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Decomposition Spaces in Combinatorics

CAP'24, IHÉS, November 20, 2024

Motivated by joint work with **Joachim Kock**

UAB Barcelona & University of Copenhagen

Motivation

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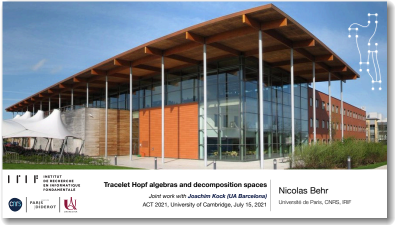

- In joint work with **Joachim Kock (UAB Barcelona & U Copenhagen)** [1], we provided a formalization of the concept of **tracelet Hopf algebras** utilizing the at the time (very) recent developments of **decomposition spaces in combinatorics** [2] and **free decomposition spaces** [3].

[1] N. Behr and J. Kock. "Tracelet Hopf Algebras and Decomposition Spaces (Extended Abstract)". In: *Proceedings of ACT 2021*. Vol. 372. EPTCS. 2022, pp. 323–337. DOI: 10.4204/EPTCS.372.23.

[2] I. Gálvez-Carrillo, J. Kock, and A. Tonks. *Decomposition Spaces in Combinatorics*. Oct. 2024. DOI: 10.48550/arXiv.1612.09225. arXiv: 1612.09225.

[3] P. Hackney and J. Kock. *Free Decomposition Spaces*. May 2024. DOI: 10.48550/arXiv.2210.11192. arXiv: 2210.11192 [math].

[4] B. Cooper and M. B. Young. *Hall Algebras via 2-Segal Spaces*. 2024. DOI: 10.48550/ARXIV.2409.19384.

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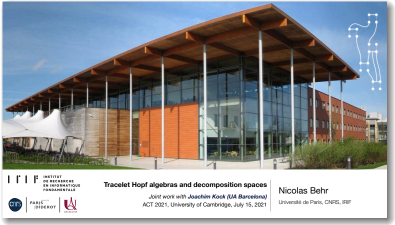

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- In a long series of works by **I. Gálvez-Carrillo**, **J. Kock**, and **A. Tonks** (c.f. [2] and references theroein), **decomposition spaces** have been demonstrated to provide a fundamental principle for reasoning in **objective combinatorics** fashion, especially about **algebraic structures** such as **incidence (co-/bi-)algerbas**.
- Slogan: **"Decomposition is often easier than composition"** — **decomposition spaces** are capable in particular of modeling generalizations of associative composition operations!
- Aside: **2-Segal spaces = decomposition spaces** (but not much more on the former in this talk — see the excellent recent review article [4] though!)

Overview

- Important conceptional observation: originally, **incidence coalgebras** were constructed for **1-Segal spaces** (e.g., for **posets**); but it is easy to find combinatorial structures that naturally give rise to **incidence coalgebras**, but are only **2-Segal spaces**!
- The simplest way to define **2-Segal spaces** is as a presheaf $S : \Delta^{op} \to \mathbf{Grp}$ that takes **active-inert pushouts** to **pullbacks** (more details later in this talk).
- Interesting technical point: in all of the decomposition space framework, algebraic structures are considered with **groupoid coefficients**. Concretely, **homotopy slices of groupoids** $\mathbf{Grpd}/_{X_1}$ will provide the basis for the algebraic constructions (with X_1 playing the role of the combinatorial structure in question).
- Slogan ([2], Sec. 1.2):

  incidence coalgebra of $\mathbf{X}_\bullet$:= comonoid object in the symmetric monoidal 2-category LIN

  Important technical ingredient here: **LIN** — **symmetric monoidal 2-category of groupoid slices and linear functors** ([2], App. A.3). Globally, this relies upon **homotopy theory of groupoids**
- Conceptually, the **decomposition space axioms** precisely guarantee **incidence coalgrba coassociativity abd counitality**.

Plan of the talk

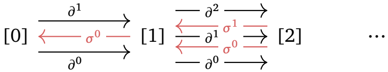
The simplex category $\triangle$ (“topologists’ Delta”)

Definition ([2], Appendix B.1)

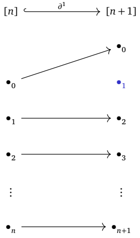
The **simplex category** $\triangle$ has

- **finite non-empty standard ordinals** $[n] = \{0 \leq 1 \leq \ldots \leq n\}$ as objects,
- **monotone (i.e., order-preserving) maps** as morphisms.

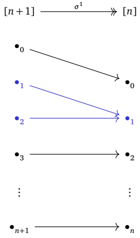
- The morphisms of $\triangle$ are generated by the following classes of maps:
  - **coface maps** — **injections** $\partial^i : [n-1] \to [n]$ that skip the value i;
  - **codegeneracy maps** — **surjections** $\sigma^i : [n+1] \to [n]$ that repeat the value i.
- These generators satisfy some obvious relations (called **cosimplicial identities**).



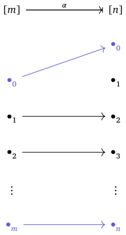
Generating maps for Δ and active/inert maps



injections

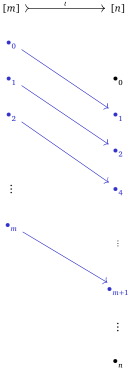


surjections



active maps

$a(0) = 0 \wedge a(m) = n$



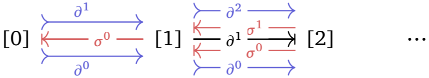
inert maps

$a(i+1) = a(i) + 1$

Active-inert factorization system on Δ ([3], Sec. 1.1.1)

- The simplex category Δ has an **active-inert factorization system**, i.e., every map of Δ factors uniquely as an **active map** followed by an **inert map**, with
  - **active maps** — $g : [k] \twoheadrightarrow [m]$ such that $g(0) = 0$ and $g(k) = m$ (**“endpoint-preserving”**)
  - **inert maps** — $f : [m] \rightarrowtail [n]$ such that $f(i+1) = f(i) + 1$ for $0 \leq i \leq m-1$ (**“distance-preserving”**)

- In terms of **generating maps** of Δ, one finds that all generators are **active maps**, except for the outer coface maps, which are **inert maps**:



- Restriction of Δ to **inert maps** (= blue arrows in the above diagram) defines a subcategory Δ_{inert} and an embedding $j : \Delta_{inert} \to \Delta$ (which will play a crucial rôle in the construction of **free decomposition spaces**).

Simplicial groupoids

- **groupoid** — small category in which all the arrows are invertible (heuristic interpretation: “sets with built-in symmetries”); **map of groupoids** — a **functor** between groupoids

  $\rightsquigarrow$ category **Grpd** of groupoids and groupoid maps
- **homotopy of groupoid maps** — a **natural transformation** of groupoid functors

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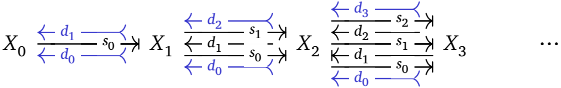
- **homotopy of groupoid maps** — a **natural transformation** of groupoid functors
- **simplicial groupoid** — a functor of the form $X : \mathbb{\Delta}^{op} \to \mathbf{Grpd}$, with $\mathbb{\Delta}$ the **simplex category** of non-empty finite standard ordinals $[n] = \{0 \leq 1 \leq \ldots \leq n\}$ and monotone maps.
- Via the generators-and-relations description of $\mathbb{\Delta}$, the previous yields (keeping in mind the **op-ing**) a diagram as below, where **active maps** ("end-point-preserving" maps) are denoted as $\rightarrow\!\!|$, and **inert maps** ("distance-preserving" maps) are denoted as $\rightarrowtail$:

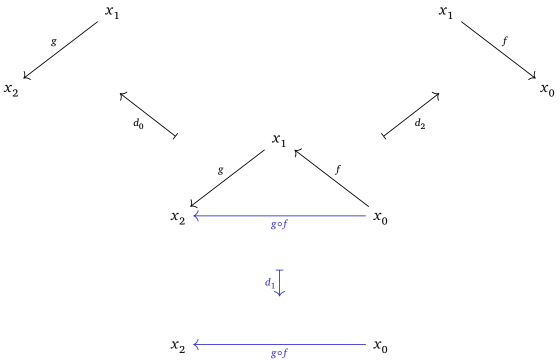


A **face map** d_i (a **degeneracy map** s_i) deletes (repeats) the i-th vertex, and the generators satisfy the relations

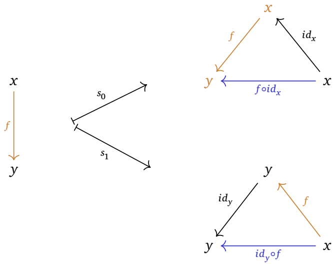
$$d_i s_i = d_{i+1} s_i = 1\,, \quad d_i d_j = d_{j-1} d_i\,, \quad d_{j+1} s_i = s_i d_j\,, \quad d_i s_j = s_{j-1} d_i\,, \quad s_j s_i = s_i s_{j-1} \qquad (i < j).$$

Example: nerve of a category

- X_0 — **objects of the category**
- X_1 — **morphisms of the category**
- $X_{n\geq 2}$ — **length n sequences of composable morphisms** (and their composites)



face maps $d_i : X_2 \to X_1$



degeneracy maps $s_i : X_1 \to X_2$

Decomposition spaces [5]

Definition

A **simplicial groupoid** $X_\bullet : \Delta^{op} \to \mathbf{Grpd}$ is a **decomposition space** if it maps **active-inert pushouts** to pullbacks.

$$X_\bullet\left(\begin{array}{ccc}[n'] & \longleftarrow & [n]\\ \uparrow & \lrcorner & \uparrow\\ [m'] & \longleftarrow & [m]\end{array}\right) = \begin{array}{ccc}X_{n'} & \rightarrowtail & X_n\\ \downarrow & \lrcorner & \downarrow\\ X_{m'} & \rightarrowtail & X_m\end{array}$$

Definition (Equivalent form)

A **simplicial groupoid** $X_\bullet : \Delta^{op} \to \mathbf{Grpd}$ is a **decomposition space** if the following commutative squares are all **homotopy pullbacks** (for all $n > 1$ and $0 < i < n$):

$$\begin{array}{ccc}X_{n+1} & \xrightarrow{d_{n+1}} & X_n\\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i}\\ X_n & \xrightarrow[d_n]{} & X_{n-1}\end{array} \qquad \begin{array}{ccc}X_{n+1} & \xrightarrow{d_0} & X_n\\ {\scriptstyle d_{i+1}}\downarrow & & \downarrow{\scriptstyle d_i}\\ X_n & \xrightarrow[d_0]{} & X_{n-1}\end{array}$$

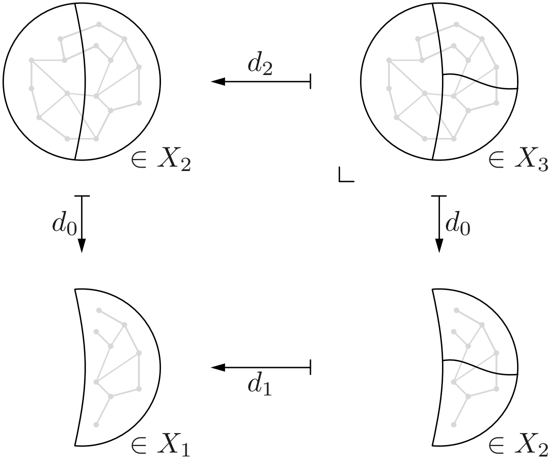
Example: For $n = 2$, the equations imply that a 3-simplex can be **reconstructed** (up to homotopy equivalences) by gluing two 2-simplices along a 1-simplex (i.e., the long edge of one along a short edge of the other).

[5] I. Gálvez-Carrillo, J. Kock, and A. Tonks. "Decomposition Spaces, Incidence Algebras and Möbius Inversion I: Basic Theory". In: *Advances in Mathematics* 331 (June 2018), pp. 952–1015. DOI: 10.1016/j.aim.2018.03.016.

Decomposition space example

Schmitt's Hopf algebra of graphs [2]

- Let X be the **simplicial groupoid** with X_k the groupoid of **directed multi-graphs with an ordered k-part vertex-induced partition** (with parts possibly empty, and X_0 the 1-element groupoid containing only the empty graph).



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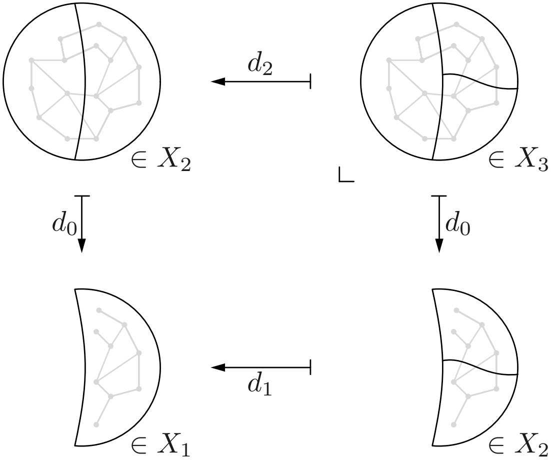
- The **decomposition space axiom** is given by the pullback diagram on the right:
  - **horizontal maps** join the last two layers
  - **vertical maps** forget the first layer
  - the diagram expresses the fact that the triple partition (top right) can be reconstructed by the information contained in the cospan



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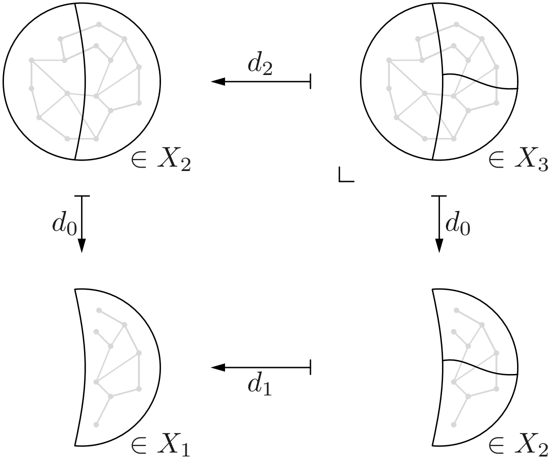
  - **horizontal maps** join the last two layers
  - **vertical maps** forget the first layer
  - the diagram expresses the fact that the triple partition (top right) can be reconstructed by the information contained in the cospan
- This is **not** an example of a **1-Segal space**, since a graph with a two-part partition cannot be reconstructed from knowing only the two parts (cf. [2], Sec. 1.1.5)



CULF functors and the category of decomposition spaces

Definition (CULF functors, [2], Sec. 1.5.1)

A **simplicial map** $F : Y \to X$ is **CULF** ("conservative and having unique lifting of factorizations") if it is **Cartesian on active maps**.

- If X is a **decomposition space** and $F : Y \to X$ a **CULF map**, then Y is a decomposition space, too.
- This motivates to define the **∞-category Decomp** of **decomposition spaces** and **CULF maps**.
- Crucially, the **incidence coalgebra constructions** are **(covariantly) functorial** in **CULF maps**.

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- This motivates to define the **∞-category Decomp** of **decomposition spaces** and **CULF maps**.
- Crucially, the **incidence coalgebra constructions** are **(covariantly) functorial** in **CULF maps**.

For the special case of **decomposition spaces** of the form $S : \mathbb{\Delta}^{op} \to \mathbf{Set}$ (but conjectured also to be true for generic decomposition spaces), one has the following result:

Theorem ([3], Thm. 4.5)

*For every **decomposition space** $\mathscr{D} : \mathbb{\Delta}^{op} \to \mathbf{Set}$, there exists an **equivalence of categories***

$$\mathbf{Decomp}/_{\mathscr{D}} \simeq \mathbf{Psh}(tw_{\mathscr{D}}),$$

*where $tw_{\mathscr{D}}$ denoted the **twisted arrow category** of $\mathscr{D}$.*

$\Rightarrow$ This opens up the possibility to use techniques from **topos theory** to study **decomposition space constructions**: defining new decomposition spaces from old (using the **internal language** of topoi), investigating notions such as **subobject classifiers** in the combinatorial setting, …

Groupoid homotopy theory

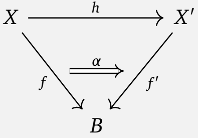
Motivation

- **Key conceptual point:** most interesting examples suffer from/feature some form of **algebraic structure on isomorphism classes of combinatorial objects**; the **decomposition space framework** thus aims to formulate algebraic structures in a **"representative-independent"** fashion
- Another interesting technical point concerns **objective combinatorics** vs. **"concrete" combinatorics**: many results at the objective level are available **without any finiteness constraints** (i.e., in the form of **bijective proofs**), but in order to recover results from traditional combinatorics, one requires notions of **finiteness** of the underlying groupoids and compatible notions of **cardinalities**
- There are very subtle issues regarding **finiteness notions**. For instance, there is a variant of the **numerical convolution algebra** which only works for finite categories; albeit this seems to be a rather strong restriction, these cases are very important in practice (as, for instance, they cover the **Hall algebras** as **numerical convolution algebras** for the **Waldhausen $S_\bullet$ construction**; in that case, one needs homological finiteness conditions, i.e., that Ext^0 and Ext^1 are finite).

Slices of simplicial groupoids and fundamental equivalence; [2], App. A.2

Definition

For every groupoid $B \in \mathbf{Grpd}$, let $\mathbf{Grpd}/_B$ denote the **homotopy slice category over** B, with objects groupoid maps $f : X \to B$, and morphisms triangles such as below, where $\alpha : f \overset{\cdot}{\cong} f' \circ h$ is a **homotopy equivalence** (i.e., a natural isomorphism):



Note: For 1 the **terminal groupoid**, we have that $\mathbf{Grpd}/_1 \simeq \mathbf{Grpd}$.

Theorem (Fundamental equivalence; [2], Thm. A.2.3)

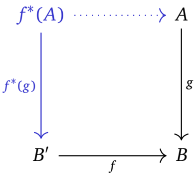
*For a fixed groupoid $B \in \mathbf{Grpd}$, there exists a **canonical equivalence***

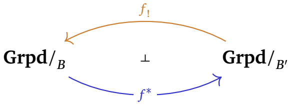
$$\mathbf{Grpd}/_B \simeq \mathbf{Grpd}^B$$

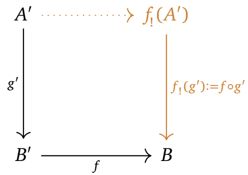
*between the **homotopy slice category** $\mathbf{Grpd}/_B$ of groupoids over B, and the category $\mathbf{Grpd}^B$ of **B-indexed families of groupoids**. This equivalence is given by taking **homotopy fibers** and via the **Grothendieck construction**.*

Linear functors (I); [2], App. A.3

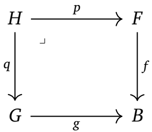
For every map of groupoids $f : B' \to B$, letting $f^* : \mathsf{Grpd}/_B \to \mathsf{Grpd}/_{B'}$ denote the functor defined by taking **homotopy pullback along f**, and $f_! : \mathsf{Grpd}/_{B'} \to \mathsf{Grpd}/_B$ the functor defined by **postcomposition with f**, one obtains the following **homotopy adjunction**:







Lemma (Beck-Chevalley)
For any ***homotopy pullback square*** *as below,*



the functors $p_! q^*, g^* f_! : \mathbf{Grpd}/_G \to \mathbf{Grpd}/_F$ *are* ***naturally homotopy equivalent***.

Linear functors (II); [2], App. A.3

Definition

Any **span** $A \xleftarrow{r} G \xrightarrow{f} B$ of **groupoid maps** yields a **functor**

$$f_! r^* : \mathbf{Grpd}/_A \to \mathbf{Grpd}/_B \,.$$

- A functor homotopy equivalent to one arising from a span is called **linear**.
- By the **Beck-Chevalley** lemma, **compositions of linear functors are linear**.
- Let **LIN** denote the **monoidal 2-category** of all **slice categories** $\mathbf{Grpd}/_B$ and **linear functors** between them, and with **monoidal product** defined as

$$\mathbf{Grpd}/_A \otimes \mathbf{Grpd}/_B := \mathbf{Grpd}/_{A\times B} \,.$$

- The **neutral object** for $\otimes$ is $\mathbf{Grpd}/_1 \simeq \mathbf{Grpd}$ ($\hat{=}$ **ground field** in **homotopy linear algebra**).
- $\mathbf{Grpd}^B$ is the **linear dual** of $\mathbf{Grpd}/_B$, since $\mathbf{Grpd}^B \simeq \mathbf{LIN}(\mathbf{Grpd}/_B, \mathbf{Grpd})$.
- There exists a **canonical pairing** $\mathbf{Grpd}/_B \times \mathbf{Grpd}^B \to \mathbf{Grpd}$:

$$\langle \ulcorner t \urcorner, h^s \rangle = Hom_{\mathbf{Grpd}}(s,t) = \begin{cases} \Omega_s(B) & (s \cong t) \\ \varnothing & (s \ncong t) \end{cases} \qquad \text{with:} \qquad \begin{array}{ll} \ulcorner t \urcorner & : 1 \to B : 1 \mapsto t \in B \\ h^s & := Hom_{\mathbf{Grpd}}(s,-) : B \to \mathbf{Grpd} \end{array}$$

Here, $\Omega_s(B)$ is the **loop groupoid of B at object s**, given by **homotopy pullback** of $\ulcorner s \urcorner : 1 \to B$ along itself.

Finiteness notions for groupoids

Definition (Connectedness and discreteness)

- A **groupoid** G is **connected** if $\mathsf{obj}(G)$ is non-empty and $Hom_G(x,y)$ is non-empty for all $y,z \in G$.
- A **component** G is a **maximally connected sub-groipoid**, denoted $[x]$ or $G_{[x]}$ for x in the **component**.
- $\pi_0(G)$ is defined as the **set of components of G**.
- $\pi_1(G,x) := Aut_G(x) = Hom_G(x,x)$ (**automorphism group of x**).

- A groupoid G is **homotopy discrete** if $\pi_1(G,x)$ is trivial for all x, and **contractible** (i.e., homotopy equivalent to the terminal groupoid 1) if it is **connected** and **homotopy discrete**.

Definition (Finiteness)

- A **groupoid** G is **locally finite** if $\pi_1(G,x)$ for every x.
- It is **(homotopy) finite** if in addition $\pi_0(G)$ is finite.
- We denote by **grpd** the **category of finite groupoids**.

Cardinality of groupoids; [2], App. A.4

- The **cardinality** $|B|$ of a **finite groupoid** B is defined as

$$|B| := \sum_{[x]\in\pi_0(B)} \frac{1}{|\pi_1(B,x)|} = \sum_{[x]\in\pi_0(B)} \frac{1}{|Aut_B(x)|} \in \mathbb{Q}$$

- For any function $q : \pi_0(B) \to \mathbb{Q}$, we introduce the **notation**

$$\int^{x\in B} q(x) := \sum_{[x]\in\pi_0(B)} \frac{q(x)}{|\pi_1(B,x)|}\,.$$

Cardinality of finite linear functors; [2], App. A.4

Lemma ([2], Prop. A.1.3)

*Any span $A \xleftarrow{r} G \xrightarrow{f} B$ of **locally finite groupoids** A, G, B, and where r has **finite homotopy fibers** induces a **finite linear functor** $\mathbf{Grpd}/_A \to \mathbf{Grpd}/_B$ that extends to $\mathbf{grpd}/_A \to \mathbf{grpd}/_B$.*

- Let $\underrightarrow{\mathbf{lin}}$ be denote the category of **slice categories** $\mathbf{grpd}/_A$ (for A finite) and **finite lienar functors**.

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- Let $\underrightarrow{\mathbf{lin}}$ be denote the category of **slice categories** $\mathbf{grpd}/_A$ (for A finite) and **finite lienar functors**.
- **global cardinality** $\|\,\| : \underrightarrow{\mathbf{lin}} \to \mathbf{Vect}$ is defined via

$$\|\mathbf{grpd}/_A\| := \mathbb{Q}_{\pi_0(A)} \quad \text{with basis } \{\delta_a\}_{a\in\pi_0(A)}$$

$$\|\mathbf{grpd}/_A \to \mathbf{grpd}/_B\| := (\mathbb{Q}_{\pi_0(A)} \to \mathbb{Q}_{\pi_0(B)}) \,:\, \delta_a \mapsto \sum_{[b]\in\pi_0(B)} |B_{[b]}|\,|G_{a,b}|\delta_b = \int^{b\in B} |G_{a,b}|\delta_b$$

Here, $G_{a,b}$ are the fibers of the map $G \to A \times B$ induced by the span $A \leftarrow G \to B$.

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- Let $\underrightarrow{\mathbf{lin}}$ be denote the category of **slice categories** $\mathbf{grpd}/_A$ (for A finite) and **finite lienar functors**.
- **global cardinality** $\| \, \| : \underrightarrow{\mathbf{lin}} \to \mathbf{Vect}$ is defined via

$$\|\mathbf{grpd}/_A\| := \mathbb{Q}_{\pi_0(A)} \quad \text{with basis } \{\delta_a\}_{a\in\pi_0(A)}$$

$$\|\mathbf{grpd}/_A \to \mathbf{grpd}/_B\| := (\mathbb{Q}_{\pi_0(A)} \to \mathbb{Q}_{\pi_0(B)}) \; : \; \delta_a \mapsto \sum_{[b]\in\pi_0(B)} |B_{[b]}| \, |G_{a,b}| \delta_b = \int^{b\in B} |G_{a,b}| \delta_b$$

  Here, $G_{a,b}$ are the fibers of the map $G \to A \times B$ induced by the span $A \leftarrow G \to B$.
- For any object $p : G \to B$ in $\mathbf{grpd}/_B$, one may define the **local cardinality** $|p|$ of p as the **global cardinality** of the **linear finite functor** $L(p)$ induced by the span $1 \leftarrow G \xrightarrow{p} B$:

$$|p| := \|L(p)\| = \int^{b\in B} |G_b| \delta_b$$

- These notions may also be **dualized** to **cardinalities** for $\mathbf{grpd}^X$ (cf. [2], A.4.5), requiring the notion of **profinite-dimensional vector spaces**.

Incidence (co-/bi-)algebras

Incidence coalgebras [2]

- Taking **homotopy pullback** along a groupoid morphism $f : B' \to B$ yields a **functor**

$$f^* : \mathbf{Grpd}_{/B} \to \mathbf{Grpd}_{/B'}$$

- f^* has a **homotopy left adjoint** $f_!$ defined by postcomposition,

$$f_! : \mathbf{Grpd}_{/B'} \to \mathbf{Grpd}_{/B}$$

- A **span** of groupoid maps $A \xleftarrow{r} G \xrightarrow{f} B$ thus induces a functor (referred to as **linear**)

$$f_! r^* : \mathbf{Grpd}_{/A} \to \mathbf{Grpd}_{/B}$$

Definition

For a **decomposition space** X, the **incidence coalgebra** $(\mathbf{Grpd}_{/X_1}, \Delta, \varepsilon)$ is defined via

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1 \qquad\qquad X_1 \xleftarrow{s_0} X_0 \xrightarrow{z} 1$$

$$\Delta := (d_2, d_0)_! \circ d_1^* \qquad\qquad \varepsilon := (s_0)_! \circ z^*$$

Incidence coalgebras [2]

- This construction is intended to generalize the **incidence coalgebra of posets**, with the idea that (for $f \in X_1$)

$$\Delta(f) = \sum_{\substack{\sigma \in X_2 \\ d_1(\sigma)=f}} d_2(\sigma) \otimes d_0(\sigma)$$

  computes all ways how $f \in X_1$ can arise as the "long edge" of some 2-simplices σ.

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- More precisely, for a **basis element** $\ulcorner t \urcorner : 1 \to X_1$ of $\mathbf{Grpd}_{/X_1}$ (i.e., a functor that picks out a particular element $t \in X_1$),

$$\Delta(\ulcorner t \urcorner) = (d_2, d_0)_! \circ d_1^*(\ulcorner t \urcorner) = \int^{(a,b)\in X_1 \times X_1} (X_2)_{t_{a,b}} \ulcorner a \urcorner \otimes \ulcorner b \urcorner \in \mathbf{Grpd}_{/X_1} \otimes \mathbf{Grpd}_{/X_1}$$

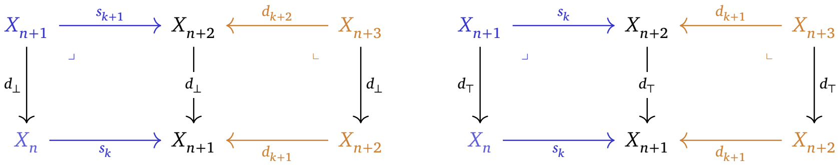
where $(X_2)_{t_{a,b}}$ denotes the **homotopy fiber** of $(d_1, d_2, d_0) : X_2 \to X_1 \times X_1 \times X_1$ over (t, a, b).

- **Compare:** in **Schmitt's construction**, for a directed multigraph G with vertex set V_G, and with $G|X$ the **restriction** of G to vertex set $X \subseteq V_G$, denoting in a slight abuse of notations by G also the isomorphism class of G, we find

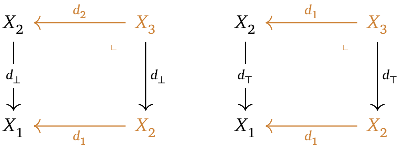
$$\Delta(G) := \sum_{A+B=V_G} G|A \otimes G|B$$

Coassociativity via decomposition space axioms

- Originally, incidence coalgebras were constructed for **1-Segal space**; but it is easy to find combinatorial structures that naturally give rise to incidence coalgebras, but are only **2-Segal spaces**
- Reacap: a **2-Segal spaces** is a functor $S : \Delta^{op} \to \mathbf{Grp}$ that takes **active-inert pushouts** to **pullbacks**.
- Parsing out the previous definition in detail, one may demonstrate [5] that one only needs to verify that the following squares are pullbacks (for all $n \geq 0$ and $0 \leq k \leq n$):



- An important special case are the following squares, which guarantee **coassociativity** of the incidence coalgebra:



From objective to numerical coalgebras; [2], Sec. 1.2.8 & [5], Sec. 7

Conceptual challenge

The notions of **decomposition spaces** and **incidence coalgebras** are inherently **objective**, in the sense that they deal directly with **combinatorial objects** rather than with **vector spaces** spanned by these objects and (co-)algebraic structures thereon. In particular, while the theory at the **objective** level is well-posed **without finiteness conditions**, to recover **numerical** results in "classical" combinatorics, one must require suitable **finiteness conditions** in order to apply **cardinality** constructions to the **objective** theory.

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- A **decomposition space** $X_\bullet : \mathbb{\Delta}^{op} \to \mathbf{Grpd}$ is **locally finite** if X_1 is a **locally finite groupoid** (i.e., $\pi_1(X,x) = Aut_X(x)$ is finite for all x), and if in addition the maps $X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2$ are **(homotopy) finite** (i.e., have finitie homotopy fibers).
- For a **locally finite decomposition space** $X_\bullet$, the **comultiplication** and **counit** maps are **finite linear functors**, and thus descend to **slices of finite groupoids**:

$$\Delta : \mathbf{grpd}/_{X_1} \to \mathbf{grpd}/_{X_1} \otimes \mathbf{grpd}/_{X_1}, \quad \varepsilon : \mathbf{grpd}/_{X_1} \to \mathbf{grpd}$$

From objective to numerical coalgebras; [2], Sec. 1.2.8 & [5], Sec. 7

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$\Rightarrow$ Taking **cardinality** yields **comultiplication** and **counit** maps on **vector spaces**

$$|\Delta| : \mathbb{Q}_{\pi_0(X_1)} \to \mathbb{Q}_{\pi_0(X_1)} \otimes \mathbb{Q}_{\pi_0(X_1)}\,, \quad |\varepsilon| : \mathbb{Q}_{\pi_0(X_1)} \to \mathbb{Q}$$

which form the **coassociative** and **counital numerical coalgebra** $I_X = (\mathbb{Q}_{\pi_0(X_1)}, |\Delta|, |\varepsilon|)$.

Examples of objective and numerical coalgebras in combinatorics

- **q-combinatorics** ([2], Sec. 2.3)
- **Faà di Bruno algebra** ([2], Sec. 2.4)
- **"Operadic" examples** (graphs, trees, ...) ([2], Sec. 2.5)
- current research topic: objective combinatorics for **symmetric functions** ([2], Sec. 2.6)

- **Free decomposition spaces** ([2], Sec. 3.3.7) $\rightarrow$ many combinatorial coalgebras of deconcatenation type are incidence coalgebras of free decomposition spaces!

- Possibly also of interest: link between **Möbius inversion and renormalization** [6]

[6] J. Kock. "From Möbius Inversion to Renormalisation". In: *Communications in Number Theory and Physics* 14.1 (2020), pp. 171–198. DOI: 10.4310/CNTP.2020.v14.n1.a3.

Incidence bialgabras via monoidal decomposition spaces; [2], Sec. 1.5.6

Recap: the appropriate notion of **functors between decomposition spaces** are **CULF functors** (cf. [2], Sec. 1.5)

Definition

A **monoidal decomposition space** is a **decomposition space** Z equipped with an **associative unital monoid structure** given by **CULF functors** $m : Z \times Z \to Z$ and $e : 1 \to Z$.

Lemma

*If Z is a **monoidal decomposition space**, then $\mathbf{Grpd}/_{Z_1}$ carries the structure of a **bialgebra**, called **incidence bialgebra**. Moreover, **monoidal CULF functors** induce **bialgebra homomorphisms**.*

Example (Schmitt Hopf algebra of graphs)

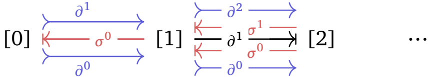
Taking as a **monoidal structure** the one induced by taking **disjoint union** of graphs with partitions, one may verify that this indeed yields a **monoidal decomposition space** (cf. [2], Sec. 1.5.10 for further details).

Free decomposition spaces

Free decomposition spaces [3]

- The simplex category Δ has an **active-inert factorization system**, i.e., every map of Δ factors uniquely as an **active map** followed by an **inert map**, with
  - **active maps** — $g : [k] \twoheadrightarrow\!| [m]$ such that $g(0) = 0$ and $g(k) = m$ (**“endpoint-preserving”**)
  - **inert maps** — $f : [m] \rightarrowtail [n]$ such that $f(i+1) = f(i)+1$ for $0 \leq i \leq m-1$ (**“distance-preserving”**)

- In terms of **generating maps** of Δ, one finds that all generators are **active maps**, except for the outer coface maps, which are **inert maps**:



- Restriction of Δ to **inert maps** (= blue arrows in the above diagram) defines a subcategory Δ_{inert} and an embedding $j : \Delta_{inert} \to \Delta$.

Corollary (Free decomposition spaces are Möbius; [3], Cor. 2.3.3)

*For any $A : \Delta_{inert}^{op} \to$ **Grpd**, the **left Kan extension** $j_!(A) : \Delta^{op} \to$ **Grpd** is a **(Möbius) decomposition space**, called the **free decomposition space associated to** A.*

Long (!) list of examples of free decomposition spaces in combinatorics

Key insight:

Essentially all examples with monoidal structure of **deconcatenation type** are **free decomposition spaces**

Prototypical examples from [3]:

- Quasi-symmetric functions (Secs. 5.1.4, 5.2.3, 5.3.3)
- WQSym and FQSym (Sec. 5.1.5)
- Parking functions (Sec. 5.1.6)
- Noncrossing partitions (Sec. 5.2.1)
- **Dyck paths** (Sec. 5.2.2; cf. also **next slide**)
- Layered posets (Sec. 5.2.3)
- Heap orders, scheduling, and sequential processes (Sec. 5.2.4)
- **Decomposition space of nondegenerate simplices** (Sec. 5.3.2) → plays a crucial rôle in the construction of **tracelet decomposition spaces** [1] (cf. last part of the talk!)

Free decomposition space example: Dyck paths [3], Sec. 5.2.2

- **Dyck path** — integer lattice path from $(0,0)$ to $(2\ell, 0)$ (for some $\ell \in \mathbb{N}$) taking only steps $\diagup = (1,1)$ and $\diagdown = (1,-1)$
- **height** of a **Dyck path** — maximal second coordinate of the Dyck path
- A_n — **set of Dyck paths of height** n (for $n \geq 0$)

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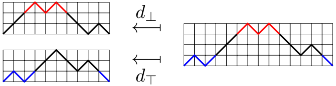
- **top and bottom face maps** $d^{\top}, d_{\perp} : A_{n+1} \to A_n$ — **clip the top-most/bottom-most level** of the Dyck paths:



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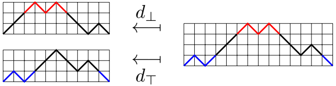
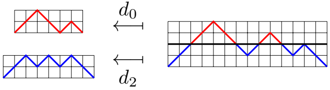


- The **free decomposition space of Dyck paths** $X_\bullet := j_!(A_\bullet)$ has
  - X_1 — set of all Dyck paths (i.e., all lengths and heights)
  - $X_{k>1}$ — the set of all Dyck paths with k **marked levels** (without affecting the path)

In particular, the **inner face map** $d_1 : X_2 \to X_1$ forgets the level marking, while the **outer face maps** $d_0, d_2 : X_2 \to X_1$ (involved in the **coproduct** definition) act as follows:



Traceelet Decomposition Spaces

Joint work with Joachim Kock

Joint work with **Joachim Kock** (UAB Barcelona and U Copenhagen).

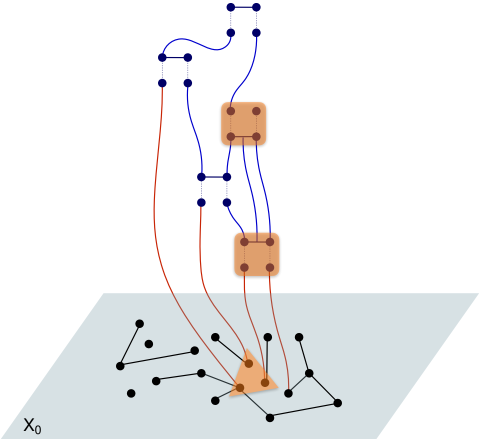
The project is based upon two **main ingredients**:

- Joachim's long line of work on the **theory of decomposition spaces** (cf. e.g. [2,3,5])
- my notion of **tracelet theory** [7]

First results of our collaboration were presented in [1].

[7] N. Behr. "Tracelets and Tracelet Analysis of Compositional Rewriting Systems". In: *Proceedings of ACT 2019*. Vol. 323. EPTCS. 2020, pp. 44–71. DOI: 10.4204/EPTCS.323.4.

A sketch of “dynamic combinatorics”



- Consider a **rewriting system** over **(undirected multi-) graphs**.
- Starting from some graph X_0, we may consider applying a **sequence of rewriting operations** (here: edge deletions and creations).

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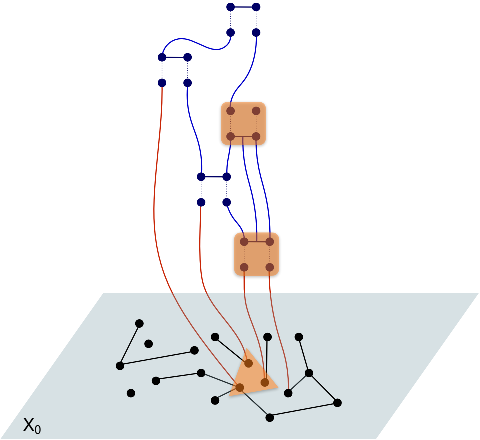

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- The diagram on the left captures the **causal structure** in such a sequence. The **blue** part is called a **tracelet**.

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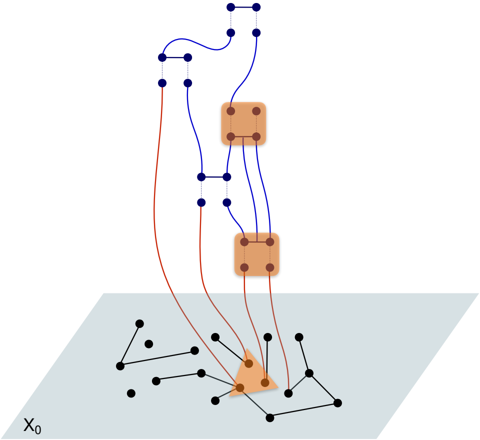

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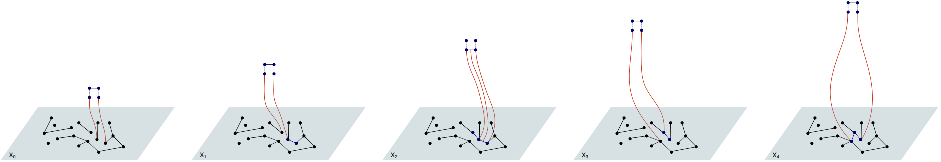
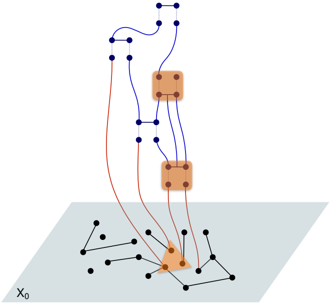
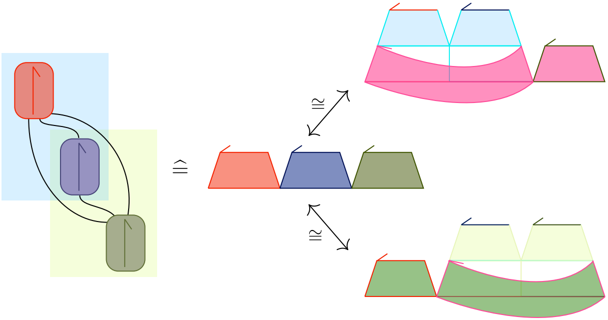
- Starting from some graph X_0, we may consider applying a **sequence of rewriting operations** (here: edge deletions and creations).
- The diagram on the left captures the **causal structure** in such a sequence. The **blue** part is called a **tracelet**.
- Typical **"dynamic combinatorics"** questions:
  - # of ways to create a triangle in n steps?
  - Dito **up to sequential commutativity**?
  - # of ways n rewrite steps can interact?



(a) A **rewriting sequence** of length 5 (with edge creation/deletion rules, and where “wires” indicate matches).



(b) (equivalence)Tracelet and **shift equivalence** example.

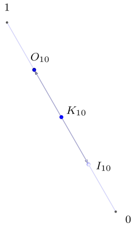


(c) Defining property of **tracelets** (here of length 3).

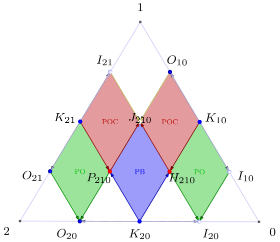
Figure: An illustration of graph **rewriting sequences** (top) and of the **tracelet** picture (bottom).

Tracelet decomposition spaces [1]

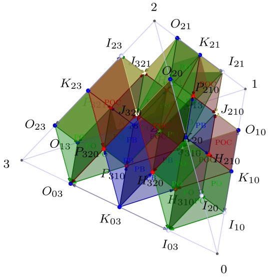
Let $X_\bullet : \Delta^{op} \to \mathbf{Grpd}$ be a **simplicial groupoid** with X_0 trivial, and with



1-simplices
rules

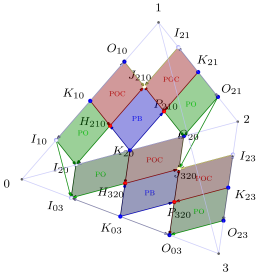


2-simplices
rule compositions

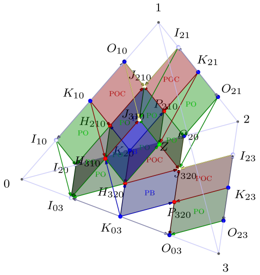


3-simplices
nested rule compositions

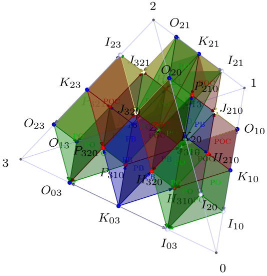
Tracelet decomposition spaces [1]



From adjacent 2-simplices…



…via the **concurrency theorem**…



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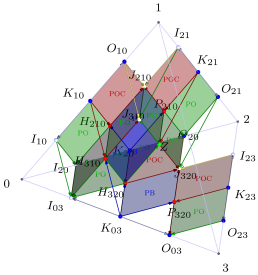

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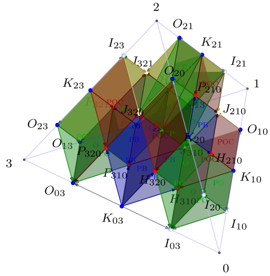

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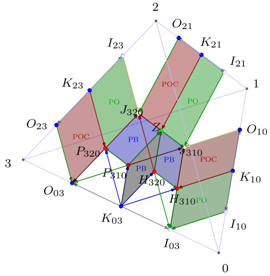
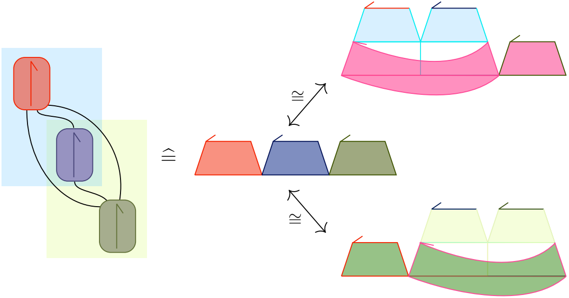


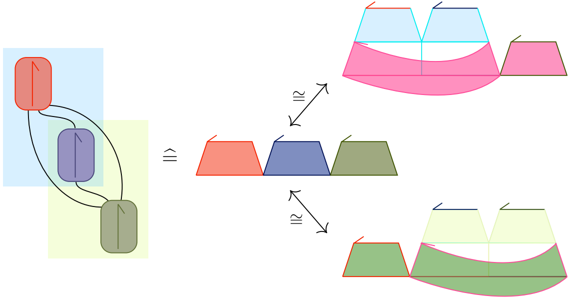
…to **tracelets**

Tracelet decomposition spaces [1]



Interpretation of the properties encoded in **3-simplices**

Tracelet decomposition spaces [1]



Interpretation of the properties encoded in **3-simplices**

Caveat:

In fact, the construction above presents for brevity merely the **decomposition space of rewrite rules**, which gives rise to a **categorification of the rule algebra**. To obtain the actual **tracelet decomposition space**, one needs additional refinements and constructions (essentially taking into account sequential commutativity and unitality), which amount to taking a **free decomposition space** construction ([1].

Conclusioon and outlook

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- **Decomposition space theory** provides a (vast!) generalization of the notion of **incidence (co-)algebras** to most known combinatorial structures.
- The framework is **objective** in nature — algebraic structures are defined utilizing the **homotopy theory of groupoids** (with **basis** given by $\mathbf{Grpd}/_{X_1}$ or its linear dual $\mathbf{Grpd}^{X_1}$, respectively.)
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- The framework yields a powerful **organizational principle** of **combinatorial structures**, permitting to compare structures via inherently **bijective correspondences**.
- **CULF maps** of **decomposition spaces** are precisely the kind of functors that preserve coalgebraic structure, and as such can serve as a **construction principle** for many combinatorially interesting examples of decomposition spaces.
- The **∞-category Decomp** of **decomposition spaces** and **CULF maps** has the important technical property that its **slice categories $\mathbf{Decomp}/_{\mathscr{D}}$** (for $\mathscr{D} \in \mathsf{obj}(\mathbf{Decomp})$) are **toposses**, which yields another versatile methodology for constructing and comparing decomposition spaces.

Recent trends in decomposition space theory: BANFF 2024 workshop

5-day workshop on **2-Segal (aka decomposition) spaces**[1] — many **talk recordings** available, and with a **conference proceedings volume** forthcoming!

[1] https://www.birs.ca/events/2024/5-day-workshops/24w5266

References

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