Combinatorial topological quantum field theories and geometrical constructions of integers in finite group representation theory

Sanjaye Ramgoolam

Queen Mary, University of London

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Topological quantum field theories (TQFTs) which have a simple physical formulation as lattice gauge theory with finite gauge group G admit elegant expressions for partition functions on closed higher genus Riemann surfaces. There are expressions for the partition functions in terms of the combinatorial counting of flat G-bundles and in terms of dimensions of irreducible representations (irreps). Consideration of the partition functions of these G-Flat-TQFTs across different genuses gives finite algorithms which start from group multiplications and yield the spectrum of dimensions of irreps. The input into the algorithms is formed by identities which generalise the classic formula for the order of a group as a sum of squares of the dimensions of irreps. Considering the partition functions of the G-Flat-TQFTs for surfaces with boundaries leads to the derivation of integrality properties of certain partial sums along columns of the character table of G. Analogous considerations starting from a topological field theory based on the fusion ring of a finite group (denoted G-Fusion-TQFT) allows the proof of analogous integrality properties for partial sums along rows of the character table. These row-column relations between integrality properties of characters can be viewed as a mathematical reflection of a physical row-column duality between the G-flat TQFTs and the G-fusion TQFTs.

Based on

[1] Robert de Mello Koch, Yang-Hui He, Garreth Kemp, Sanjaye Ramgoolam "**Integrality, Duality and Finiteness in Combinatoric Topological Strings,**" arXiv[hep-th; 2106.05598 J. High Energ. Phys. 2022, 71 (2022)

[2] S. Ramgoolam and E. Sharpe "**Combinatoric topological string theories and group theory algorithms,**" J. High Energ. Phys. 2022, 147 (2022)

[3] Adrian Padellaro, Rajath Radhakrishnan, Sanjaye Ramgoolam, "**Row-Column duality and combinatorial topological strings,**" arXiv:2304.10217 [hep-th]

Introduction : TQFTs on 2D surfaces from physics

Topological quantum field theories in two dimensions - defined according to Atiyah's axioms (or subsequent developments) or by physical method (e.g. lattice gauge theory) - based on finite groups G.

Lattice gauge theory - defined by discretising (triangulating) a the two-dimensional surface of genus G. Partition function defined by a sum over group variables associated to the edges. The summand is a product of weights of the 2-cells.

The weights imposed a flatness and are topological: invariant under refinement of the triangulation. This defines G-Flat-TQFT (Dijkgraaf-Witten TQFT2).

Introduction : Partition functions

The partition functions on genus G have two kinds of expressions :

1. Sums over group elements (close to the lattice gauge theory definition).

2. Sums over irreducible representations(irreps)

The latter are derived from the former using finite group representation theory identities.

Introduction : Combinatorial representation theory (CRT)

In CRT we are interested in combinatorial constructions of integer quantities defined using representation theory. Some classic examples :

1. Enumeration of irreps of the symmetric groups S_n using Young diagrams having n boxes.

2. Dimensions of these irreps using a product of weights associated with the Young diagram boxes (hook formula).

3. The multiplicities of the reduction of irreps R of S_n into a direct sum of irreps of $S_{n-k} \times S_k$. (Littelwood-Richardson rule)

4. Burnside algorithm for irreducible characters of any finite group G - combinatorial data of group multiplications.

5. The identity : $|G| = \sum_R d_R^2$.

Lattice TQFTs and Combinatorial representation theory (CRT)

The expressions for partition functions of G-Flat-TQFTs on surfaces,combining information from a range of genuses of the surfaces (in the spirit of string theory), allow us to generalise the last two constructions.

Interesting integers in rep theory from TQFTs

1. All the d_R for a finite group.

2. Integer partial sums of characters along columns of the character table of a finite group G.

What about partial row sums

It is known in CRT that row sums of characters are integers

$$\sum_C \chi_C^R = \sum_S N_{SS}^R$$

where N_{SS}^R are the multiplicities of decomposition

$$V_S \otimes V_S = \bigoplus_R V_R \otimes V_{mult:SSR}$$

i.e.

$$N_{SS}^R = \mathrm{Dim}(V_{mult:SSR})$$

What about partial row sums

This gives a hint : Use TQFTs (this time defined axiomatically using Atiyah's axioms) based on fusion coefficients (G-Fusion-TQFTs).

We find that analogous computations which replace G-Flat-TQFTs with G-Fusion-TQFTs give results on integrality of partial row sums of character tables.

OUTLINE

1. Using G-Flat-TQFT to obtain d_R
2. Using G-Flat TQFT : Burnside algorithm
3. G-Flat TQFT : Integrality of partial column sums
4. G-Fusion TQFT : Integrality of partial row sums.

PART 1: Partition functions on closed surfaces

Z_h, the genus h partition function of G-Flat TQFT, is given by

$$Z_h = \frac{1}{|G|} \sum_{g_1, g_2, \cdots, g_{2h-1}, g_{2h} \in G} \delta([g_1, g_2][g_3, g_4] \cdots [g_{2h-1}, g_{2h}])$$
$$= \sum_R \left(\frac{|G|}{d_R} \right)^{2h-2}$$

The LHS is combinatorial - counting numbers of $2h$-tuples of group elements which multiply to identity. The RHS gives a power sum of dimensions d_R.

PART 1: Partition functions

A special case is $h = 0$

$$\frac{1}{|G|} = \sum_R \frac{d_R^2}{|G|^2}$$

$$|G| = \sum_R d_R^2$$

Useful known fact from finite group rep theory : $\frac{|G|}{d_R}$ are known to be integers.

PART 1: Partition functions and power sums

Consider $h = 2, \cdots, (K+1)$ where K is the number of conjugacy classes. Define

$$a_R = \left(\frac{|G|}{d_R}\right)$$

This sequence gives us

$$\sum_R a_R^2 = Z_2$$
$$\sum_R a_R^4 = Z_3$$
$$\vdots$$
$$\sum_R a_R^{2K} = Z_{K+1}$$

PART 1: USeful matrix X

It is useful to define a $K \times K$ matrix

$$X = Diag\left(\frac{|G|^2}{d_R^2}\right) = Diag\left(a_R^2\right)$$

The Z_h partition functions for $h \in \{2, 3, \cdots, K+1\}$ give us $\mathrm{tr}(X), \mathrm{tr}(X^2), \cdots, \mathrm{tr}(X^K)$ as combinatorial input.

PART 1: Characteristic polynomial of X

Consider the polynomial

$$
\begin{aligned}
F(x,X) &\equiv \det(x - X) = (x - a_1^2)(x - a_2^2)\cdots(x - a_K^2) \\
&= x^K - (\mathrm{tr}X)x^{K-1} + \frac{1}{2}((\mathrm{tr}X)^2 - \mathrm{tr}X^2)x^{n-2} + \cdots + (-1)^K(\det X) \\
&= x^K - e_1(X)x^{K-1} + e_2(X)x^{K-2} + \cdots + (-1)^K e_K(X)
\end{aligned}
$$

The $e_i(X)$ are elementary symmetric functions which can be expressed in terms of products of traces (newton's identities give a recursive formula).

It is also useful to define $E_l(X) = (-1)^l e_l(X)$ which leads to

$$
\begin{aligned}
&F(X,x) \\
&= x^n + E_1(X)x^{n-1} + E_2(X)x^{n-2} + \cdots + E_{K-1}(X)x + E_K(X) \\
&= \sum_{l=0}^{K} x^{K-l} E_l(X)
\end{aligned}
$$

PART 1: Solving for integer solutions of characteristic polynomial of X

The a_R^2 are the zeroes of $F(X, x)$, which is viewed as a polynomial in x with coefficients constructed from G-TQFT2 partition functions as above.

The a_R are known to be integers. There are simple algorithms using this integrality for finding the roots of the polynomial.

The numbers $(a_1^2, a_2^2, \cdots, a_K^2)$ are divisors of $\det X$ since $F(X, x = 0) = (-1)^K \det X = (-1)^K \prod_{i=1}^K a_i^2$. Let

$$\mathrm{Div}_0 = \text{ Set of divisors of } (-1)^K F(X, x = 0)$$

Each of the a_R^2 is a divisor of $(-1)^K F(X, x = 0)$, i.e. an element of Div_0.

PART 1: Solving for integer eigenvalues of characteristic polynomial of X

Next note that $(-1)^K F(X, x = 1) = \prod_R (a_R^2 - 1)$. Let r_i be the divisors of $(-1)^K F(X, x = 1)$. The a_i^2 are among the $(r_i + 1)$.

$$\mathrm{Div}_1 = \text{ Set of divisors of } F(X, x = 1) \text{ shifted up by } 1$$

Each element in the list $\{a_1^2, a_2^2, \cdots, a_K^2\}$ is in the intersection

$$\mathrm{Div}_0 \cap \mathrm{Div}_1 \cap \mathrm{Div}_2 \cdots \cap \mathrm{Div}_{K-1}$$

and the list satisfies

$$\prod_R (a_R^2 - l) = (-1)^{K-l} F(X, l)$$

for all $l \in \{0, 1, \cdots, K - 1\}$.
Search among the elements of (1) sets of K integers (candidate $\{a_1^2, a_2^2, \cdots, a_K^2\}$ which satisfy the condition in (1). This suffices to ensure that we get the correct a's. As explained in **[1]** this ensures that the set of resulting elements is indeed the set of $\{a_1^2, a_2^2, \cdots, a_K^2\}$

PART 2: Partition functions for surfaces with boundaries and characters

Let $\mathcal{C}_p$ be a conjugacy class of G and let $|\mathcal{C}_p|$ be the number of elements in the conjugacy class. We will denote by T_p the sum of group elements, in the group algebra $\mathbb{C}(G)$

$$T_p = \sum_{g \in \mathcal{C}_p} g$$

T_p is a central element of $\mathbb{C}(G)$, i.e. commutes with all elements in $\mathbb{C}(G)$. The normalized characters for $\mathcal{C}_p$ are, for $g \in \mathcal{C}_p$,

$$\frac{|\mathcal{C}_p|\chi^R(g)}{d_R} = \frac{\chi^R(T_p)}{d_R}$$

The set of T_p for all conjugacy classes spans the centre $\mathcal{Z}(\mathbb{C}(G))$. Another basis for $\mathcal{Z}(\mathbb{C}(G))$ is given by the projectors labelled by irreducible representations R

$$P_R = \frac{d_R}{|G|} \sum_g \chi^R(g) g^{-1}$$

The first basis $\{T_p\}$ corresponds to conjugacy classes. The second basis set $\{P_R\}$ is labelled by irreps. The projectors satisfy

$$P_R P_S = \delta_{RS} P_S$$

A useful property is

$$T_p P_R = \frac{\chi^R(T_p)}{d_R} P_R$$

The normalised characters are the eigenvalues of T_p viewed as an operator on $\mathcal{Z}(\mathbb{C}(G)$.
The product in $\mathcal{Z}(\mathbb{C}(G))$ in the T_p basis is

$$T_p T_q = \sum_r C_{pq}^{\ r} T_r = \sum_r \frac{\delta(T_p T_q T_{r'})}{|T_r|} T_r$$

The coefficients C_{pq}^r are integers. The matrix $(C_p)_q^r$ is an integer matrix. Its eigenvalues are algebraic integers. It follows that $\frac{\chi^R(T_p)}{d_R}$ are algebraic integers (for $G = S_n$ they are integers).

PART 2: Partition functions for surfaces with boundaries and power sums of normalised characters

A standard result in G-Flat-TQFT2 is that the amplitude for a genus h surface with r distinct boundaries, where the group element at the boundary is constrained to be in a conjugacy class $\mathcal{C}_p$ is

$$\begin{aligned}
&\sum_R \left(\frac{|G|}{d_R}\right)^{2h-2} \left(\frac{\chi_R(T_p)}{d_R}\right)^r \\
&= \mathrm{tr}\left(X^{2h-2} X_p^r\right) \\
&= \frac{1}{|G|} \sum_{s_i, t_i} \sum_{\sigma_1, \cdots, \sigma_r \in \mathcal{C}_p} \delta\left((\prod_{i=1}^{h} s_i t_i s_i^{-1} t_i^{-1}) \sigma_1 \cdots \sigma_r \right)
\end{aligned}$$

X is a diagonal matrix with diagonal entries equal to $\frac{|G|}{d_R}$. We have defined X_p to be the diagonal matrix with matrix entries $\left(\frac{\chi_R(T_p)}{d_R}\right)$. Fixing $h = 1$, we get power sums of the normalised character from combinatoric data.

$$\begin{aligned}
\mathrm{tr}(X_p^r) &= \sum_R \left(\frac{\chi_R(T_p)}{d_R}\right)^r \\
&= \frac{1}{|G|} \sum_{s_1,t_1} \sum_{\sigma_1,\cdots,\sigma_r \in \mathcal{C}_p} \delta(s_1 t_1 s_1^{-1} t_1^{-1} \sigma_1 \cdots \sigma_r)
\end{aligned}$$

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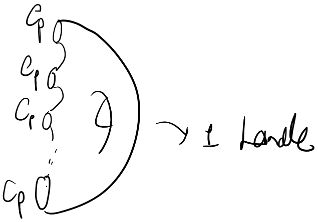
multiple boundaries with C_p observables



This gives the combinatoric data reproducing $\mathrm{tr}(X_p^r)$. It is the counting of G bundles on genus one surfaces with r punctures where the monodromy around each puncture has the specified conjugacy class. In this way we can construct the characters of all conjugacy classes, using the same algorithm as for the dimensions. The problem reduces to solving the polynomial equation

$$F(X_p, x) = \det(X_p - x) = 0$$

PART 2: Connection to Burnside algorithm

Burnside algorithm proceeds by finding the eigenvalues of $(C_p)^r_q$

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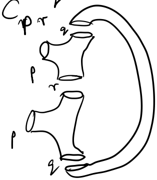
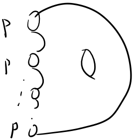
$C_{pq}{}^{r} \longmapsto$

$tr(C_p) = C_{pq}{}^{q} \rightarrow$

$tr(C_p^2) = C_{pq}{}^{r}\, C_{pr}{}^{q}$



<u>in general:</u>



PART 3: Partial sums along columns of character tables

For the Burnside algorithm, we used genus one with general number of boundaries labelled by $\mathcal{C}_p$.

What do we get if consider one boundary labelled by $\mathcal{C}_p$, and a multiple boundaries labelled by $\mathcal{C}_q$.

First observe that higher genus partition functions are related to the handle creation operator

$$\Pi = \sum_{g_1,h_1\in G} ghg^{-1}h^{-1} = \sum_R \frac{|G|^2}{d_R^2} P_R = \sum_R a_R^2 P_R$$

$$Z_h = \frac{1}{|G|}\delta\left(\Pi^h\right),$$

PART 3: Powers of Π

Theorem The powers of Π, that is, $\{1, \Pi, \Pi^2, \cdots\}$ span a subspace of $\mathcal{Z}(\mathbb{C}(G))$ which has dimension D_0 - the number of distinct values of the dimensions d_R of irreducible representations.

Proof

$$\Pi = \sum_R a_R^2 P_R^2 = \sum_{R'} a_{R'}^2 \tilde{P}_{R'}$$

R' runs over a maximal set of irreps which have distinct dimensions. $\tilde{P}_{R'}$ is the sum of projectors for the irreps which have the same dimension as R'.

$$\tilde{P}_{S'} = \prod_{R' \neq S'} \frac{(\Pi - a_{R'}^2)}{(a_{S'}^2 - a_{R'}^2)}$$

PART 3: Powers of Π and characters

Let D_0 be the number of distinct values of the dimension d_R as R ranges over the irreps of G.

$$
\begin{aligned}
\frac{1}{|G|}\delta(\Pi^h T_p) &= \sum_R \left(\frac{|G|^2}{(\dim R)^2}\right)^{h-1} \frac{\chi^R(T_p)}{\dim R}, \\
&= \sum_{R'} \left(\frac{|G|^2}{(\dim R')^2}\right)^{h-1} \sum_{\{R:R'\}} \frac{\chi^R(T_p)}{\dim R}
\end{aligned}
$$

for the range $h \in \{1, 2, \cdots, D_0\}$

The primed sum runs over a maximal set $\{R'\}$ of irreducible representations R' having distinct dimensions. The sum over $\{R : R'\}$ is a sum over the distinct irreducible representations R with the same dimension as R'. Let us define $\tilde{R}'$ to be the direct sum of irreducible representations R with the same dimension as R'. Then we can write

$$
\frac{1}{|G|}\delta\left(\Pi^h T_p\right) = \sum_{R'} \left(\frac{|G|^2}{(\dim R')^2}\right)^{h-1} \frac{\chi^{\tilde{R}'}(T_p)}{\dim R'}
$$

PART 3: Partial sums along columns of character tables

Then we can write

$$\frac{1}{|G|}\delta\left(\Pi^h T_p\right) = \sum_{R'} \left(\frac{|G|^2}{(\dim R')^2}\right)^{h-1} \frac{\chi^{\tilde{R}'}(T_p)}{\dim R'}$$

As h runs over the set $\{1, \cdots, D_0\}$, we have a linear system of equations of size $D_0 \times D_0$ for the normalized characters $\chi^{\tilde{R}'}(T_p)/\dim R'$. As R' and l range over the D_0 possibilities, we have a matrix

$$\mathcal{V}_{R',h} = \left(\frac{|G|^2}{(\dim R')^2}\right)^{h-1}$$

of size $D_0 \times D_0$.

PART 3: Partial sums along columns of character tables

The equation (1) takes the form

$$Y = \mathcal{V} \cdot X$$

where

$$Y_h = \frac{1}{|G|} \delta \left(\Pi^h T_p \right) ,$$
$$X_{R'} = \frac{\chi^{\tilde{R}'}(T_p)}{\dim R'}$$

and we recognize $\mathcal{V}$ as a Vandermonde matrix.
Since the R' have been chosen to run over a set of irreducible representations with distinct dimensions, the integers $\left(\frac{|G|^2}{(\dim R')^2} \right)$ are distinct. This ensures that $\mathcal{V}$ is invertible. The inverse matrix can thus be used to construct the normalized characters $X_{R'}$ from the combinatoric G-CTST data Y_h.

These ratios are known to be algebraic integers.

$$\frac{\chi^{\tilde{R}'}(T_p)}{\dim R'}$$

1.e solutions to a polynomial equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

with integer coefficients. They are eigenvalues of the integer (structure constant) matrices $(\mathcal{C}_p)^r_q$.

The Van der Monde matrix has integer entries. The inverse has rational entries. Applying this integer matrix to the vector of partition functions gives rational numbers.

An elementary fact from number theory says that: Algebraic integers which are rational are actually integers. Hence

Theorem These sums of normalised characters over irreps of fixed dimension $d_{R'}$ are integers

$$\frac{\chi^{\tilde{R}'}(T_p)}{\dim R'} = \sum_{\{R;R'\}} \frac{\chi^R(T_p)}{d_R}$$

If the sum over $\{R; R'\}$ of

$$\sum_{\{R;R'\}} \frac{\chi^R(T_p)}{d_R} = \sum_{\{R;R'\}} \frac{|\mathcal{C}_p|\chi_p^R}{d_R} = \frac{|\mathcal{C}_p|}{d_{R'}} \sum_{\{R;R'\}} \chi_p^R$$

is integer, then $\sum_{\{R;R'\}} \chi_p^R$ is rational.

The individual χ_p^R are known to be algebraic integers. The sum is therefore also algebraic integers (algebraic integers form a ring).

Again using the fact that an algebraic integer which is rational must be an integer, we conclude that

Theorem The sums

$$\sum_{\{R;R'\}} \chi_p^R$$

are integers.

EXAMPLE :

Character table of A_5

A_5: Alternating group on 5 letters; $= SL_2(\mathbb{F}_4) = L_2(5) = L_2(4) =$ i

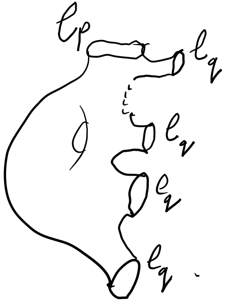
class	1	2	3	5A	5B	
size	1	15	20	12	12	
ρ_1	1	1	1	1	1	trivial
ρ_2	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	orthogonal faithful
ρ_3	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	orthogonal faithful
ρ_4	4	0	1	-1	-1	orthogonal faithful
ρ_5	5	1	-1	0	0	orthogonal faithful

Sums of characters in any column, for fixed value in first column (Dimension) - is integer.

PART 3: Partial sums along columns of character tables : for fixed integer value of an integer column

Consider a conjugacy class $\mathcal{C}_q$ where all the characters are integer, entries for any other column say for $\mathcal{C}_p$, taken over subsets where $\frac{\chi^R(T_q)}{d_R}$ is fixed, is integer.
Similar reasoning as above except the partition functions considered are

$$\frac{1}{|G|}\delta(\Pi T_q^l T_p) = \sum_R \left(\frac{\chi^R(T_q)}{d_R}\right)^l \frac{\chi^R(T_p)}{d_R}$$



Similar steps :

- Write the sum as a sum over distinct level sets for $\frac{\chi^R(T_q)}{d_R}$
- Recognise an invertible integer Van-der-Monde matrix
- Invert and apply to rational data on the left.
- Infer that the sums of $\frac{\chi^R(T_p)}{d_R}$ over the distinct level sets are rational.
- Use the algebraic number property of the sums above, along with the rationality, to conclude that the sums over level sets must be intger.

Along the same lines, partial sums of χ^R_p along level sets of $d_R, \chi^R_{q_1}, \cdots$ where $q_i \neq p$.

PART 4: Row sums

Two classsical identities (proved using expansion into characters and orthogonality relations):

$$\sum_R \frac{\chi^R(T_p)}{d_R} = \sum_q \frac{1}{|T_q|}\delta(T_q T_q T_q)$$

$$\sum_p \chi_p^R = \sum_S N_{SS}^R$$

N_{SS}^R is the multiplicity of R in the tensor product $S \otimes S$.

PART 4: A dual 2D topological field theory where projectors are labelled by conjugacy classes

The G-Flat-TQFT can be defined, from an axiomatic point of view, by associated conjugacy classes (p, q, r) to 3-holed spheres and the class algebra structure constants to the 3-holed sphere.

The algebra is

$$T_p T_q = \sum_r C^r_{pq} T_r$$

Fourier transforming the states to representation basis states :

$$P_R = \frac{d_R}{|G|} \sum_p \chi^R_p T'_p$$

$$P_R P_S = \delta_{RS} P_R$$

PART 4: A dual 2D topological field theory where projectors are labelled by conjugacy classes

$$a_R a_S = \sum_T N_{RS}^T a_T$$

$$A_p = \frac{1}{Sym\ p} \sum_R \chi_p^{\bar{R}} a_R$$

$$A_p A_q = \delta_{pq} A_p$$

$$a_R A_p = \chi_p^R A_p$$

In this G-Fusion-TQFT, the handle creation operator is calculated to be

$$\Theta = \sum_{p} (Sym\ p) A_p$$

The genus h partition function

$$Z_h = \sum_{p} (\mathrm{Sym} p)^{h-1}$$

Can use the genus-h partition functions as input (some polynomials in fusion coefficients), get the power sums of the $\mathrm{Sym} p$ and hence recover the conjugacy class sizes.
By considering general genus partition functions, with one boundary labelled by irrep R, we can prove integrality of partial sums of characters running over conjugacy classes of equal size.

Let S be an irrep such that all the characters in the corresponding row of the character table are integer, i.e. $\chi_p^S \in \mathbb{Z}$ for conjugacy classes p.
Consider the set $\langle S, q \rangle$ which is the complete set of conjugacy classes with a fixed value of the character equal to χ_q^S Then

$$\sum_{p \in \langle S,q \rangle} \chi_p^R \in \mathbb{Z}$$

$$Z_{h=1,S^l,R} = \sum_p (\chi_p^S)^l \chi_p^R$$

PART 4: Generalised partitions

Row sum is a positive integer. A question in CRT/complexity : is there an efficient combinatorial construction for this integer)e.g. $G = S_n$?

We are learning that this integer has the structure $z_R = z_R^+ - z_R^-$, where z_R^+ is a sum of positive integers, over level sets of a subset of integer rows $S \neq R$ and z_R^- is a sum of positive integers over level sets for the complementary subset of integer rows $S \neq R$.

PART 4: Generalised partitions

Row sum is a positive integer. FOr some groups e.g. symmetric groups, all the entries in the character table are integer ; and the positive number is the sum of a greater positive number with a smaller negative number. Something analogous is true for general G.

We are learning that this integer has the structure $z_R = z_R^+ - z_R^-$, where z_R^+ is a sum of positive integers, over level sets of a subset of integer rows $S \neq R$ and z_R^- is a sum of positive integers over level sets for the complementary subset of integer rows $S \neq R$.

In **[2]** we obtained analogous integrality results for partial column sums, using twisted Dijkgraaf-Witten theory, for projective irreducible reps of G.

PART 5: OUTLOOK

More standard approach to integrality properties of finite group character tables is based on Galois theory. For any finite group there is a minimal integer E (the exponent) such that $g^E = 1$. The Galois extension of the rationals by $e^{2\pi i/E}$ - the cyclotomic field $\mathbb{Q}(e^{2\pi i/E})$ - can be used to study the rep theory of G (over $\mathbb{C}$). There is a Galois action on character tables.

PART 5: OUTLOOK

We did not find the proofs of the partial row/colum sum integrality results in the literature. We gave these proofs in **[3]**.

Interplay between the Galois methods and the constructive TQFT methods will be interesting to study.

Better understanding of the duality between G-Flat-TQFT and G-Fusion-TQFT. Duality invariant observables ? Can we extend this in a meaningful way to $\mathbb{C}(G)$? (and derive results about $D^R_{ij}(g)$) ?

PART 5: OUTLOOK

The study of constructive algorithms for rep theory questions using objects arising in physics (here lattice TQFT, fusion TQFT on surfaces) is also relevant to more refined aspects of rep thery – e.g. LR coefficients, or Kronecker coefficients.

J Ben Geloun, S. Ramgoolam, arXiv-2020 ; Algebraic Combinatorics 2023

Bipartite ribbon graphs, Integrality play an important role there ... Understanding the classical/quantum complexity of these algorithms raises some precise questions about symmetric groups...

J Ben Geloun, S. Ramgoolam, https://arxiv.org/abs/2303.12154 ; JHEP-2023