New Initial Approximation in the Loop Vertex Expansion

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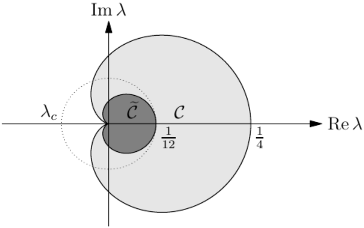
What is known by Loop Vertex Expansion (LVE)

The domain of analyticity for the free energy of the quartic matrix model,

$$F[\lambda, N] = -\frac{1}{N^2} \log \int dM \exp\Big\{ -\mathrm{Tr}(MM^\dagger) - \frac{\lambda}{2N} \mathrm{Tr}([MM^\dagger]^2) \Big\}$$

is proven to be



[V. Rivasseau, 2007; R. Gurau, T. Krajewski 2014].

Loop Vertex Expansion

To prove the main result we apply and develop the LVE machinery, which, in contrast with traditional constructive methods, is not based on cluster expansions nor involves small/large field conditions.

- Like Feynman's perturbative expansion, the LVE allows the computation of connected quantities at a glance: log(forests) = trees.
- The LVE is an explicit repacking of infinitely many subsets of pieces of Feynman amplitudes.
- The convergence of the LVE implies analyticity in the domain uniform in N and Borel summability of the usual perturbation series.

Main steps of LVE

1. The divergence of the standard perturbation theory is caused by the too-singular growth of the interaction potential at large fields. Therefore, we derive an effective action $\mathcal{S}(M)$, providing
   polynomial interaction ====> Log-type interaction.
2. Taylor expansion

$$e^{\mathcal{S}(M)} = \sum_{n=0}^{\infty} \frac{\mathcal{S}^n(M)}{n!}$$

3. Replication of fields, by introducing a degenerate Gaussian measure, so

$$\mathcal{S}^n(M) ====> \prod_i^n \mathcal{S}(M_i)$$

4. Application of the BKAR forest formula + taking the log by reducing the sum over forests to the sum over trees.
5. Derivation of the bounds for the LVE tree amplitudes.

Log-type action and the analyticity

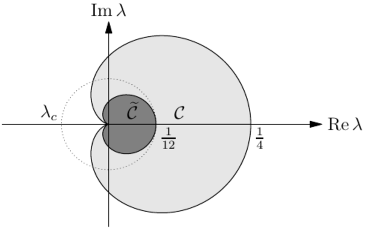
In the LVE the log-type action has the form

$$\mathcal{S}(M) = \log(\mathbf{1} + i\sqrt{\lambda}M),$$

and its derivatives, appearing after Gaussian integration look like

$$\frac{\sqrt{\lambda}}{\mathbf{1} + i\sqrt{\lambda}M} \otimes \ldots \otimes \frac{\sqrt{\lambda}}{\mathbf{1} + i\sqrt{\lambda}M}$$

Hence, the domain of analyticity



Variational Perturbation Theory

[Seznec, Zinn-Justin (1979); Halliday, Suranyi (1980); Feynman, Kleinert (1986); Guida, Konishi, Suzuki (1996);...]

$$\int dx\, e^{-x^2-gx^4} = \int dx\, e^{-x^2-gx^4 \pm ax^2}$$

$$\approx \sum_n^N \int dx\, \frac{(-1)^n (gx^4 - (a(g,N)-1)x^2)^n}{n!} e^{-a(g,N)x^2} =: I_N(g,a)$$

For instance, one can try to find $a(g, N)$ by requiring

$$\partial_a I_N(g,a) = 0$$

New effective action

Starting with the partition function

$$\mathcal{Z}[\lambda, N] = \frac{1}{Z_0} \int dM \exp\left\{ -\operatorname{Tr}(MM^\dagger) - \frac{\lambda}{2N} \operatorname{Tr}([MM^\dagger]^2) \right\}$$

we rewrite it as

$$\begin{aligned} \mathcal{Z}[\lambda, N] &= \frac{1}{Z_0} \int dM \int dA \exp\Big\{ -\frac{1}{2} \operatorname{Tr}(A^2) - a \operatorname{Tr}(MM^\dagger) \\ &+ \mathrm{i} \operatorname{Tr}\Big(A\Big[\sqrt{\frac{\lambda}{N}} MM^\dagger + \frac{(1-a)\sqrt{N}}{2\sqrt{\lambda}}\Big]\Big) + \operatorname{Tr} \frac{(1-a)^2 N}{8\lambda} \Big\} \end{aligned}$$

Integrating out initial fields, rescaling, etc., we obtain effective action

$$N\mathcal{S}(A) = N\left(\operatorname{Tr}\log\left(1 - \mathrm{i}\frac{\sqrt{\lambda}}{a\sqrt{N}} A\right) + \frac{\mathrm{i}}{\sqrt{N}} \operatorname{Tr}\left(A \frac{(1-a)}{2\sqrt{\lambda}} \right)\right)$$

How to compute $\log \mathcal{Z}$

The effective action provides a way to generate convergent expansion for the partition function

$$\mathcal{Z} = K[\lambda, N, a] \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\mu(A) \left[N\mathcal{S}(A) \right]^n$$

To compute the logarithm we apply the forest/tree expansion:
forests ====> log ====> trees

Theorem 8 (Brydges-Kennedy-Abdesselam-Rivasseau). *Let $\phi : \mathbb{R}^{\frac{n(n-1)}{2}} \to \mathbb{C}$ be a smooth, sufficiently derivable function. Then:*

$$\phi(1, \ldots, 1) = \sum_{F \text{ forest}} \int_0^1 \prod_{(i,j)\in F} du_{ij} \left(\frac{\partial^{|E(F)|}\phi}{\prod_{(i,j)\in F} \partial x_{ij}} \right) (v_{ij}^F) \, , \tag{128}$$

where v_{ij}^F is given by:

$$v_{ij}^F = \begin{cases} \inf_{(k,l)\in P_{i\leftrightarrow j}^F} u_{kl} & \text{if } P_{i\leftrightarrow j}^F \text{ exists} \\ 0 & \text{if } P_{i\leftrightarrow j}^F \text{ does not exist} \end{cases} \, , \tag{129}$$

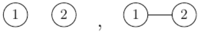
and $|E(F)|$ is the number of edges in the forest F.

BKAR forest formula

$n = 2$

$$\phi(1) = \phi(0) + \int_0^1 dt_{12} \left(\frac{\partial \phi}{\partial x_{12}} \right) (t_{12}) \tag{130}$$

The first term corresponds to the empty forest ($|E(F)| = 0$) and the second one to the full forest ($|E(F)| = 1$).



Preparing the application of the forest formula

Starting with

$$\mathcal{Z} = K[\lambda, N, a] \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d\mu(A) \left[N\mathcal{S}(A) \right]^n ,$$

we generate replicas replacing (for the order n) the integral over the single $N \times N$ matrix A by an integral over an n-tuple of such $N \times N$ matrices A_i , $1 \leq i \leq n$.

$$d\mu \; ====> \; d\mu_C$$

with a degenerate covariance $C_{ij} = 1$, $\forall i, j$.

$$\int d\mu_C(A) \, A_{i|ab} A_{j|cd} = C_{ij} \, \delta_{ad} \delta_{bc}$$

$A_{i|ab}$ is the matrix element in the row a and column b of the matrix A_i.

$$d\mu_C(A) = d\mu(A) \, \delta(A_1 - A_2) \cdot ... \cdot \delta(A_{n-1} - A_n)$$

Application of the forest formula

Now we replace the covariance $C_{ij} = 1$ by $C_{ij}(x) = x_{ij}$, $(x_{ij} = x_{ji})$ evaluated at $x_{ij} = 1$ for $i \neq j$, and $C_{ii}(x) = 1$, $\forall i$, and can apply the BKAR formula

$$\phi(1, \ldots, 1) = \sum_{F \ forest} \int_0^1 \prod_{(i,j) \in F} du_{ij} \left(\frac{\partial^{|E(F)|} \phi}{\prod_{(i,j) \in F} \partial x_{ij}} \right) (v_{ij}^F)$$

Then, we treat derivatives with respect to x_{ij}, as

$$\frac{\partial}{\partial x_{ij}} \left(\int d\mu_{C(x)}(A) F(A) \right) = \int d\mu_{C(x)}(A) \ \mathrm{Tr} \left[\frac{\partial}{\partial A_i} \frac{\partial}{\partial A_j} \right] F(A)$$

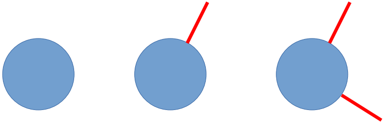
Derivatives of the effective action and corner operators

First derivative

$$\frac{\partial}{\partial A_{i|cd}}\mathcal{S}(A) = \frac{\sqrt{\lambda}}{a\sqrt{N}}\left(1 - \mathrm{i}\frac{\sqrt{\lambda}}{a\sqrt{N}}A_i\right)^{-1}_{cd} + \frac{\mathrm{i}}{\sqrt{N}}\frac{(1-a)}{2\sqrt{\lambda}}\mathbf{1}_{cd}$$

Second derivative

$$\frac{\partial}{\partial A_{i|ab}}\frac{\partial}{\partial A_{i|cd}}\mathcal{S}(A) = \mathrm{i}\frac{\lambda}{a^2 N}\left(1 - \mathrm{i}\frac{\sqrt{\lambda}}{a\sqrt{N}}A_i\right)^{-1}_{ca}\left(1 - \mathrm{i}\frac{\sqrt{\lambda}}{a\sqrt{N}}A_i\right)^{-1}_{bd}$$



$$\mathcal{C} = \begin{cases} \frac{\sqrt{\lambda}}{a}\left(1 - \mathrm{i}\sqrt{\frac{\lambda}{a^2 N}}A_i\right)^{-1}_{cd} + \mathrm{i}\frac{(1-a)}{2\sqrt{\lambda}}\mathbf{1}_{cd}\,, \text{only one corner in the vertex} \\ \\ \frac{\sqrt{\lambda}}{a}\left(1 - \mathrm{i}\sqrt{\frac{\lambda}{a^2 N}}A_i\right)^{-1}_{cd}\,, \text{ if there are more corners} \end{cases}$$

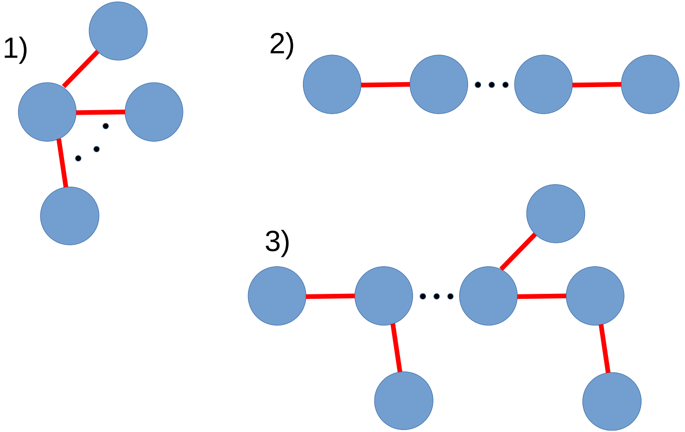
LVE for $\log \mathcal{Z}$

Therefore, the logarithm of $\mathcal{Z}$ can be expressed, as

$$\log \mathcal{Z} = \log K[\lambda, N, a] + \sum_{\substack{T \\ \text{LVE tree}}} \mathcal{A}_T[\lambda, N] \,,$$

$$\mathcal{A}_T[\lambda, N] = \frac{N^{|V(T)|-|E(T)|}}{|V(T)|!} \int_0^1 \prod_{e \in E(T)} dt_e \times \int d\mu_{C_T}(A) \; \mathrm{Tr} \left[\overrightarrow{\prod_{c \in \partial T \text{ corner}}} \mathcal{C}_c(i_c) \right] , \qquad (1)$$

LVE trees



Bounds for the corner operators

Let $\frac{\sqrt{\lambda}}{a} = \rho e^{i\frac{\theta}{2}}$, then

$$\left\| \left(1 - \mathrm{i}\frac{\sqrt{\lambda}}{a\sqrt{N}} A\right)^{-1} \right\| \leq \frac{1}{\cos\frac{\theta}{2}} \ ,$$

and

$$\|\mathcal{C}\| \leq \begin{cases} \left|\frac{\sqrt{\lambda}}{a}\frac{1}{\cos\frac{\theta}{2}}\right| + \left|\frac{(1-a)}{2\sqrt{\lambda}}\right|, \text{only one corner in the vertex} \\ \\ \left|\frac{\sqrt{\lambda}}{a}\frac{1}{\cos\frac{\theta}{2}}\right|, \ \text{ if there are more corners} \end{cases}$$

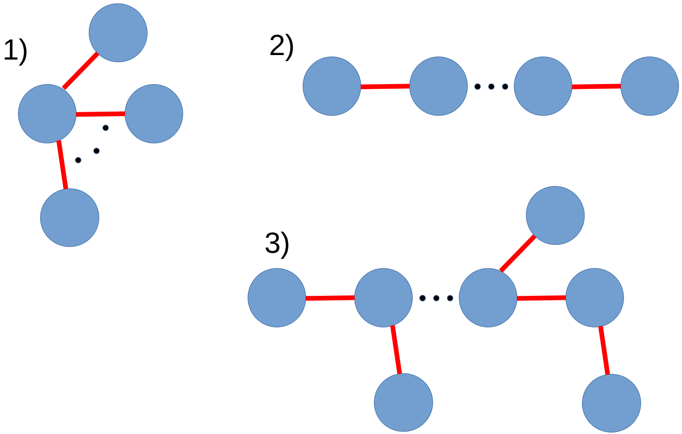
Bounds again

Let $a = x\sqrt{\lambda}e^{i\psi}$, $x > 0$ and $\frac{\sqrt{\lambda}}{a} = \rho e^{i\frac{\theta}{2}}$.
By taking x large, we can ensure that

$$\left|\frac{\sqrt{\lambda}}{a}\frac{1}{\cos\frac{\theta}{2}}\right| \leq \left|\frac{(1-a)}{a\cos\frac{\theta}{2}}\right| \leq \frac{\sqrt{2}}{\cos\frac{\theta}{2}}$$

$$\|\mathcal{C}\| \leq \begin{cases} \left|\frac{\sqrt{\lambda}}{a}\frac{1}{\cos\frac{\theta}{2}}\right| + \left|\frac{(1-a)}{2\sqrt{\lambda}}\right|, \text{only one corner in the vertex} \\ \left|\frac{\sqrt{\lambda}}{a}\frac{1}{\cos\frac{\theta}{2}}\right|, \quad \text{if there are more corners} \end{cases}$$

There are not so many LVE trees with many leaves!



$$\sum_{\substack{T\\ \text{LVE tree}}} = \sum_{|V(T)|<30} + \sum_{T_{\geq}} + \sum_{T_{<}}$$

How many trees have more than $\alpha|V(T)|$ leaves?

Lemma

The number of trees with $|V(T)|$ vertices and $\alpha|V(T)|$ or more leaves with $1/2 < \alpha < 1$, $|T_{\geq}|$ is bounded by

$$
\begin{aligned}
|T_{\geq}| &\leq (|V(T)| - \lceil \alpha|V(T)| \rceil + 1)\big(|V(T)|!\big) \\
&\times 2^{|V(T)|-3} e^{\lceil \alpha|V(T)| \rceil} \left(\frac{1-\alpha}{\alpha}\right)^{\lceil \alpha|V(T)| \rceil}
\end{aligned}
$$

Bound for trees $\in T_{\geq}$

Lemma

For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\alpha = \frac{59}{60}$, the sum of absolute values of amplitudes of the trees $T_{\geq}$ is smaller than the sum of an absolutely convergent series,

$$\sum_{T_{\geq}} \left|\mathcal{A}_{T_{\geq}}\right| \leq \frac{N^2 e}{472} \sum_{v=60}^{\infty} \left(\frac{1}{60} v + 1\right) \left(\frac{7}{8}\right)^{v} .$$

Bound for trees $\in T_<$

Lemma

For $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and $\alpha = \frac{59}{60}$, the sum of absolute values of amplitudes of the trees $T_<$ is smaller than the sum of an absolutely convergent series,

$$\sum_{T_<} |\mathcal{A}_{T_<}| \quad \leq \quad N^2 \frac{2(x_4 \cos\frac{\theta}{2})^5 \sqrt{\lambda}}{3(x_4\sqrt{\lambda} - 1)} \sum_{v=60}^{\infty} \frac{1}{v^2} \left(\frac{1}{2}\right)^v ,$$

where

$$x_4 = \max\{x3, \frac{2^{30} e^{30}}{\cos\frac{\theta}{2}} \Big(\frac{3\sqrt{2}}{2\cos\frac{\theta}{2}}\Big)^{59/30}\} + 1 \, .$$

Result

Let's define a cardioid domain of the coupling constant λ,

$$\mathcal{C} = \left\{ \lambda \in \mathbb{C} \,\middle|\, \arg \lambda = \phi \, , 4|\lambda| < \cos^2\left(\frac{\phi}{2}\right) \right\} ,$$

then

Theorem

For any $\lambda \in \mathcal{X}$, where $\mathcal{X}$ is defined as

$$\mathcal{X} = \mathcal{C} \cup \left\{ \lambda \in \mathbb{C} \,\middle|\, \lambda \neq 0 \, , |\phi| < \frac{3\pi}{2} \right\} ,$$

the free energy

$$F[\lambda, N] = -\frac{1}{N^2} \log \mathcal{Z}[\lambda, N] \, .$$

of the quartic matrix model, is analytic uniformly in N.

Outlook

Loop Vertex Expansion was successfully applied to many problems related to quartic models:

- J. Magnen and V. Rivasseau, "Constructive φ^4 field theory without tears," (2008)
- T. Delepouve and V. Rivasseau, "Constructive Tensor Field Theory: The T_3^4 Model," (2014)
- V. Lahoche, "Constructive Tensorial Group Field Theory I: The $U(1)$ T_4^4 Model," (2015)
- R. Gurau, "The 1/N Expansion of Tensor Models Beyond Perturbation Theory," (2014)
- D. Benedetti, R. Gurau, H. Keppler, and D. Lettera, "The small-N series in the zero-dimensional O(N) model: constructive expansions and transseries.", (2022).

It might be interesting to try to extend these results with the Variational Loop Vertex expansion.