Limit shapes from skew Howe duality

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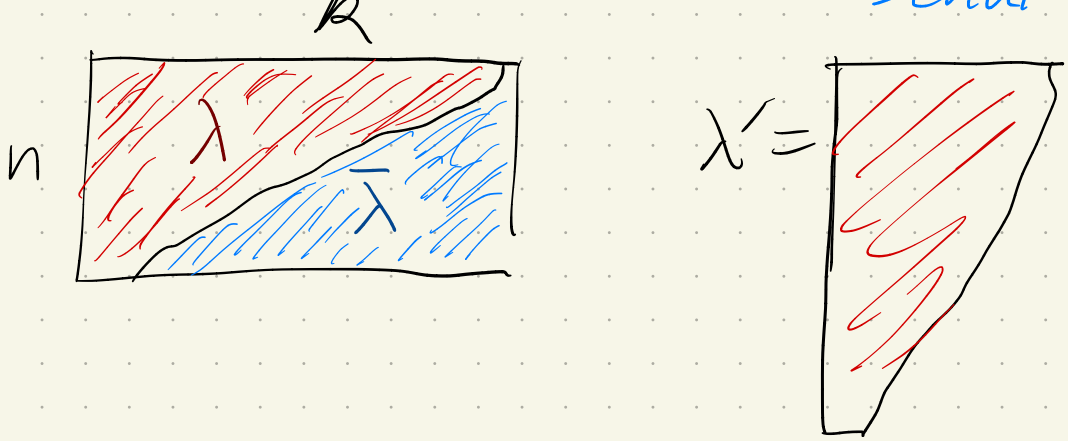
$(GL_n \times GL_k)$-reps $\Lambda(V_n \boxtimes V_k) \cong \bigoplus_{\lambda \subseteq k^n} V(\lambda) \boxtimes V(\bar{\lambda})$

Skew Howe duality

Taking characters give dual Cauchy eq.

$$\prod_{i,j=1}^{n,k} (1 + x_i y_j) = \sum_{\lambda \subseteq k^n} s_\lambda(x_n) \underbrace{s_{\lambda'}(y_k)}_{\text{schur functions}}$$



This gives a measure on partitions inside $n \times k$ rectangle by

$$\mu_{n,k}(\lambda) = \frac{s_\lambda(x) s_{\lambda'}(y)}{\prod_{i,j} (1 + x_i y_j)}$$

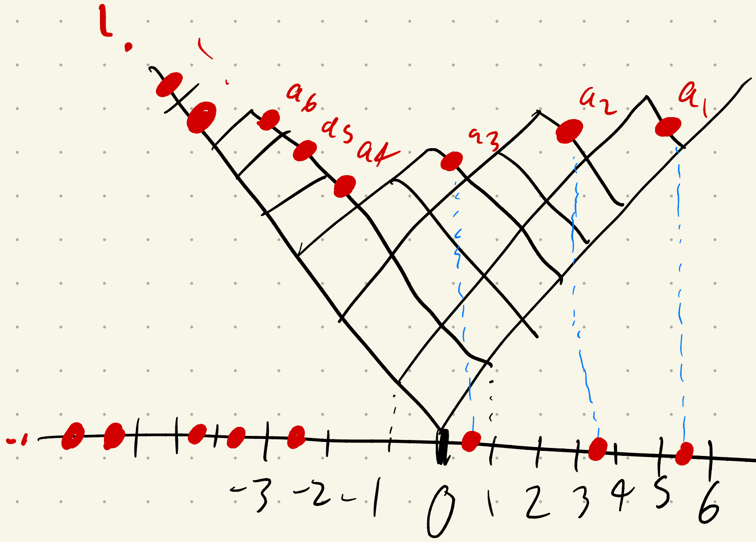
For $x = 1, y = 1$ Krawtchouk

Limit shape

Boundary fluctuations

For $x=1, y=\alpha$ oriented digital boiling λ_1 distribution
Gravner–Tracy–Widom ~'01

Free fermions Okounkov–Reshetikhin
(forthcoming work with D. Betea, P. Nikitin, D. Saratannikov)



particles will be on half integer points

Maya diagram

λ partition, i-th particle $a_i = \lambda_i - i + 1/2$

We can realize partitions as elements in $\Lambda\mathbb{C}^\infty$ $\mathbb{C}^\infty = \{v_{i+1/2} \mid i \in \mathbb{Z}\}$

$|\lambda\rangle = v_{a_1} \wedge v_{a_2} \wedge v_{a_3} \wedge \cdots$

$\mathcal{F} = \operatorname{span}_\lambda |\lambda\rangle \subseteq \Lambda\mathbb{C}^\infty$, (Fermionic) Fock space

Fact/ $\{|\lambda\rangle\}_\lambda$ basis $\mathcal{F}$

There is a Clifford algebra action given by ψ_i creates a particle at i
ψ_i^* annihilates a particle. If we cannot do this, then result is 0.

$$\psi_i v = v_i \wedge v$$

$$\psi_i \psi_j + \psi_j \psi_i = 0 \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0$$

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}$$

Current operators: $a_0 F = 0 \cdot F$

$$a_\ell = \sum_{j \in \mathbb{Z}+1/2} :\psi_{j-\ell} \psi_j^*:$$

Moves particles ℓ steps to the left. (if possible)

These satisfy the relations of the infinite dimensional Heisenberg alg.

$$a_m a_\ell - a_\ell a_m = [a_m, a_\ell] = m\,\delta_{m+\ell,0}$$

anti-involution $*$ $\quad \psi_i \longleftrightarrow \psi_i^*$

$$a_i^* = a_{-i}$$

F^* dual space $\langle\lambda|$ dual of $|\lambda\rangle$

$$\langle\mu|\lambda\rangle = \delta_{\lambda\mu}$$

natural pairing $F^* \times F \to \mathbb{C}$

Let $X \in$ Clifford algebra

$\langle\mu|X|\lambda\rangle$ is well-defined.

$$\big(\langle\mu|X\big)|\lambda\rangle = \langle\mu|\big(X|\lambda\rangle\big)$$

$$\big(a_\ell|\lambda\rangle\big)^* = \langle\lambda|a_{-\ell}$$

The fact we have both a Clifford a'gebra and Heisenbeg algebra action on the same space is called the boson-fermion correspondence.

Half-vertex operators

$$\Gamma_\pm(x) = \exp\Big(\sum_{\ell=1}^{\infty} \frac{\overbrace{p_\ell(x)}^{\text{powersum sym. funcs.}}}{\ell} a_{\pm\ell}\Big) \qquad p_\ell(x) = x_1^\ell + x_2^\ell + \cdots$$

Fact/ $\langle\mu|\Gamma_+(x)|\lambda\rangle = s_{\lambda/\mu}(x)$

$$\Gamma_\pm(x,y) = \Gamma_\pm(x)\,\Gamma_\pm(y)$$

$$(\Gamma_+(x))^* = \Gamma_-(x)$$

$$\Gamma_+(x)\,\Gamma_-(y) = H(x;y)\,\Gamma_-(y)\,\Gamma_+(x)$$

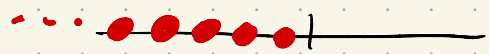
where $H(x;y) = \prod_{i,j} \frac{1}{(1-x_i y_j)}$

$\ell > 0$

$a_\ell|0\rangle = 0$

Ex/ $\langle 0|\Gamma_+(x)\Gamma_-(y)|0\rangle$

$|0\rangle$



Cauchy identity $= H(x;y)\langle 0|\Gamma_-(y)\Gamma_+(x)|0\rangle$

$\Gamma_+(x)|0\rangle = |0\rangle$

$\Gamma_+(x) = 1 +$ terms w/ a_ℓ, $\ell>0$

$$= H(x;y)\langle 0|0\rangle = H(x;y)$$

$$= \sum_\lambda \langle 0|\Gamma_+(x)\underbrace{|\lambda\rangle\langle\lambda|}_{\text{identity operator}}\Gamma_-(y)|0\rangle$$

$$= \sum_\lambda s_\lambda(x)\,s_\lambda(y)$$

Let $\Gamma'_\pm(x) = \exp\left(\sum_{\ell=1}^{\infty} (-1)^{\ell-1}\frac{p_\ell(x)}{\ell}\alpha_{\pm\ell}\right)$

$$= \omega\,\Gamma_\pm(x)$$

$\omega p_\ell = (-1)^{\ell-1} p_\ell$ Fact/ $\omega s_\lambda = s_{\lambda'}$

Ex/ $\langle 0|\Gamma_+(x)\,\Gamma_-'(y)|0\rangle$ dual Cauchy identity

$$= \sum_{\lambda} \langle 0|\Gamma_+(x)|\lambda\rangle \cdot \langle \lambda|\Gamma_-'(y)|0\rangle$$

$$= \sum_{\lambda} s_\lambda(x)\, \underbrace{s_{\lambda'}(y)}_{\omega s_\lambda}$$

$$= E(x;y)\langle 0|\Gamma_-'(y)\Gamma_+(x)|0\rangle$$

Note/

$$\Gamma_-'(y) = \Gamma_-^{-1}(-y)$$

$$H(x;-y)^{-1} = E(x;y) = \prod_{i,j}(1+x_i y_j)$$

Define formal power series

$$\psi(z) = \sum_{j\in\mathbb{Z}+1/2} \psi_j z^j \qquad \psi^*(w) = \sum_{j\in\mathbb{Z}+1/2} \psi_j^* w^{-j}$$

$$\Gamma_\pm(x)\psi(z) = H(x;z^{\pm})\psi(z)\Gamma_\pm(x)$$

$$\Gamma_\pm(x)\psi^*(w) = H(x;w^{\pm})^{-1}\psi^*(w)\Gamma_\pm(x)$$

Note $\psi_i\psi_i^*$ is a projection operator checking if there is a particle at i.

$\langle 0|\Gamma_+(x)\psi_i\psi_i^*\Gamma_-'(y)|0\rangle$ this checks if there is a particle at i.

This gives the kernel of a det. pt. process by using Wick's theorem.

$$\langle 0|\Gamma_+(x)\psi_{i_1}\psi_{i_1}^*\cdots\psi_{i_k}\psi_{i_k}^*\Gamma_-'(y)|0\rangle$$

$$= \det\left[\langle 0|\Gamma_+(x)\psi_{i_\alpha}\psi_{i_\beta}^*\Gamma_-'(y)|0\rangle\right]_{\alpha,\beta=1}^{k}$$

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$$\langle 0|\Gamma_+(x)\psi_k\psi_{k'}^*\Gamma_-'(y)|0\rangle$$

$$= \oint\oint_{|w|<|z|} \frac{dz}{2\pi i}\frac{dw}{2\pi i} z^{-1}w^{-1}\cdot z^{k}w^{-k'}\langle 0|\Gamma_+(x)\psi(z)\psi^*(w)\Gamma_-'(y)|0\rangle$$

$$= K(k,k')$$

comes from

$\swarrow \langle 0|\psi(z)\psi^*(w)|0\rangle$

$$= \oint\oint_{|-y_j|<|w|<|z|<|x_j^{-1}|} \frac{dz}{2\pi i z}\frac{dw}{2\pi i w}\frac{F(z)}{F(w)} z^{k}w^{-k'}\frac{\sqrt{zw}}{z-w}$$

where $F(z) = \prod_{j=1}^{n}\frac{1}{1-x_j z}\prod_{j=1}^{k}\frac{1}{1+\frac{y_j}{z}}$

However, this does not work for other dual reductive pairs (G_1, G_2).

$(G_1,G_2) = (Sp_{2n}, Sp_{2k}), (SO_{2n+1}, Pin_{2k}), (SO_{2n+1}, Sp_{2k}), (O_{2n}, SO_k)$

We focus on the measure given by dim.

$$\mu_{n,k}(\lambda) = \dim V(\lambda)_{G_1} \boxtimes V(\bar{\lambda}')_{G_2} / 2^{nk}$$

To get product formulas, we compute the multiplicity of $V(\lambda)_{G_1}$ in $V^{\otimes k}$, where

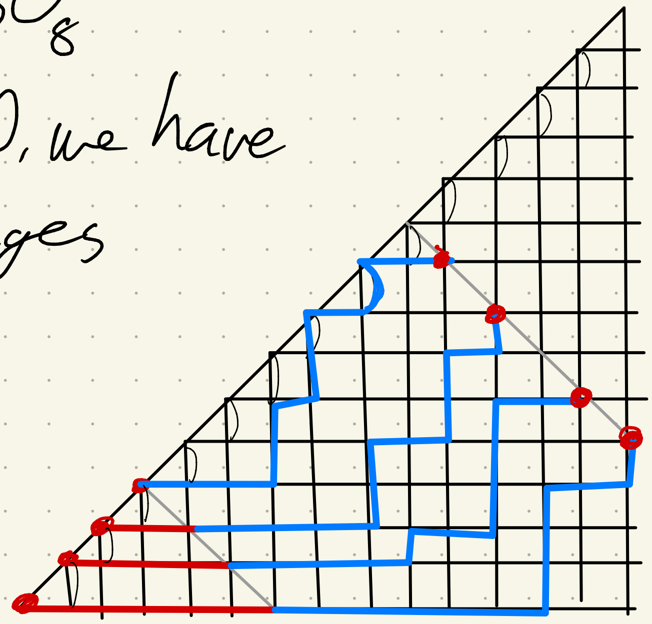
$V = \Lambda V(\square)$ for $G_1 = Sp_{2n}$, $V =$ spinor otherwise

We count the multiplicity using the theory of Kashiwara crystals. Here, we translate this into families of non-intersecting Dyck paths: paths of $U \uparrow, R \rightarrow$ steps that stay below the diagonal

The LGV lemma gives determinat formulas that we then evaluate.

Ex/ $G_1 = SO_8$

For type D, we have "double" edges along the diagonal



$\lambda = (2,2,1,1)$

$= \Lambda_2 + 2\Lambda_4$

$V = V(\Lambda_3) \oplus V(\Lambda_4)$

$V^{\otimes 12}, \; k = 6$

Thm/ There is a natural q-analog of these paths that give the q-char of $V(\bar{\lambda}')_{G_2}$.

Cor/

$$\mu_{n,k}(\lambda) = \frac{\dim V(\lambda)_{G_1} \cdot \dim V(\lambda)_{G_2}}{2^{nk}} = \frac{\dim V(\lambda)_{G_1} \, \mathrm{Mult}(\lambda)}{2^{nk}}$$

$$= \underbrace{C_{n,k}}_{\text{normalization constant}} \underbrace{\prod_{i<j} (\nu_i^2 - \nu_j^2)^2}_{\text{Vandermonde determinant}} \underbrace{\prod_i W(\nu_i)}_{\nu_i^{*} \binom{k+n-1}{\nu_i}}$$

To compute the limit shape, we rewrite

$$\mu_{n,k}(\lambda) = \frac{1}{Z_n} \exp\left(-n^2 J[\rho_n] + O(n \ln n)\right),$$

where $k = cn$ and $J[\rho_n]$ is a functional in terms of the density $\rho_n(x)$.

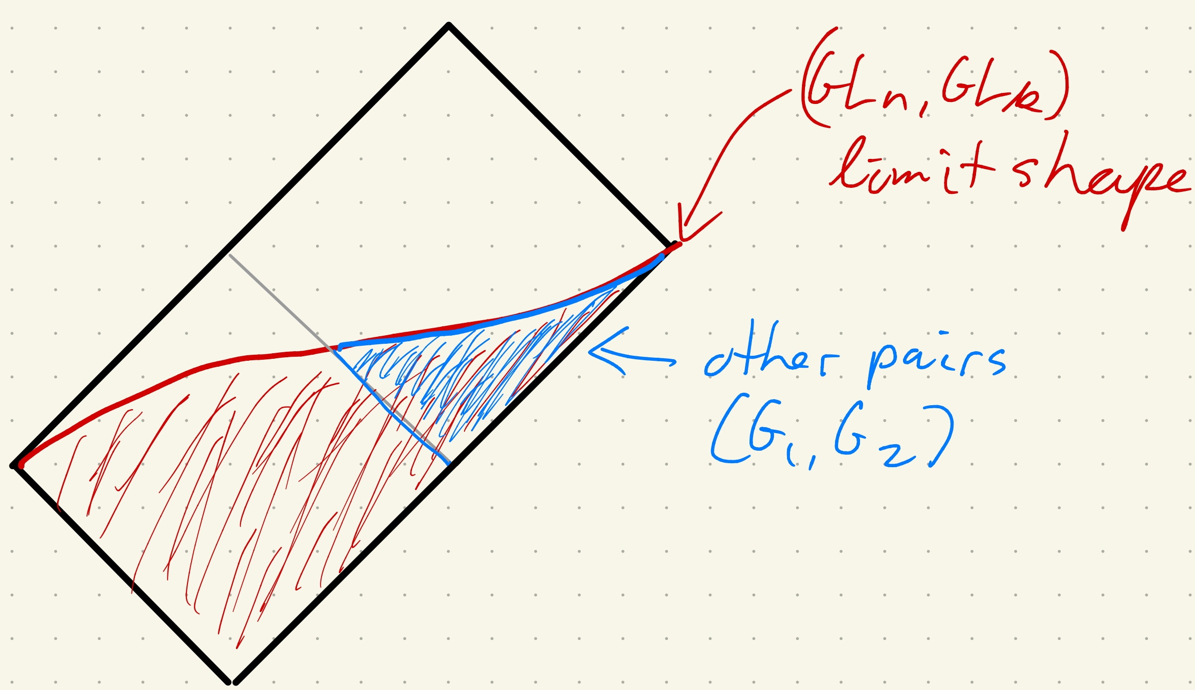
We then solve a Riemann-Hilbert problem to show the minimizer of $J[\rho_n]$ is

$$\rho(x) = \frac{1}{2\pi}\left[\tan^{-1}\left(\frac{-(c+1)x+2c}{(c-1)\sqrt{c-x^2}}\right) + \tan^{-1}\left(\frac{(c+1)x+2c}{(c-1)\sqrt{c-x^2}}\right)\right]$$

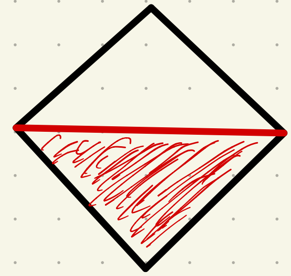
Thm/ The limit shape is given by

$$f(x) = \begin{cases} 1 + \int_0^x (1 - 2\rho(t))\,dt & \text{if } c \geq 1, \\ 1 + \int_0^x (2\rho(t) - 1)\,dt & \text{if } c < 1. \end{cases}$$

Ex/



The case $c=1$ is special



For (GL_n, GL_k) case, the boundary fluctuations are given by the Tracy-Widom GUE distribution

Thm [BNNSS '23+]/ When the (GL_n, GL_k) limit shape approches the corner with rate $\tilde{s}/\sqrt{n}$, then the fluctuations are given by the discrete Hermite kernel.

Conj/ This holds for other skew Howe duality measures

Thank you!