Counting partitions by genus

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IHÉS, November 2023

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J.-B. Z., "*Counting partitions by genus. I. Genus 0 to 2* ", arxiv:2303.05875 ,

Robert Coquereaux and J.-B. Z., "*Counting partitions by genus. II. A compendium of results*" , arxiv:2305.011005

What is the problem ?

α a **set** partition, of the set $\{1,\cdots,n\}$:

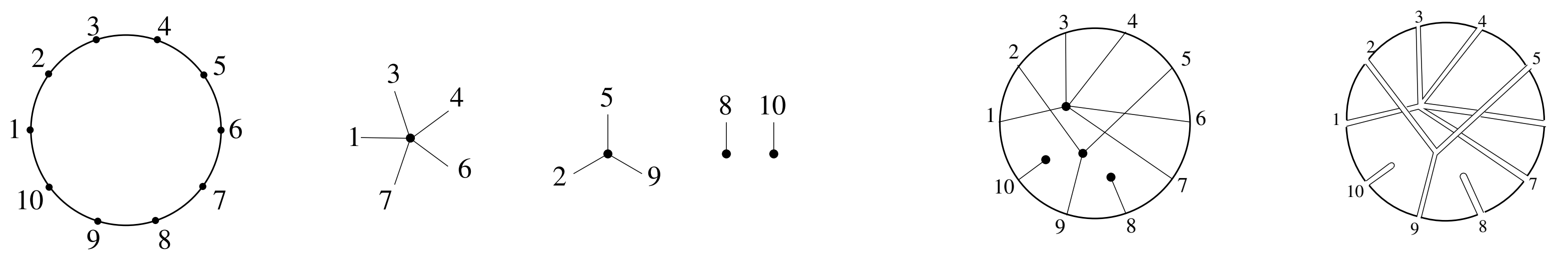
α of *type* $[\alpha] = [1^{\alpha_1},\cdots,n^{\alpha_n}]$ if $\alpha_\ell = \#$blocks of size ℓ. (Note $[\alpha] \vdash n$)

Represent it diagrammatically by a circle with n points on it numbered from 1 to n and α_ℓ ℓ-vertices inside disk, and a map between vertices and circle **that respects the order**(*).

Then *genus* g given by Euler formula

$$2-2g = \#\text{vertices} - \#\text{edges} + \#\text{faces} = \sum \alpha_\ell + 1 - n + f$$

For example the partition $(\{1,3,4,6,7\},\{2,5,9\},\{8\},\{10\})$ of $\{1,\cdots,10\}$: type $[1^2,3,5]$ represented by the map



Here $\sum \alpha_\ell + 1 = 5$, $n = 10$, $f = 3$ hence $g = 2$.

Note 1: it is the constraint (*) that makes the counting non trivial.

Note 2: since $f \geq 1$, $g \leq g_{\text{max}} := \left\lfloor \frac{1}{2}(n - \sum \alpha_k) \right\rfloor$

Problem: compute the number $C^{(g)}_{n,[\alpha]}$ of partitions of type $[\alpha]$ and genus g.

Their sum over g is known $C_{n,[\alpha]} = \sum_g C^{(g)}_{n,[\alpha]} = \frac{n!}{\prod_{\ell=1}^{n} \alpha_\ell!(\ell!)^{\alpha_\ell}}$

(Faà di Bruno coefficients: n-th derivatives of a composition of two functions $\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(g(t))$.)

Introduce generating functions (GF)

$$W(x) = \sum_{n\geq 1} \kappa_n x^n$$

for a set of indeterminates κ_n, $n \in \mathbb{N}_+$, and

$$\begin{aligned} Z(x) &= 1 + \sum_{n\geq 1} \sum_{[\alpha]\vdash n} C_{n,[\alpha]} \kappa_{[\alpha]} x^n = \sum_g Z^{(g)}(x) \\ Z^{(g)}(x) &= \delta_{g0} + \sum_{n\geq 1} \sum_{[\alpha]\vdash n} C^{(g)}_{n,[\alpha]} \kappa_{[\alpha]} x^n \end{aligned}$$

where $\kappa_{[\alpha]} := \prod_{\ell=1}^{n} \kappa_\ell^{\alpha_\ell}$.

Side remark: in probability theory or statistical mechanics : $\kappa_\ell = \ell$-th cumulant, $m_n = n$-th moment of r.v. X.

$m_n = \sum_{[\alpha]\vdash n} C_{n,[\alpha]}\kappa_{[\alpha]}$: ordinary "cumulant expansion"

$m_n = \sum_{[\alpha]\vdash n} C^{(0)}_{n,[\alpha]}\kappa_{[\alpha]}$: "expansion on *non-crossing* (aka planar or free) *cumulants*", in (large) matrix integrals or free probability.

Thus knowledge of the $C^{(g)}_{n,[\alpha]}$, $g \neq 0$, would yield an *interpolation* between ordinary and free cumulants expansions:

$$m_n(\epsilon) = \sum_{[\alpha]\vdash n} \sum_{g=0}^{g_{\max}([\alpha])} C^{(g)}_{n,[\alpha]} \epsilon^g \kappa_{[\alpha]} \,.$$

For example, $m_4(\epsilon) = \kappa_4 + 4\,\kappa_3\,\kappa_1 + (2+\epsilon)\kappa_2^2 + 6\,\kappa_2\,{\kappa_1}^2 + {\kappa_1}^4$.

Outline of this talk

1. A few exact results

2. Genus 0, planar (aka non-crossing) partitions. Kreweras' formula and Cvitanovic's equation

3. Reduction to "primitives" [Cori and Hetyei]

4.-5. Dressing primitives: the case of genus 1 and 2

6. Final remarks.

1. A few exact results

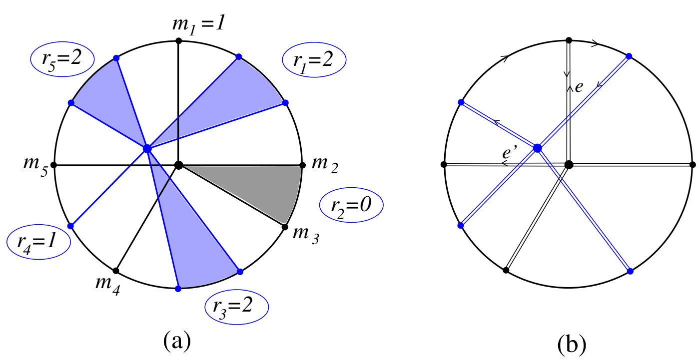
– partitions in pairs, *i.e.*,of type $[2^k]$.

An old problem [Walsh–Lehmann '72, Harer–Zagier '86, . . .]

$$C^{(g)}_{2k,[2^k]} = \frac{(2k)!}{(k+1)!(k-2g)!}\left[\left(\frac{u/2}{\tanh u/2}\right)^{k+1}\right]_{u^{2g}}$$

– partitions into two parts, *i.e.*,of type $[n-p,p]$ [Z'23]

$$C^{(g)}_{n,[p,n-p]} = \frac{n}{g+1}\binom{p-1}{g}\binom{n-p-1}{g}$$



– genus 0, any type: see below

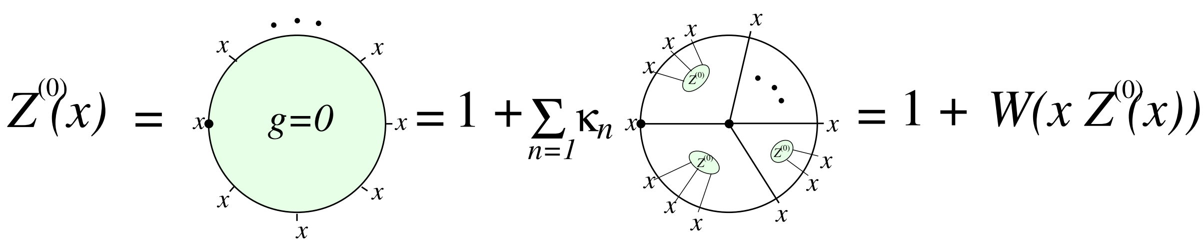
2. Genus 0. Non-crossing partitions

Kreweras' result (1972)

$$C^{(0)}_{n,[\alpha]} = \frac{n!}{(n+1-\sum \alpha_k)!\ \prod_k \alpha_k!}\,, \tag{1}$$

$$Z^{(0)}(x) = 1 + W(xZ^{(0)}(x))\,. \tag{2}$$

Reappeared later in [Brézin-Itzykson-Parisi-Z '78, Cvitanovic '81, Speicher '94]



Equivalently, $X(y) := y^{-1}(1+W(y))$, $Y(u) := u^{-1}Z^{(0)}(u^{-1})$ satisfy $X \circ Y = \mathrm{id}$ ("inverse relation").

$R(z) = Y(z) - \frac{1}{z}$ is Voiculescu's R function.

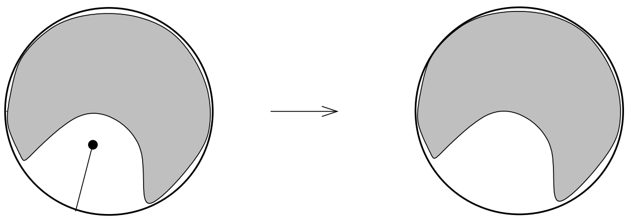
3. Reduction to primitives

[Cori–Hetyei 2017]

4 operations that do not affect the genus of a diagram. Recall that $2 - 2g = \sum \alpha_\ell + 1 - n + f$

– removal of singletons:

$\alpha_1 \to 0, n \to n - \alpha_1$



Reduction to primitives

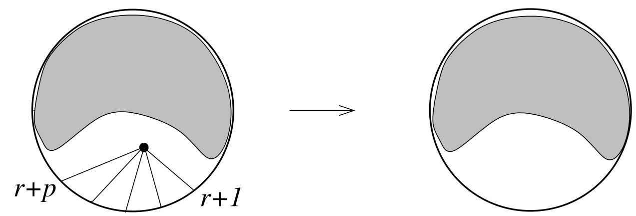
4 operations that do not affect the genus of a diagram. Recall

$2 - 2g = \sum \alpha_\ell + 1 - n + f$

– removal of singletons

– removal of "centipedes":

$\alpha_p \to \alpha_p - 1, n \to n - p, f \to f - (p - 1)$



Reduction to primitives

4 operations that do not affect the genus of a diagram. Recall

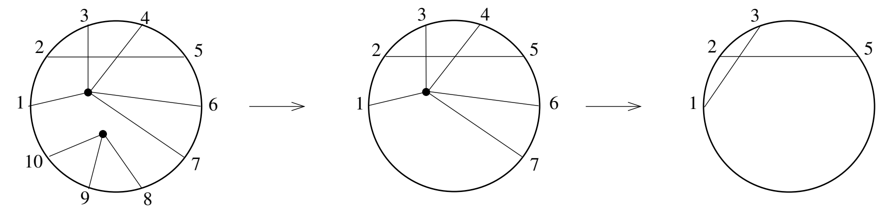
$2-2g=\sum\alpha_\ell+1-n+f$

– removal of singletons

– removal of "centipedes"

– removal of "adjacent pairs"

$\sum\alpha_\ell$ unchanged, $n\to n-1$, $f\to f-1$



(a) (b) (c)

Reduction to primitives

4 operations that do not affect the genus of a diagram. Recall

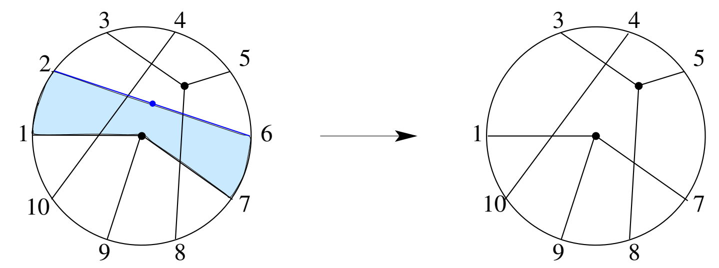
$2-2g=\sum\alpha_\ell+1-n+f$

– removal of singletons:

– removal of "centipedes":

– removal of adjacent pairs

– removal of parallel lines



(one of which at least attached to a 2-vertex)

$\alpha_p\to\alpha_p-1, n\to n-2, f\to f-1$

Reduction to primitives

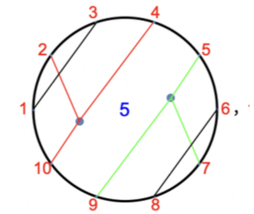
4 operations that do not affect the genus of a diagram.

– removal of singletons

– removal of "centipedes"

– removal of adjacent pairs

– removal of parallel lines



a semi-primitive diagram

Primitive diagrams are those in which all these reductions have been carried out.

Result independent of the order of reductions.

(Need some removal convention for later reconstruction)

Another subtlety: primitive have no parallel pairs; "semi-primitive" diagrams may still have some, attached to vertices of valency > 2

Theorem [Cori–Hetyei, 2017] For given g, there are only a *finite* number of primitives and semi-primitives.

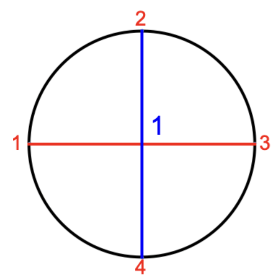
Hint: for a primitive, no 2-cycle, hence $f \leq n/3$ and $\sum \alpha_\ell \leq n/2$, hence $n \leq 6(2g-1)$.

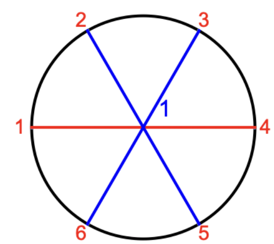
Idea: Reconstruct all diagrams by "dressing" the primitive ones.

"dressing" = reintroduce the lines removed above

4. Genus 1.

Cori–Hetyei's result (2013, 2017). There are two primitive diagrams of genus 1:





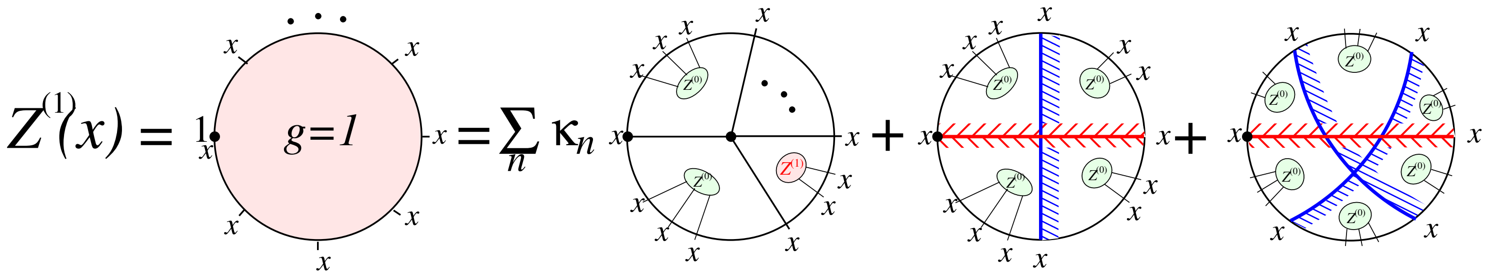
$(g = 1)$

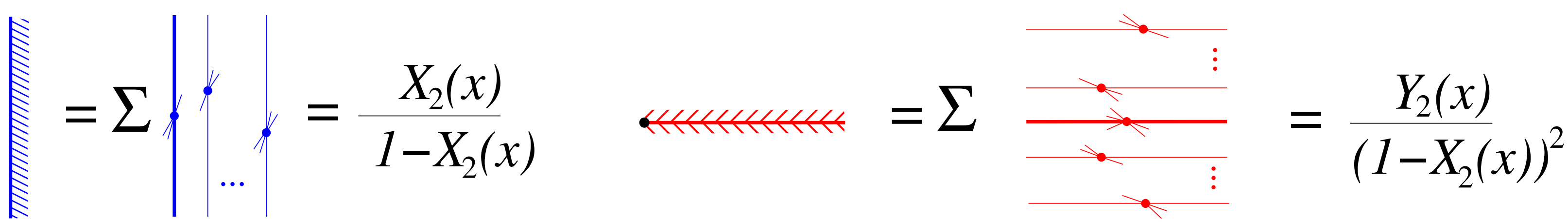
The two "primitive" diagrams of genus 1. The blue figure in the middle is the length of its orbit under rotations.

Then write a Cvitanovic-like relation:

$$Z^{(1)}(x) = \sum_{n\geq 2} \kappa_n n x^n (Z^{(0)})^{n-1} Z^{(1)}(x) + \text{sum of dressed diagrams of Fig. } (g = 1)$$

$$Z^{(1)}(x) = \underbrace{\sum_{n\geq 2}\kappa_n n x^n (Z^{(0)})^{n-1}}_{xW'(xZ^{(0)})}\, Z^{(1)}(x) + \text{sum of dressed diagrams of Fig. } {\color{red}(g=1)}$$





where

$$X_2(x) := \sum_{k\geq 2}(k-1)\kappa_k x^k \qquad Y_2(x) = \sum_{k\geq 2}\frac{k(k-1)}{2}\kappa_k x^k$$

Dressing of the two primitive diagrams gives

$X_2 Y_2/(1-X_2)^3 + X_2^2 Y_2/(1-X_2)^4 = X_2 Y_2/(1-X_2)^4\big|_{\tilde{x}=xZ^{(0)}(x)}$

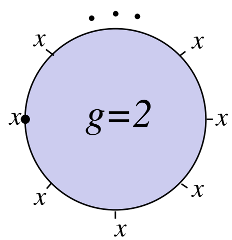
Theorem 1. *If $\tilde{x} = xZ^{(0)}(x)$, the generating function of genus 1 partitions is given by*

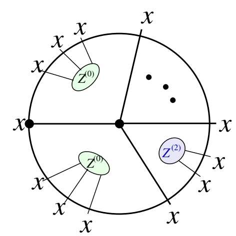
$$Z^{(1)}(x) = \frac{X_2(\tilde{x})Y_2(\tilde{x})}{(1-X_2(\tilde{x}))^4\,(1-V(x))}\,. \qquad (3)$$

with

$$X_2(x) := \sum_{k\geq 2}(k-1)\kappa_k x^k \qquad Y_2(x) = \sum_{k\geq 2}\frac{k(k-1)}{2}\kappa_k x^k \qquad V(x) = \sum_k k\kappa_k x^k Z^{(0)\,k-1} = xW'(\tilde{x})\,.$$

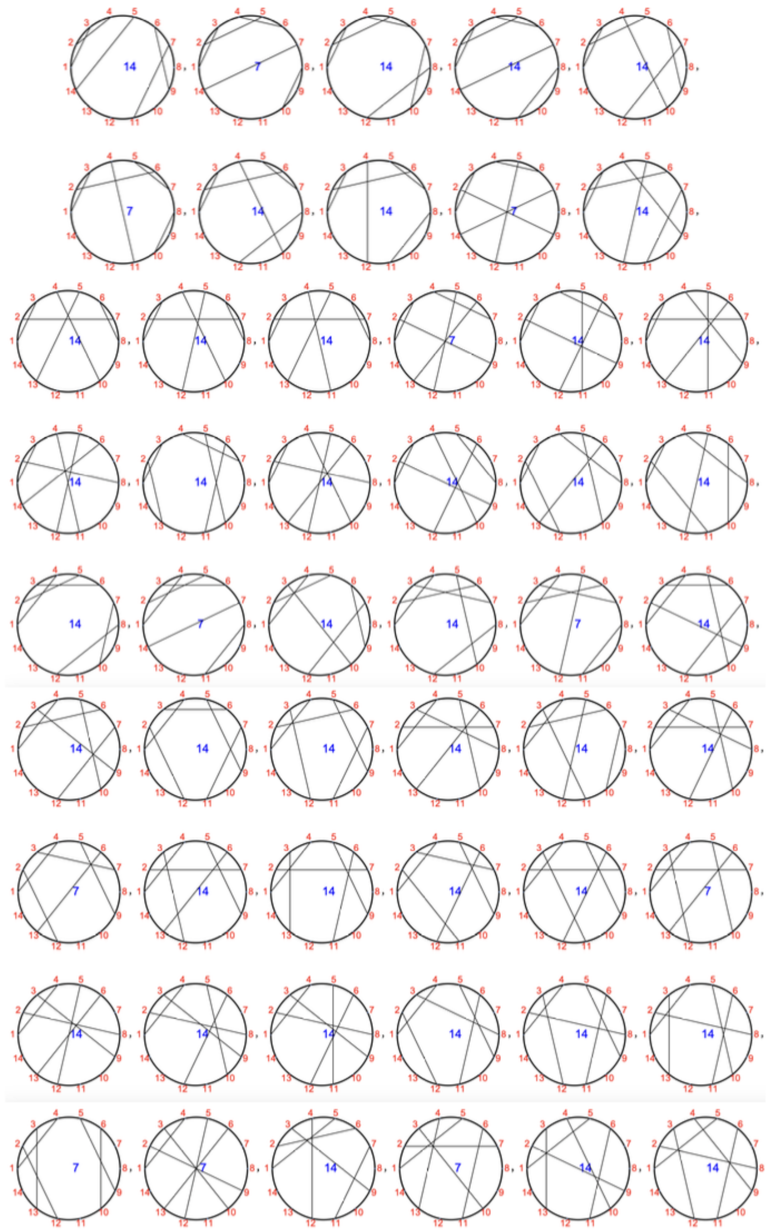
5. Genus 2. Same idea . . . but more complicated !!

$$Z^{(2)}(x) =$$ 

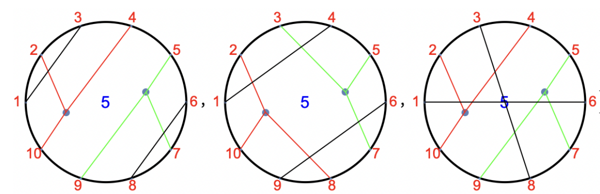
$$= \sum_n \kappa_n$$ 
$$+ \sum_n$$ dressed (semi−)primitive diagrams of genus 2

n	2-vertices	one 3-vertex	two 3-vertices	two 3-vertices semi-prim.	one 4-vertex
6	0	0	1	0	0
7	0	14	0	0	0
8	21	0	20	0	6
9	0	141	0	0	0
10	168	0	65	15	15
11	0	407	0	0	0
12	483	0	52	36	9
13	0	455	0	0	0
14	651	0	0	21	0
15	0	175	0	0	0
16	420	0	0	0	0
17	0	0	0	0	0
18	105	0	0	0	0

Table 1. Number of (semi-)primitive diagrams of genus 2.



The 52 primitive diagrams of type $[2^7]$ and genus 2, of total weight 651



The 3 semi-primitive diagrams of type $[2^2\ 3^2]$ and genus 2, of total weight 15

Theorem 2. *The generating function of genus 2 partitions is given by*

$$Z^{(2)}(x)(1 - V(x)) = z_2 + z_3 + z_{33} + z_{33_s} + z_4 \qquad (4)$$

where $V(x) = xW'(\tilde{x})$, $\tilde{x} = xZ^{(0)}(x)$ *as before and*

$$\begin{aligned}
z_2 &= \tilde{Y}_2(21\tilde{X}_2^3 + 168\tilde{X}_2^4 + 483\tilde{X}_2^5 + 651\tilde{X}_2^6 + 420\tilde{X}_2^7 + 105\tilde{X}_2^8)\,; \\
z_3 &= \tilde{X}_3\tilde{Y}_2(8\tilde{X}_2 + 94\tilde{X}_2^2 + 296\tilde{X}_2^3 + 350\tilde{X}_2^4 + 140\tilde{X}_2^5) \\
&\quad + \tilde{X}_2(6\tilde{X}_2 + 47\tilde{X}_2^2 + 111\tilde{X}_2^3 + 105\tilde{X}_2^4 + 35\tilde{X}_2^5)\Big(\tilde{Y}_3 + \tilde{X}_3\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\Big)\,; \\
z_{33} &= \tilde{X}_3^2\tilde{Y}_2(5 + 26\tilde{X}_2 + 26\tilde{X}_2^2) \\
&\quad + \tilde{X}_3(1 + 15\tilde{X}_2 + 39\tilde{X}_2^2 + 26\tilde{X}_2^3)\Big(\tilde{Y}_3 + \tilde{X}_3\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\Big)\,; \\
z_{33_s} &= \tilde{Y}_2\tilde{X}_3^2\tilde{X}_2(6 + 18\tilde{X}_2 + 12\tilde{X}_2^2)(1 - X_2(\tilde{x})) \\
&\quad + \tilde{Y}_3\tilde{X}_3\tilde{X}_2^2(9 + 18\tilde{X}_2 + 9\tilde{X}_2^2)(1 - (X_2(\tilde{x})) + \tilde{X}_3^2\tilde{X}_2^2(15 + 30\tilde{X}_2 + 15\tilde{X}_2^2)Y_2(\tilde{x})\,; \\
z_4 &= \tilde{Y}_2\tilde{X}_4(3\tilde{X}_2 + 9\tilde{X}_2^2 + 6\tilde{X}_2^3) + (3\tilde{X}_2^2 + 6\tilde{X}_2^3 + 3\tilde{X}_2^4)\Big(\tilde{Y}_4 + \tilde{X}_4\frac{2Y_2(\tilde{x})}{(1 - X_2(\tilde{x}))}\Big),
\end{aligned}$$

and

$$X_\ell(x) = \sum_{k\geq\ell}\binom{k-1}{\ell-1}\kappa_k x^k; \quad Y_\ell(x) = \sum_{k\geq\ell}\binom{k}{\ell}\kappa_k x^k; \quad \text{if } \ell > 2 \quad \tilde{X}_\ell(x) := \frac{X_\ell(\tilde{x})}{(1 - X_2(\tilde{x}))^\ell} \quad ; \quad \tilde{Y}_\ell(x) := \frac{Y_\ell(\tilde{x})}{(1 - X_2(\tilde{x}))^\ell}.$$

An example: Higher genus Fuss-Catalan

Case $\kappa_i = \delta_{i,3}$. Partitions into triplets, type $[3^k]$

Genus 0: $Z^{(0)}$ satisfies $(xZ)^3 - Z + 1 = 0$: this is the GF of Fuss–Catalan numbers;

$$Z^{(0)}(x) = \frac{2}{\sqrt{3x^3}} \sin\left(\frac{1}{3}\text{Arcsin}\left(\frac{3}{2}\sqrt{3x^3}\right)\right).$$

Then

$$Z^{(1)}(x) = \frac{1152\, x^3 \sin^6\left(\frac{1}{3}\text{Arcsin}\left(\frac{3\sqrt{3x^3}}{2}\right)\right)}{\left(2\cos\left(\frac{1}{3}\text{Arccos}\left(1-\frac{27x^3}{2}\right)\right)-1\right)\left(9\sqrt{x^3}-4\sqrt{3}\sin\left(\frac{1}{3}\text{Arcsin}\left(\frac{3\sqrt{3x^3}}{2}\right)\right)\right)^4}$$

and

$$Z^{(2)}(x) = \frac{192 s^6 x^6 \left(8s^3\left(128\left(11264 s^9 + 8676\sqrt{3}s^6 x^{3/2} + 3105 s^3 x^3\right) + 9315\sqrt{3}x^{9/2}\right) + 729x^6\right)}{\left(2\cos\left(\frac{1}{3}\text{Arccos}\left(1-\frac{27x^3}{2}\right)\right)-1\right)\left(9\sqrt{x^3}-4\sqrt{3}\sin\left(\frac{1}{3}\text{Arcsin}\left(\frac{3\sqrt{3x^3}}{2}\right)\right)\right)^{10}}$$

Also: reproduce former results of Cori–Hetyei on "genus dependent Bell or Stirling numbers": total number of partitions of order n and genus 0,1, 2, with/without fixed number of parts. . .

A curious observation [math.CO:2306.16237]

Inspired by the inversion relation Alex Hock (Oxford) made some amazing observations that simplify these results a great deal.

Let $y := Y(x) = x^{-1}Z^{(0)}(x^{-1})$, $X(y) = y^{-1}(1 + W(y))$ its (functional) inverse. Reexpress Theorems 1 and 2 in terms of $x = X(y)$

$$x^{-1}Z^{(1)}(x^{-1}) = \frac{\partial}{\partial x}\left(\frac{1}{4y^4X'(y)^2} + \frac{1}{6y^6X'(y)^3}\right)$$

and

$$\begin{aligned} x^{-1}Z^{(2)}(x^{-1}) = \quad & \frac{\partial}{\partial x}\Bigg(\frac{21}{8y^8X'(y)^4} + \frac{74}{5y^{10}X'(y)^5} + \frac{24}{y^{12}X'(y)^6} + \frac{12}{y^{14}X'(y)^7} \\ & -\frac{X^{(3)}(y)}{8y^8X'(y)^6} - \frac{X^{(3)}(y)}{4y^{10}X'(y)^7} - \frac{X^{(3)}(y)}{8y^{12}X'(y)^8} \\ & +\frac{(X^{(2)}(y))^2}{24y^6X'(y)^6} + \frac{(X^{(2)}(y))^2}{y^8X'(y)^7} + \frac{19(X^{(2)}(y))^2}{8y^{10}X'(y)^8} + \frac{35(X^{(2)}(y))^2}{24y^{12}X'(y)^9} \\ & +\frac{X^{(2)}(y)}{y^7X'(y)^5} + \frac{23X^{(2)}(y)}{3y^9X'(y)^6} + \frac{29X^{(2)}(y)}{2y^{11}X'(y)^7} + \frac{8X^{(2)}(y)}{y^{13}X'(y)^8}\Bigg). \end{aligned}$$

Conjecture generalization to higher genus with undetermined coefficients.

But what is the combinatorial interpretation of these coefficients?

Topological relation at work ?

6. Final remarks

– Singularities of the Generating Functions

Some evidence of a universal singular behaviour of all GF

$$Z^{(g)}(x) \sim (x_0 - x)^{\frac{1}{2}-3g}$$

implying a large n behaviour of coefficients $C^{(g)}_{n,[\alpha]}$ (for appropriately rescaled patterns α)

$$C^{(g)}_{n,[\alpha]} \sim \text{const}\ x_0^{-n-3g+\frac{1}{2}} n^{3g-\frac{1}{2}} \qquad \text{as } n, [\alpha] \text{ grow large.}$$

Also encountered in enumeration of unicellular maps [Chapuy], and in boundary loop models and Wilson loops [Kostov]. . .

– Topological Recursion [Chekov–Eynard–Orantin]

Does it apply to the enumeration of higher genus partitions? [Hock] ?

Thank you !