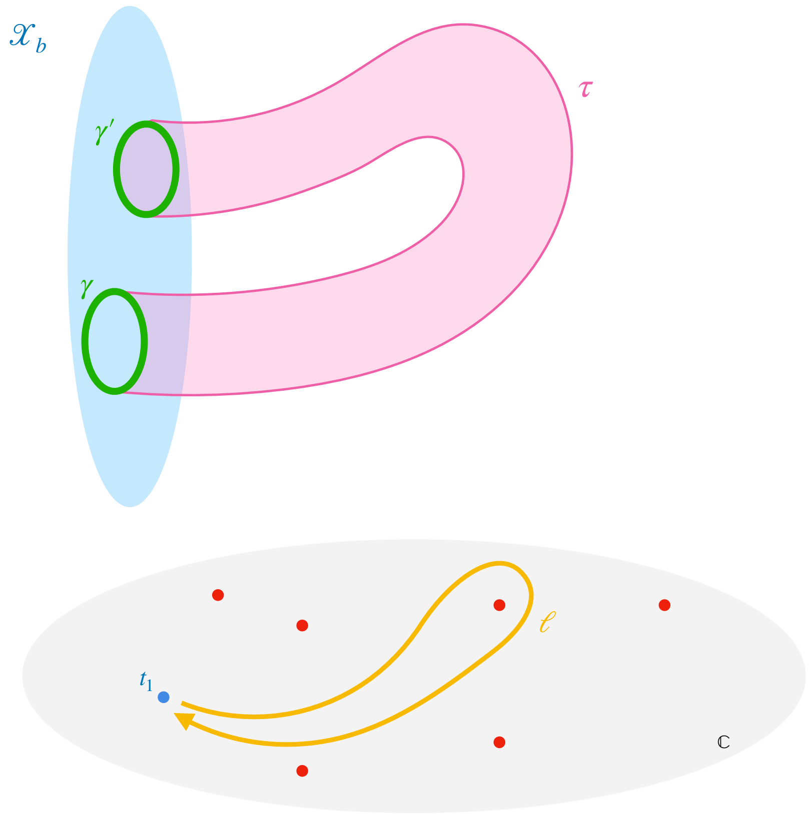


Periods of the surface

$$\int_\tau f(x,y)\mathrm{d}x\mathrm{d}y = \int_\ell \left(\int_{\gamma_y} f(x,y)\mathrm{d}x \right) \mathrm{d}y$$

Periods of an algebraic curve

Comparison with dimension 1

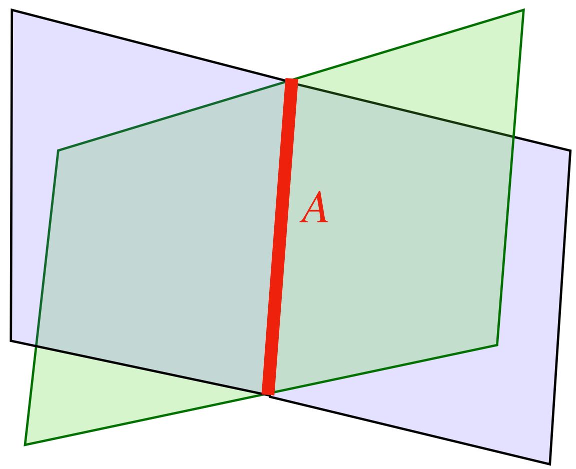


	Dimension 1	**Dimension 2**
	Looking for 1-cycles of $\mathscr{X}$	Looking for 2-cycles of $\mathscr{X}$
	Thimbles are paths obtained as extensions of points along loops.	Thimbles are "tubes" (pink) obtained as extensions of 1-cycles (green) along loops.
	Monodromy along ℓ is an isomorphism of $H_0(\mathscr{X}_b)$ (induced by a permutation of $\mathscr{X}_b$)	Monodromy along ℓ is an isomorphism of $H_1(\mathscr{X}_b)$
There is one thimble per critical value	Complex analysis	Picard-Lefschetz theory
	We obtain all 1-cycles by gluing thimbles.	We obtain **almost** all 2-cycles by gluing thimbles.

Periods are given as integrals along paths.

Some complications

The fibration of $\mathscr{X}$ is given by a hyperplane pencil $\{H_t\}_{t\in\mathbb{P}^1}$, with $\mathscr{X}_t = \mathscr{X} \cap H_t$.



In dimension ≥ 2, this pencil has an axis $A = \cap_{t\in\mathbb{P}^1} H_t$ that intersects $\mathscr{X}$. Therefore each fibre contains a copy of $\mathscr{X}' = \mathscr{X} \cap A$.

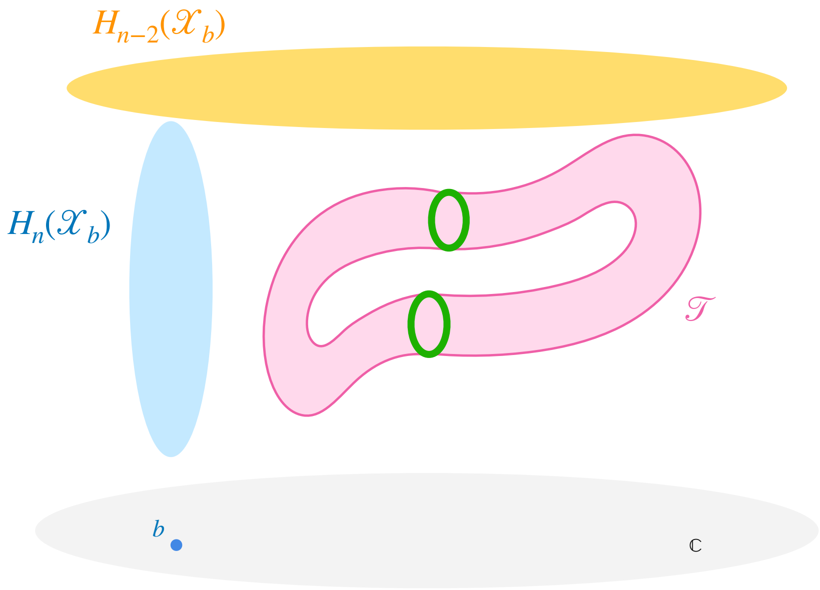
The fibration is thus not isomorphic to $\mathscr{X}$, but to the blow up $\mathscr{Y}$ of $\mathscr{X}$ along $\mathscr{X}'$.

What we compute is in fact $H_n(\mathscr{Y})$, which contains the homology classes of the exceptional divisors. To recover $H_n(\mathscr{X})$ we need to be able to identify these classes.

$$0 \to H_{n-2}(\mathscr{X}') \to H_n(\mathscr{Y}) \to H_n(\mathscr{X}) \to 0$$

Some complications

Not all cycles of $H_n(\mathscr{Y})$ are lift of loops, and thus not all are combinations of thimbles.

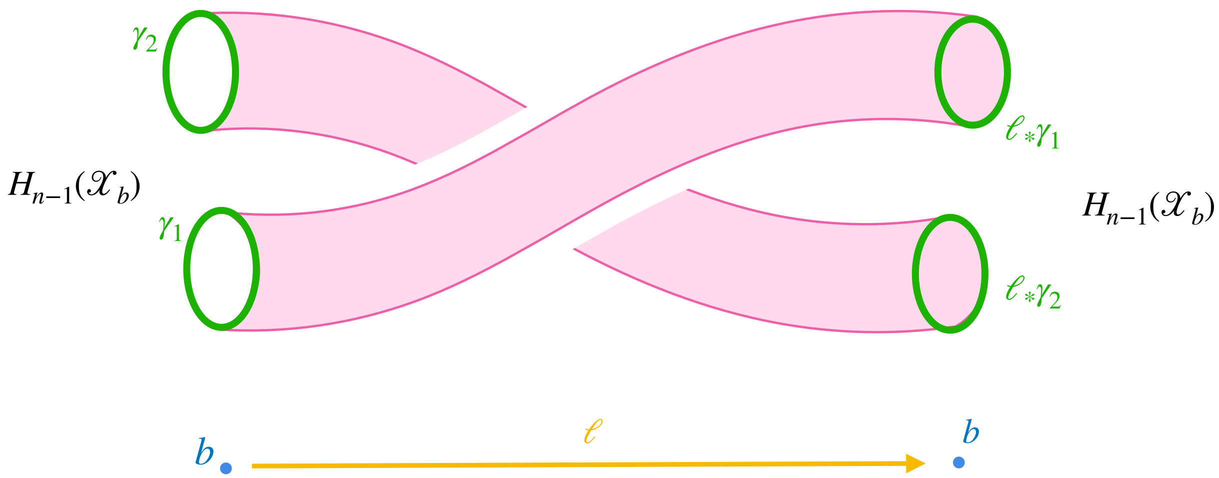


More precisely, we are missing the homology class of the fibre $H_n(\mathscr{X}_b)$
and a section (an extension to $H_{n-2}(X_b)$ to all of $\mathbb{P}^1$).

We have a (non-canonical) decomposition
$$H_n(\mathscr{Y}) \simeq H_n(\mathscr{X}_b) \oplus \mathscr{T} \oplus H_{n-2}(X_b)$$

Computing monodromy

$$\pi_1(\mathbb{C}\setminus\{\text{critical values}\}) \to GL(H_{n-1}(\mathscr{X}_b))$$



Given by periods

DEMO

Tools used:

- Induction on dimension — we know the cycles of $H_{n-1}(\mathscr{X}_b)$
- Isomorphism between homology and De Rham cohomology → we obtain analytical structure!
- Monodromy of differential operators (Picard-Fuchs equation / Gauss-Manin connexion)

[Mezzarobba]

Results and perspectives

Holomorphic periods of quartic surfaces in an hour (previously unattainable in most cases).

A specific example: the Tardigrade family (a family of singular quartic K3 surfaces).
[Doran, Harder, PP, Vanhove 2023]
→ explicit embedding of the Néron-Severi lattice in the standard K3 homology lattice

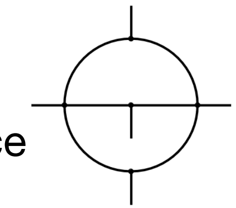


FIGURE 13. The tardigrade graph

Quartic surfaces of $\mathbb{P}^3$ with Picard rank 2, 3, 5

$$\mathcal{X} = V\left(\begin{array}{c} X^4 - X^2Y^2 - XY^3 - Y^4 + X^2YZ + XY^2Z + X^2Z^2 - XYZ^2 + XZ^3 \\ -X^3W - X^2YW + XY^2W - Y^3W + Y^2ZW - XZ^2W + YZ^2W - Z^3W + XYW^2 \\ +Y^2W^2 - XZW^2 - XW^3 + YW^3 + ZW^3 + W^4 \end{array}\right)$$

This approach is generalisable to other types of varieties (complete intersections, elliptic surfaces, etc)

The bottleneck for dealing with higher dimensional/higher degree examples is still the order and degree of the Picard-Fuchs equations.

Thank you!