Thirty-six entangled officers of Euler: Quantum solution of a classically impossible combinatorial problem

Karol Życzkowski
Jagiellonian University (Cracow)
Polish Academy of Sciences (Warsaw)

in collaboration with

S. A. Rather, A. Lakshminarayan (Chennai),
A. Burchardt (Amsterdam), W. Bruzda (Cracow),
G. Rajchel–Mieldzioć (Barcelona)

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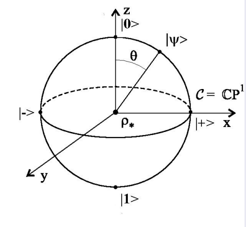
Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Qubit = quantum bit; $N = 2$

$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

Bloch sphere of $N = 2$ pure states (*isotropic*)



Space of pure states for an arbitrary N:

a complex projective space $\mathbb{C}P^{N-1}$ of $2N-2$ real dimensions.

This space is isotropic ! All states are **'equal'** !

There are no states with **extremal** properties !

situation changes if

some structure is imposed to the system

Then all quantum states are **equal**,

but some are **more equal** then others...

a simple example

a composed two-qubit system: $\mathcal{H}_2 \otimes \mathcal{H}_2$
geometric perspective – Segre embedding: $\mathbb{C}P^1 \times \mathbb{C}P^1 \subset \mathbb{C}P^3$.

separable states are distinguished
so are **maximally entangled Bell states**, e.g. $|\psi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$
as they are **more equal** then other states...

In such a case the search for states with **extremal** properties is justified:
Bell states are the most entangled *two-qubit* states,
(most distant from the set $\mathbb{C}P^1 \times \mathbb{C}P^1$ of separable states),
useful for several applications in quantum technologies...

Classical Combinatorial Designs

Latin Squares and **Greaco-Latin Squares** are well known subjects considered in *recreational mathematics*.

A **Latin square** of size d is given by d copies of d symbols arranged in a square such that each its row and each column contains different symbols.

cards example of order $d = 3$:

♠	♣	♢
♢	♠	♣
♣	♢	♠

A **Greaco-Latin** square of size d (also called **two orthogonal Latin squares** consists of two Latin squares, (one written with Greek letters one with Latin), such that all d^2 pairs of symbols are different

example of size $d = 3$ studied by **Euler**

αA	βB	γC
γB	αC	βA
βC	γA	αB

Combinatorial design: a constellation of elements of a finite set arranged with certain **symmetry** and **balance**

are related to statistics and planning of experiments

Mutually orthogonal Latin Squares (MOLS)

A classical example:

Take 4 **aces**, 4 **kings**, 4 **queens** and 4 **jacks**
and arrange them into an 4×4 array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

A classical example:

Take 4 **aces**, 4 **kings**, 4 **queens** and 4 **jacks**
and arrange them into an 4×4 array, such that

a) - in every row and column there is only a **single** card of each suit

b) - in every row and column there is only a **single** card of each rank

$A\spadesuit$	$K\clubsuit$	$Q\diamondsuit$	$J\heartsuit$
$K\heartsuit$	$A\diamondsuit$	$J\clubsuit$	$Q\spadesuit$
$Q\clubsuit$	$J\spadesuit$	$A\heartsuit$	$K\diamondsuit$
$J\diamondsuit$	$Q\heartsuit$	$K\spadesuit$	$A\clubsuit$

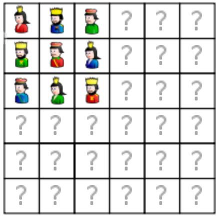
Two **mutually orthogonal** **Latin squares** of size $d = 4$

Graeco–Latin square !

Mutually orthogonal Latin Squares (MOLS)

♣) $d = 2$. There are no orthogonal Latin Square
(for **2** aces and **2** kings the problem has no solution)
♡) $d = 3, 4, 5$ (and any **power of prime**) $\Longrightarrow$ there exist $(d-1)$ MOLS.
♠) $d = 6$. Only a **single** Latin Square exists (No OLS!).

Euler's problem: **36** officers of six different ranks from six different units come for a **military parade**. Arrange them in a square such that in each row / each column all uniforms are different.

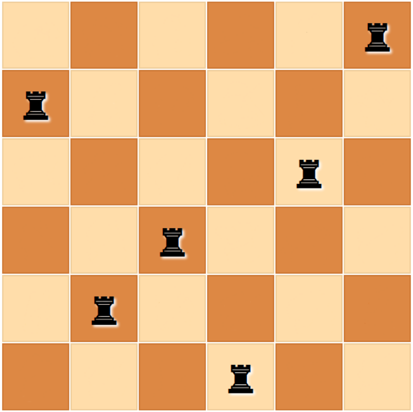


No solution exists ! (**1779** conjecture by **Euler**), proof (**121** years later)
Gaston Tarry *Le Probléme de 36 Officiers, Compte Rendu* **(1900)**.

36 officers of Euler revisited

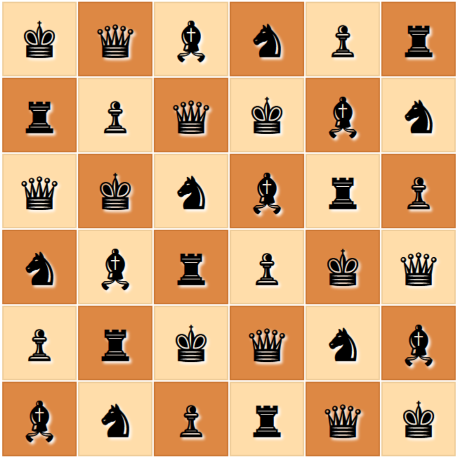
introductory exercise

Step i) Place **six rooks** on a chessboard of size six,
in such a way that no figure attacks any other:



36 officers of Euler, step two

Step ii) Take six pieces of five other figures
and place them onto the board in an analogous (**rooks–like**) way:

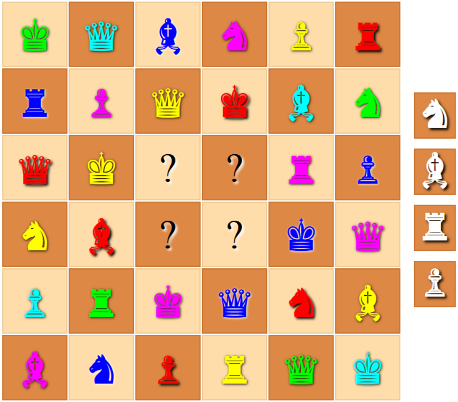


Latin Square of order six

36 officers of Euler, step three...

Step iii) Color them into six **colors**,

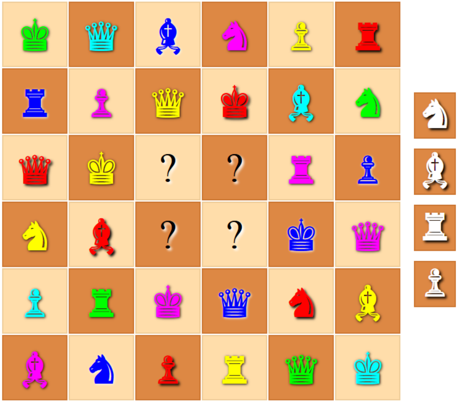
so that all **colors**, in each row and column are different...



Place the remaining four figures, two of them in **cyan** and two in **green**, so that all the rules of Euler are fulfilled

36 officers of Euler, step three $\Rightarrow$ no go !

Step iii) Color them into six **colors**, $d = 6 = 2 * 3$,
so that all **colors**, in each row and column are different...



Place the remaining four figures, two of them in **cyan** and two in **green**, so that all the rules of Euler are fulfilled – **this is not doable !** **G. Tarry**

Quantum Combinatorial Designs

Vicary, Musto (2016): a **square** of order d consisting d^2 states $|\psi_{ij}\rangle \in \mathcal{H}_d$ is called **Quantum Latin** if each of its rows and columns forms an **ortogonal basis**, $\langle\psi_{ij}|\psi_{ik}\rangle = \langle\psi_{ji}|\psi_{ki}\rangle = \delta_{jk}, \ i = 1, \ldots, d.$

Example of order $d = 4$: $$QLS(4) = \begin{vmatrix} |1\rangle & |2\rangle & |3\rangle & |4\rangle \\ |4\rangle & |3\rangle & |2\rangle & |1\rangle \\ |\chi_-\rangle & |\xi_-\rangle & |\xi_+\rangle & |\chi_+\rangle \\ |\chi_+\rangle & |\xi_+\rangle & |\xi_-\rangle & |\chi_-\rangle \end{vmatrix},$$

where $|\chi_\pm\rangle = \frac{1}{\sqrt{2}}(|2\rangle \pm |3\rangle)$, $|\xi_+\rangle = \frac{1}{\sqrt{5}}(i\,|1\rangle + 2\,|4\rangle)$
and $|\xi_-\rangle = \frac{1}{\sqrt{5}}(2\,|1\rangle + i\,|4\rangle)$ denote **superposition** states,
and give 2 column bases + 2 row bases = 4 **orthogonal bases** in $\mathcal{H}_4$

Standard **combinatorics**: **discrete** set of symbols, $1, 2, \ldots, d$,
+ **permutation** group
generalized **'Quantum' combinatorics**: **continuous** family
of states $|\psi\rangle \in \mathcal{H}_d$ + **unitary** group $U(d)$,
Gerhard Zauner, Ph.D. Thesis, Wien 1999.

Quantum Orthogonal Latin Squares (QOLS)

C) **Classical OLS** consists of d^2 pairs of d symbols such that
a) all d^2 pairs of symbols are **different**,
b,c) there is no **repetition** of any symbol in each row and each column

Q) **Quantum OLS** is formed by d^2 bipartite states

$$|\psi_{ij}\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d = \mathcal{H}_A \otimes \mathcal{H}_B \quad \text{such that}$$

a)' all d^2 states are **mutually orthogonal**, $\langle\psi_{ij}|\psi_{k\ell}\rangle = \delta_{ik}\delta_{j\ell}$,
b)', c)' all rows and columns satisfy **partial trace** relations

$$\mathrm{Tr}_B\Big(\sum_{k=0}^{d-1} |\psi_{ik}\rangle\langle\psi_{jk}|\Big) = \delta_{ij}\mathbb{I}_d, \quad \textit{the average colour is 'white'}$$

$$\mathrm{Tr}_B\Big(\sum_{k=0}^{d-1} |\psi_{ki}\rangle\langle\psi_{kj}|\Big) = \delta_{ij}\mathbb{I}_d, \quad \textit{in each row and column...}$$

and dual conditions for Tr_A.

a 'classical' $d = 3$ example of **QOLS**:

$\|A♠\rangle$	$\|K♣\rangle$	$\|Q♢\rangle$
$\|K♢\rangle$	$\|Q♠\rangle$	$\|A♣\rangle$
$\|Q♣\rangle$	$\|A♢\rangle$	$\|K♠\rangle$

is based on **product** states, e.g. $|K♣\rangle = |K\rangle \otimes |♣\rangle$.
To get **genuinely Quantum OLS** one needs to introduce **entanglement**

Composed systems & entangled quantum states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states **not** of the above product form.

Two–qubit system: $2 \times 2 = 4$

Maximally entangled Bell state $|\varphi^+\rangle := \Big(|00\rangle + |11\rangle\Big) = (|A♠\rangle + |K♣\rangle)$

useful for several applications in quantum technologies...

Maximally Entangled states of a $d \times d$ system

Any pure state from $\mathcal{H}_d \otimes \mathcal{H}_d$ can be written as

$|\psi\rangle = \sum_{ij} C_{ij} |i\rangle \otimes |j\rangle$, where $|\psi|^2 = \mathrm{Tr}\, CC^\dagger = 1$.

A state $|\psi\rangle$ is **maximally entangled** if its partial trace is maximally mixed,

$\sigma = \mathrm{Tr}_B |\psi\rangle\langle\psi| = CC^\dagger = \mathbb{1}_d / d$,

which is the case if the rescaled **matrix** $U = \sqrt{d}C$ of size d is **unitary**.

Absolutely maximally entangled state (AME)

Definition. A pure state of an even number N of qu$d$its is called **absolutely maximally entangled**, **AME(N,d)** if for any choice of $N/2$ subsystems traced out the reduced state is maximally mixed.

Scott (2004), **Facchi**+ (2008), **Helwig**+ (2012), **Arnaud**+ (2013)

An **AME state** of four parties A, B, C, D with d levels each,

$$|\psi\rangle = \sum_{i,j,l,m=1}^{d} T_{ijlm}|i,j,l,m\rangle$$

It is **maximally entangled** with respect to all **three** partitions:

$$AB|CD \text{ and } AC|BD \text{ and } AD|BC.$$

Let $\rho_{ABCD} = |\psi\rangle\langle\psi|$. Hence its three reductions are **maximally mixed**,

$$\rho_{AB} = \mathrm{Tr}_{CD}\rho_{ABCD} = \rho_{AC} = \mathrm{Tr}_{BD}\rho_{ABCD} = \rho_{AD} = \mathrm{Tr}_{BC}\rho_{ABCD} = \mathbb{1}_{d^2}/d^2$$

Thus matrices $U_{\mu,\nu}$ of order d^2 obtained by reshaping the tensor T_{ijkl} are **unitary** for three reorderings:

$$\text{a) } \mu,\nu = ij, lm, \quad \text{b) } \mu,\nu = im, jl, \quad \text{c) } \mu,\nu = il, jm.$$

Such a tensor T is called **perfect**, **Pastawski** et al. (2015),
and a matrix U **two–unitary** **Goyeneche** et al. (2015)

AME states and Quantum OLS

Theorem. Existence of QOLS(d) is equivalent to AME states of 4 systems with d levels each.

each field of a QOLS(d) encodes four data: two digits from one to d determine address of a square, two other data its content.

Let $\{|\phi_{ij}\rangle \in \mathcal{H}_d\}_{i,j=1}^{d}$ form a QOLS(d).
Then 4-partite state $|\Psi_4\rangle := \sum_{i,j=1}^{d} |i,j\rangle \otimes |\phi_{ij}\rangle = \sum_{i,j,k,\ell=1}^{d} t_{ijk\ell}|i,j,k,\ell\rangle$ forms the state $|AME(4,d)\rangle$ while $t_{ijk\ell}$ forms a **perfect tensor** as reshaped into a matrix $U_{\mu\nu}$ is unitary for all pairs of indices;

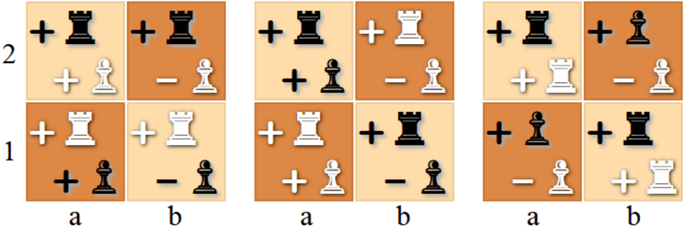
$$\mu = (i,j);\ \mu = (i,k);\ \mu = (i,\ell).$$

Maximal entanglement of a four-party state $|\Psi_4\rangle = |\Psi\rangle_{ABCD}$
with respect to all **three** partitions: $AB|CD$ and $AC|BD$ and $AD|BC$
is equivalent to the fact that d^2 bi-partite states $|\phi_{ij}\rangle$
form a **quantum ortogonal Latin square**.

No Quantum OLS of order $d = 2$

There are no **classical OLS** of size $d = 2$

There are no **quantum** **OLS** of size $d = 2$ either!



Even using entangled states (*more than a single figure in one chess field*) it is not possible to find a 2×2 square of four states which satisfies QOLS conditions **a'), b'), c')**.

Higuchi, Sudbery (2001) proved that there are no AME states of 4 qubits $\Longrightarrow$ no QOLS(2) !

Higher dimensions: AME(4,3) state of four qutrits

A **Greaco-Latin square** of size $d = 3$

each symbol encodes 4 digits: (c,r,f,s) = column, row, figure, suit

αA	βB	γC
γB	αC	βA
βC	γA	αB

=

$A\spadesuit$	$K\clubsuit$	$Q\diamondsuit$
$K\diamondsuit$	$Q\spadesuit$	$A\clubsuit$
$Q\clubsuit$	$A\diamondsuit$	$K\spadesuit$

=

$0,0$	$1,2$	$2,1$
$1,1$	$2,0$	$0,2$
$2,2$	$0,1$	$1,0$

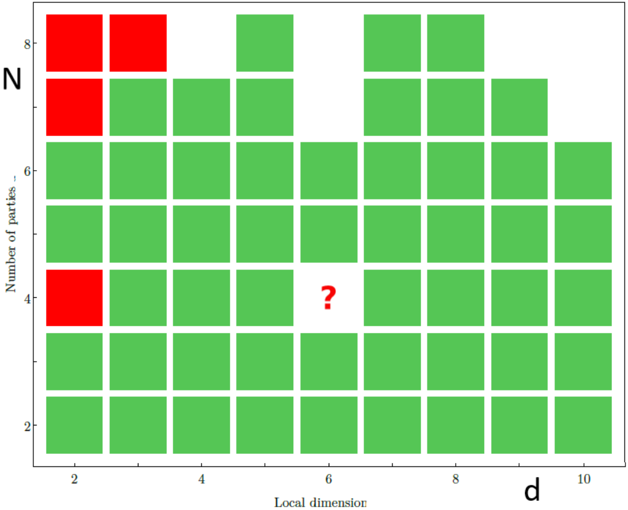
yields an **AME state** of **4 qutrits**:

$$\begin{aligned} |\Psi_3^4\rangle \quad = \quad & |0000\rangle + |0112\rangle + |0221\rangle + \\ & |1011\rangle + |1120\rangle + |1202\rangle + \\ & |2022\rangle + |2101\rangle + |2210\rangle. \end{aligned}$$

Corresponding **Quantum Code**:

$$\begin{aligned} |0\rangle \rightarrow |\tilde{0}\rangle &:= |000\rangle + |112\rangle + |221\rangle \\ |1\rangle \rightarrow |\tilde{1}\rangle &:= |011\rangle + |120\rangle + |202\rangle \\ |2\rangle \rightarrow |\tilde{2}\rangle &:= |022\rangle + |101\rangle + |210\rangle \end{aligned}$$

Existence of **Absolutely maximally entangled** states



The case: $N = 4$ subsystems with $d = 6$ levels each (*corresponding to 36 officers of Euler*) up till 2021 **remained open**!

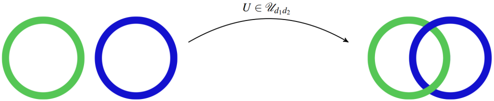
[1]see on-line table by **F. Huber & N. Wyderka**

In hunt for an $|AME(4,6)\rangle$ state of 4 quhex

To find the state

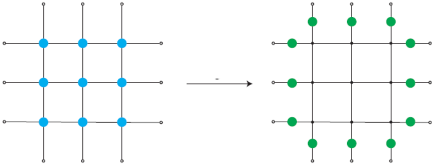
$|AME(4,6)\rangle = (U_{AB} \otimes \mathbb{I}_{CD})|\Psi^{+}_{AC|BD}\rangle = \sum_{i,j,k,\ell=1}^{6} t_{ijk\ell}|i,j,k,\ell\rangle$

we look for a 2-unitary matrix $U_{AB} \in U(36)$, which remains unitary after reorderings, maximizes the **entangling power** $e_p(U)$



(average entanglement of $U_{AB}|\psi_A\rangle \otimes |\psi_B\rangle$)

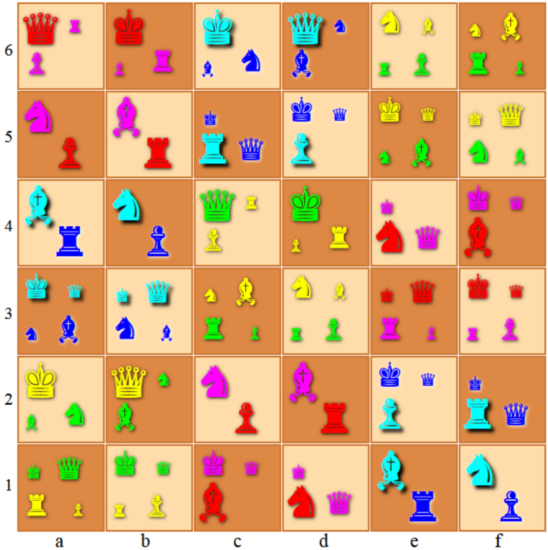
and leads to a perfect tensor $t_{ijk\ell}$ used for models of bulk/boundary duality



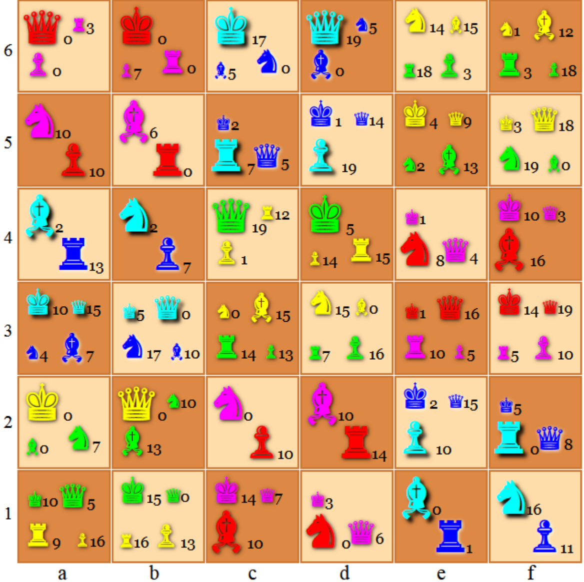
Optimization over the space $U(36)$ of dimension $36^2 - 1 = 1295$

is not easy...

No classical OLS(6). **But a quantum solution exists !**



Quantum solution of *36 entangled officers* of **Euler**. Size of the figures represents moduli of superpositions, Is field $c2$ equal to $a5$?



Quantum solution of *36 entangled officers* of **Euler**. Size of the figures represents moduli of superpositions, index k denotes the complex **phase** $\exp(i\pi k/20)$, e.g. field c2) denotes $|$♞$\rangle - |$♟$\rangle$ and is orthogonal to a5).

Full solution of the problem of 36 **entangled officers of Euler**

encoded in the chessboard of size 6 looks like this...

(each state $|\psi_{ij}\rangle$ determines a single row of a 2-unitary matrix U_{36})

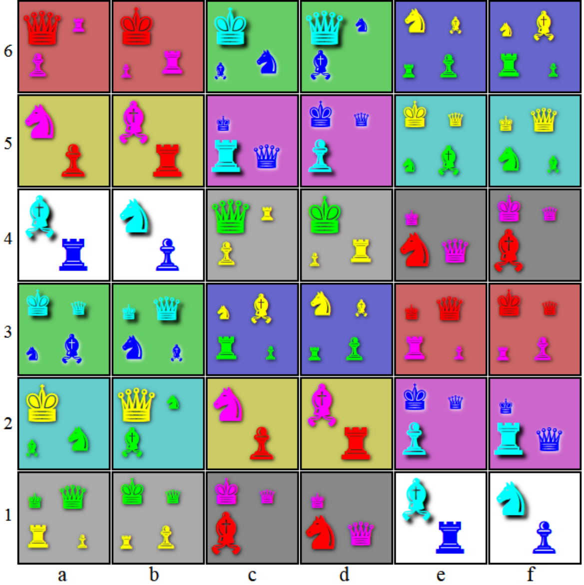
$$|\psi_{00}\rangle = c|10\rangle + a\omega^3|43\rangle + b|53\rangle = c|\text{♛}\rangle + a\omega^3|\text{♜}\rangle + b|\text{♟}\rangle$$
$$|\psi_{01}\rangle = c|00\rangle + b|43\rangle + a\omega^7|53\rangle = c|\text{♚}\rangle + b|\text{♜}\rangle + a\omega^7|\text{♟}\rangle$$
$$|\psi_{02}\rangle = c\omega^{17}|01\rangle + b|24\rangle + a\omega^5|34\rangle = c\omega^{17}|\text{♚}\rangle + b|\text{♞}\rangle + a\omega^5|\text{♝}\rangle$$
$$|\psi_{10}\rangle = c\omega^{10}|23\rangle + c\omega^{10}|50\rangle = c\omega^{10}|\text{♞}\rangle + c\omega^{10}|\text{♟}\rangle$$
$$|\psi_{11}\rangle = c\omega^{6}|33\rangle + c|40\rangle = c\omega^{6}|\text{♝}\rangle + c|\text{♜}\rangle$$
$$|\psi_{12}\rangle = a\omega^2|04\rangle + b\omega^5|14\rangle + c\omega^7|41\rangle = a\omega^2|\text{♚}\rangle + b\omega^5|\text{♛}\rangle + c\omega^7|\text{♜}\rangle$$
$$\dots \quad = \quad \dots$$
$$|\psi_{55}\rangle = c\omega^{16}|21\rangle + c\omega^{11}|54\rangle = c\omega^{16}|\text{♞}\rangle + c\omega^{11}|\text{♟}\rangle,$$

where $\omega = \exp(i\pi k/20)$, and $a^2 + b^2 = c^2 = 1/2$,

while the ratio of the two sizes of the figures is equals to the

golden mean, $b/a = (1+\sqrt{5})/2 = \varphi$.

It is easy to check that this constellation satisfies the desired conditions **a'), b'), c')** specified above and it deservs an appelation golden **square**.



Four states on background of the same colour form a basis and are **orthogonal** ! The board of size 6 with 36 fields is divided into 9 groups of 4 two-qubit orthogonal states. **9*4=6*6**

ENTANGLING POWER

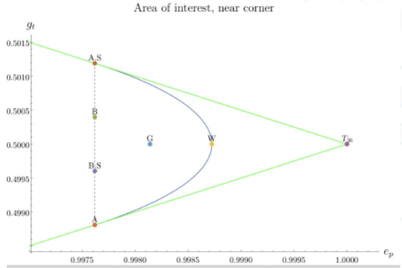
$$e_p(U) = \frac{1}{E(S)}\Big(E(U) + E(US) - E(S)\Big) \in [0,1]$$

$$E(U) = 1 - \big(\textstyle\sum_{j=1}^{d^2} \lambda_j^2\big)/d^4$$

$$g_t(U) = \frac{1}{2E(S)}\Big(E(U) - E(US) + E(S)\Big) \in [0, \tfrac{1}{2}]$$

$$S\Big(|\phi_A\rangle \otimes |\phi_B\rangle\Big) = |\phi_B\rangle \otimes |\phi_A\rangle$$

$U \in \mathcal{U}_{d^2}$ is 2-unitary $\Leftrightarrow E(U) = E(SU) = E(S) \Leftrightarrow e_p(U) = 1, g_t(U) = \frac{1}{2}$



11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
54	63	26	15	42	31
65	56	12	21	33	44

=

A♠	K♣	Q♦	J♥	10✿	9✷
K♦	A♥	J✿	Q✷	9♠	10♣
Q♣	J♠	9♥	10♦	A✷	K✿
J✷	Q✿	10♠	9♣	K♥	A♦
10♥	9♦	K✷	A✿	J♣	Q♠
9✿	10✷	A♣	K♠	Q♦	J♥

$$e_p = \frac{314}{315}$$

Numerical Search

$$U_0 \mapsto U_0^{\mathrm{R}} \mapsto \left(U_0^{\mathrm{R}}\right)^{\Gamma} := U_0^{\Gamma\mathrm{R}} \mapsto U_1$$

$\tilde{P} =$

11	22	33	44	55	66
23	14	45	36	61	52
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A♠	K♣	Q♦	J♥	10✿	9✷
K♦	A♥	J✿	Q✷	9♠	10♣
Q♣	J♠	9♥	10♦	A✷	K✿
J✷	Q✿	10♠	9♣	K♥	A♦
10♥	9♦	K✷	A✿	J♣	Q♠
9✿	10✷	A♣	K♠	Q♦	J♥

$$e_p(\tilde{P}) = \frac{314}{315} \simeq .9968 \qquad e_p(\tilde{P}e^{iH\varepsilon}) \to .9991$$

$$e_p(\tilde{P}) \to \frac{419}{420} \simeq .9976$$

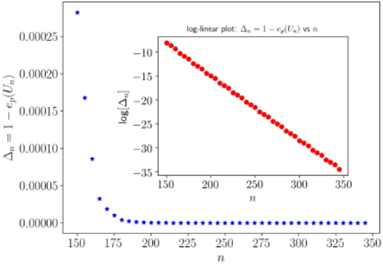
$$e_p(\widetilde{P_s}) = \frac{104}{105} \simeq .9905 \qquad e_p(\widetilde{P_S}e^{iH\varepsilon}) \to 1$$

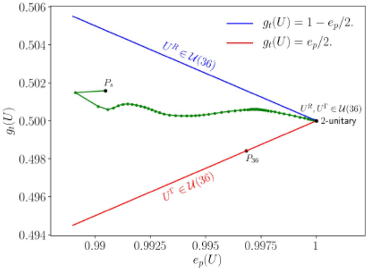
$\tilde{P}_s =$

11	22	33	44	55	66
23	14	45	36	61	52
32	41	64	53	16	25
46	35	51	62	24	13
64	56	26	15	43	31
55	63	12	21	42	34

=

A♠	K♣	Q♦	J♥	10✿	9✷
K♦	A♥	J✿	Q✷	9♠	10♣
Q♣	J♠	9♥	10♦	A✷	K✿
J✷	Q✿	10♠	9♣	K♥	A♦
9♥	10✷	K✷	A✿	J♦	Q♠
10✿	9♦	A♣	K♠	J♣	Q♥

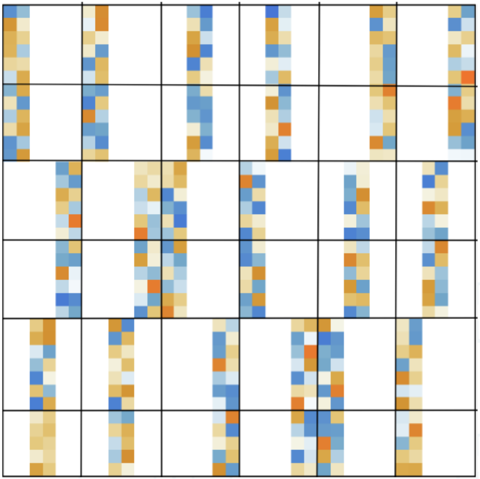




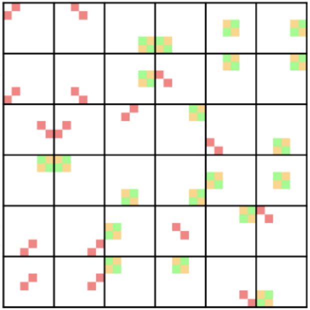
Numerical Cleaning

$$\left(U_6^{(1)} \otimes U_6^{(2)}\right) U_{36} \left(U_6^{(4)} \otimes U_6^{(3)}\right)$$

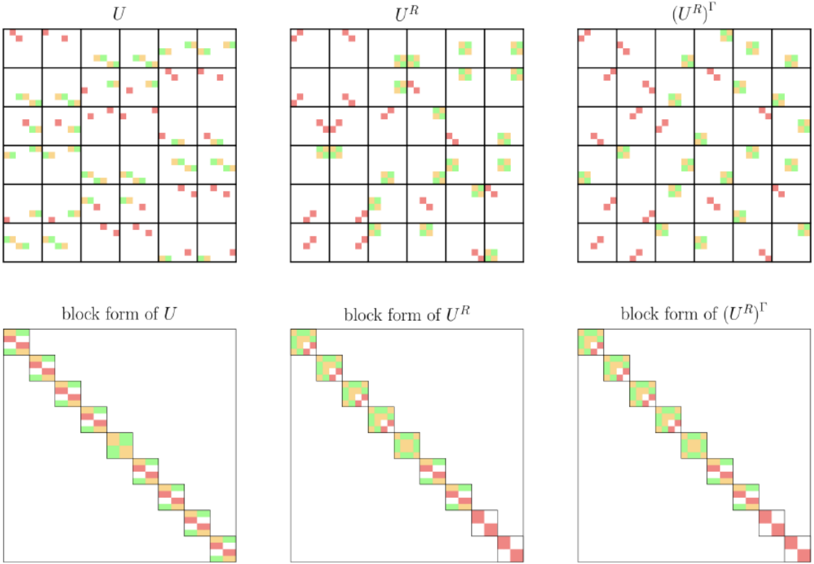
$$\left(\cancel{U_6} \otimes U_2^{\otimes 3}\right) U_{36} \left(U_2^{\otimes 3} \otimes U_2^{\otimes 3}\right)$$



0.75	0	0	0	0	0	0.25	0
0	0.25	0	0	0	0	0	0.25
0	0	0.25	0	0.25	0	0	0
0	0	0	0.75	0	0.25	0	0
0	0	0.25	0	0.25	0	0	0
0	0	0	0.25	0	0.75	0	0
0.25	0	0	0	0	0	0.75	0
0	0.25	0	0	0	0	0	0.25



Solution Found



Solution Found

(1,1)	(2,2)	(1,2)	(2,1)	
$a\,\omega^{10}$	a	$b\,\omega^{15}$	$b\,\omega^5$	(6,3)
		c	c	(1,1)
c	c			(5,6)
$b\,\omega^{10}$	b	$a\,\omega^5$	$a\,\omega^{15}$	(4,2)

(1,3)	(2,4)	(1,4)	(2,3)	
$a\,\omega^2$	$a\,\omega^{14}$	$b\,\omega$	$b\,\omega^5$	(2,5)
		$c\,\omega^5$	$c\,\omega^{19}$	(3,3)
$c\,\omega^{17}$	$c\,\omega^{19}$			(1,2)
$b\,\omega^{14}$	$b\,\omega^6$	$a\,\omega^3$	$a\,\omega^7$	(6,4)

(1,5)	(2,6)	(1,6)	(2,5)	
$a\,\omega$	$a\,\omega^{19}$	$b\,\omega^{14}$	$b\,\omega^{16}$	(4,1)
$a\,\omega$	$a\,\omega^3$	$b\,\omega^{10}$	$b\,\omega^4$	(3,4)
$b\,\omega^4$	$b\,\omega^{18}$	$a\,\omega^3$	$a\,\omega^9$	(2,6)
$b\,\omega^2$	$b\,\omega^8$	$a\,\omega^5$	$a\,\omega^{15}$	(5,5)

(3,1)	(4,2)	(3,2)	(4,1)	
$a\,\omega^4$	$a\,\omega^{10}$	$b\,\omega^{17}$	$b\,\omega^7$	(4,5)
		$c\,\omega^2$	$c\,\omega^2$	(3,2)
$c\,\omega^{10}$	$c\,\omega^6$			(2,4)
$b\,\omega^7$	$b\,\omega^{13}$	$a\,\omega^{10}$	a	(5,3)

(3,3)	(4,4)	(3,4)	(4,3)	
a	a	$b\,\omega^{15}$	$b\,\omega^{15}$	(4,6)
		c	$c\,\omega^{10}$	(6,1)
c	$c\,\omega^{10}$			(5,4)
b	b	$a\,\omega^5$	$a\,\omega^5$	(1,5)

(3,5)	(4,6)	(3,6)	(4,5)	
$a\,\omega^2$	a	$b\,\omega^{19}$	$b\,\omega^{13}$	(2,3)
		$c\,\omega^{16}$	c	(6,2)
$c\,\omega^8$	$c\,\omega^{16}$			(3,1)
$b\,\omega^{14}$	$b\,\omega^{12}$	$a\,\omega$	$a\,\omega^{15}$	(1,6)

(5,1)	(6,2)	(5,2)	(6,1)	
$a\,\omega^3$	$a\,\omega^7$	b	b	(1,4)
		c	$c\,\omega^{10}$	(2,1)
$c\,\omega^{13}$	$c\,\omega^7$			(3,5)
$b\,\omega^9$	$b\,\omega^{13}$	$a\omega^{16}$	$a\omega^{16}$	(6,6)

(5,3)	(6,4)	(5,4)	(6,3)	
$a\,\omega^{12}$	$a\,\omega^{14}$	$b\,\omega^{15}$	$b\,\omega$	(3,6)
		$c\,\omega^{14}$	$c\,\omega^{10}$	(5,1)
$c\,\omega^7$	$c\,\omega^{19}$			(2,2)
$b\,\omega^{14}$	$b\,\omega^{16}$	$a\omega^7$	$a\omega^{13}$	(4,3)

(5,5)	(6,6)	(5,6)	(6,5)	
$a\,\omega^{18}$	$a\,\omega^{18}$	$b\,\omega^3$	$b\,\omega^3$	(1,3)
		c	$c\,\omega^{10}$	(5,2)
$c\,\omega$	$c\,\omega^{11}$			(6,5)
$b\,\omega^{10}$	$b\,\omega^{10}$	$a\omega^5$	$a\omega^5$	(4,4)

AME(4,6) state

$$\frac{1}{6}\sum_{i,j,k,\ell=1}^{d} t_{i,j,k,\ell}\,|i\rangle\,|j\rangle\,|k\rangle\,|\ell\rangle$$

$$a = \frac{1}{\sqrt{2}(\omega+\overline{\omega})} = \frac{1}{\sqrt{5+\sqrt{5}}}$$

$$b = \frac{1}{\sqrt{2}(\omega^3+\overline{\omega}^3)} = \sqrt{\frac{5+\sqrt{5}}{20}}$$

$$c = \frac{1}{\sqrt{2}}$$

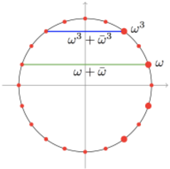
$$\omega = \exp(i\,\pi/20)$$

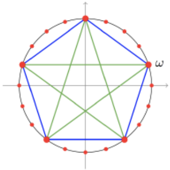
Pythagoras theorem

$$a^2 + b^2 = c^2 = \frac{1}{2}$$

Golden ratio

$$b/c = \varphi = = \frac{1+\sqrt{5}}{2}$$





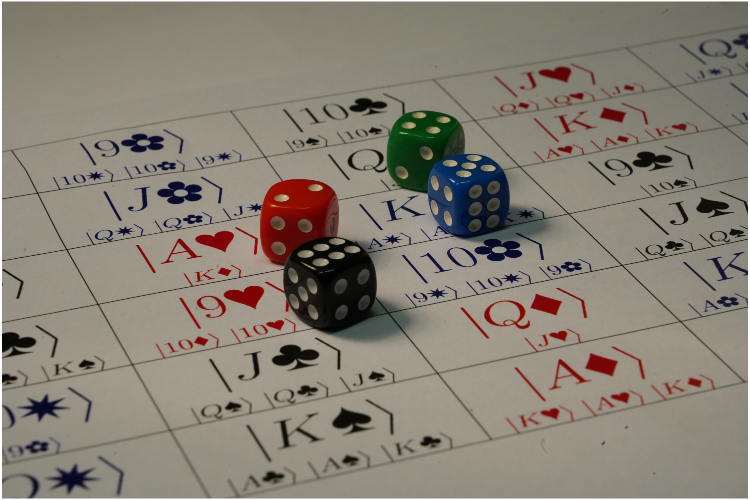
Quantum Officers of Euler

|A♣⟩ |K♠⟩	|A♦⟩ |K♥⟩	|10✿⟩ |10✱⟩ |9✿⟩ |9✱⟩	|10♠⟩ |10♣⟩ |9♠⟩ |9♣⟩	|Q♦⟩ |Q♥⟩ |J♦⟩ |J♥⟩	|Q✿⟩ |Q✱⟩ |J✿⟩ |J✱⟩
|10♣⟩ |9♠⟩	|10♦⟩ |9♥⟩	|Q✿⟩ |Q✱⟩ |J✿⟩ |J✱⟩	|Q♠⟩ |J♣⟩	|A♦⟩ |A♥⟩ |K♦⟩ |K♥⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩
|Q✿⟩ |J✱⟩	|Q♣⟩ |J♠⟩	|A♥⟩ |K♦⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|10♠⟩ |9♣⟩	|10♦⟩ |10♥⟩ |9♦⟩ |9♥⟩
|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|A♠⟩ |A♣⟩ |K♠⟩ |K♣⟩	|10♦⟩ |10♥⟩ |9♦⟩ |9♥⟩	|10✿⟩ |10✱⟩ |9✿⟩ |9✱⟩	|Q♠⟩ |Q♣⟩ |J♠⟩ |J♣⟩	|Q♦⟩ |Q♥⟩ |J♦⟩ |J♥⟩
|10♥⟩ |9♦⟩	|10✱⟩ |9✿⟩	|Q♠⟩ |Q♣⟩ |J♠⟩ |J♣⟩	|Q♦⟩ |J♥⟩	|A✿⟩ |A✱⟩ |K✿⟩ |K✱⟩	|A♠⟩ |K♣⟩
|Q♥⟩ |J♦⟩	|Q✱⟩ |J✿⟩	|A♠⟩ |A♣⟩ |K♠⟩ |K♣⟩	|A♦⟩ |A♥⟩ |K♦⟩ |K♥⟩	|10✿⟩ |9✱⟩	|10♠⟩ |10♣⟩ |9♠⟩ |9♣⟩

$A/K \to A$
$D/J \to B$
$10/9 \to C$

♠/♣ → α
♦/♥ → β
✿/✱ → γ

$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$	$B\beta$	$B\gamma$
$C\alpha$	$C\beta$	$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$
$B\gamma$	$B\alpha$	$A\beta$	$A\gamma$	$C\alpha$	$C\beta$
$A\gamma$	$A\alpha$	$C\beta$	$C\gamma$	$B\alpha$	$B\beta$
$C\beta$	$C\gamma$	$B\alpha$	$B\beta$	$A\gamma$	$A\alpha$
$B\beta$	$B\gamma$	$A\alpha$	$A\beta$	$C\gamma$	$C\alpha$



Four dice in the golden $|AME(4,6)\rangle$ state corresponding to *36 entangled officers* of **Euler**. Any pair of dice is unbiased, although their outcome determines the state of the other two.

Concluding Remarks

Strongly entangled extremal **multipartie** quantum states can be useful for quantum error correction codes, multiuser quantum communication and other protocols.

Theorem. Absolutely maximally entangled states $|AME(4,6)\rangle$ of 4 subsystems with 6 levels each **do** exist !

Rather, Burchardt, Bruzda, Rajchel, Lakshminarayan, K.Ż.
preprint arXiv:2102.07787, *April 2021.* (**121** years after Tarry) and *Phys. Rev. Lett.* (2022). This implies **existence** of

1. solution of the quantum analogue of the 36 officers problem of **Euler**,
2. optimal bi-partite unitary gate U_{36} with maximal **entangling power**
3. **perfect tensor** $t_{ijk\ell}$ with 4 indices, each running from 1 to 6, to be applied for tensor networks and bulk/boundary correspondence,
4. nonadditive **quantum error correction code** $((3,6,2))_6$ which allows one to encode a single quhex in three quhexes
5. distinguished point in $\mathbb{C}P^{36\times 36-1} \supset \mathbb{C}P^5 \times \mathbb{C}P^5 \times \mathbb{C}P^5 \times \mathbb{C}P^5$

$\Longrightarrow$ such quantum states with *extremal* properties can be useful...

an exscript from our preprint, arXiv: 14 April 2022

Thirty-six entangled officers[1] of Euler, $|\psi_{13}\rangle = (|\text{♚}\rangle + |\text{♛}\rangle)/\sqrt{2}$

[1] It is sad to note that these Russian officers recently left their parade ground in Saint Petersburg, where they belong, and went a thousand miles South...

However, explicit analytical results described in this work strongly suggest that the officers might eventually suffer a transition

into a highly *entangled* state.