Cluster algebras for spinor helicity and momentum twistor varieties

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joint work in progress with Jian-Rong Li (University of Vienna)

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Motivation: scattering amplitudes

In quantum field theory physicist probability of particle interactions are computed via scattering amplitudes.

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The *symbol* of an iterated integral captures information on the integration kernels while disregarding coefficients, e.g.

$$Li_2(x) = -\int_0^x d\log(y) \int_0^z d\log(1-z)$$

has symbol $\mathcal{S}(Li_2(x)) = -(1-x) \otimes x$, $1-x$ and x are called the *letters*.

Motivation: cluster bootstrap

In the toy model *planar $\mathcal{N} = 4$ super Yang-Mills* symbol letters of the amplitudes for $n = 6, 7$ particles are *cluster variables* Grassmannian $\mathrm{Gr}_{4,n}$.

This has lead to a *bootstrap* for the amplitude: it is the unique generalized polylogarithmic function with symbol letters given by cluster variables.

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Question: Can this be extended beyond $\mathcal{N} = 4$ sYM?

1. What are the relevant kinematic spaces?
2. Are there cluster algebras?

Overview

1. Partial flag varieties
2. Spinor helicity and momentum twistor varieties
3. Cluster algebras
4. Symbol alphabet for the spinor helicity variety
5. Summary and open questions

§1 Partial flag varieties

Consider $1 \leq d_1 < d_2 < n$ and define the (two step) *partial flag variety*

$$\mathcal{F}_{d_1,d_2;n} := \{0 \in V_1 \subsetneq V_2 \subsetneq \mathbb{C}^n : \dim_{\mathbb{C}} V_i = d_i\}$$

Associate to $\mathcal{V} \in \mathcal{F}_{d_1,d_2;n}$ a matrix $M_{\mathcal{V}} = (m_{ij}) \in \mathbb{C}^{d_k \times n}$ such that V_i is generated by the first d_i rows of M.

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$\{i_1, \ldots, i_{d_j}\} \subset \{1, \ldots, n\}$, define the *Plücker coordinate*

$$P_{i_1,\ldots,i_{d_j}}(\mathcal{V}) := \det(m_{ab})_{1 \leq a \leq d_j,\ b \in \{i_1,\ldots,i_{d_\ell}\}}.$$

Example: $M_{\mathcal{V}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & -3 \end{bmatrix}$, so that $P_{134}(\mathcal{V}) = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & -3 \end{bmatrix}\right) = -6.$

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The $P_{i_1,\ldots,i_{d_j}}$ satisfy *Plücker relations* (quadratic polynomials) and determine an embedding

$$\mathcal{F}_{d_1,d_2;n} \hookrightarrow \mathbb{P}^{\binom{n}{d_1}-1} \times \mathbb{P}^{\binom{n}{d_2}-1}$$

Let $\mathbb{C}[\mathcal{F}_{d_1,d_2;n}]$ denote the homogeneous coordinate ring.

§2 Example 1: Particles and helicity spinors

Minkowski space $\mathbb{R}^4$ with inner product

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The mass m of p satisfies $p^2 := p \cdot p = m^2$. If p is *lightlike* $m = 0$, we have

$$\det(\sigma(p)) = p_0^2 - (p_1^2 + p_2^2 + p_3^2) = p^2 = 0.$$

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$$\mathcal{SH}_n := \{(\Lambda, \tilde{\Lambda}) \in \mathrm{Gr}_{2;n} \times \mathrm{Gr}_{2;n} : \Lambda \tilde{\Lambda}^T = 0\}.$$

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satisfying *Schouten identities* and *momentum conservation*

$$\begin{aligned} 0 &= P_{ij}P_{kl} - P_{ik}P_{jl} + P_{il}P_{jk} = \tilde{P}_{ij}\tilde{P}_{kl} - \tilde{P}_{ik}\tilde{P}_{jl} + \tilde{P}_{il}\tilde{P}_{jk} \\ 0 &= \sum_{s=1}^{n} P_{is}\tilde{P}_{sj} \quad (\Leftrightarrow \Lambda\tilde{\Lambda}^T = 0) \end{aligned}$$

§2 Example 1: *Spinor helicity variety*

Consider the map

$$\langle ij\rangle \mapsto P_{ij}, \quad \text{and} \quad [ij] \mapsto (-1)^{i+j-1} P_{[n]-ij}$$

where $[n] - ij := \{1, \ldots, n\} - \{i, j\}$.

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$$\begin{aligned} P_{ij}P_{kl} - P_{ik}P_{jl} + P_{il}P_{jk} &= 0 \\ P_{[n]-ij}P_{[n]-kl} - P_{[n]-ik}P_{[n]-jl} + P_{[n]-il}P_{[n]-jk} &= 0 \\ \sum_{s=1}^{n} (-1)^{s+j-1} P_{is} P_{[n]-js} &= 0 \end{aligned}$$

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As $\dim \mathcal{SH}_n \overset{(*)}{=} 4(n-3) = \dim \mathcal{F}_{2,n-2;n}$ the map induces an isomorphism (identification) between the spinor helicity variety $\mathcal{SH}_n$ and the partial flag variety $\mathcal{F}_{2,n-2;n}$. [1]

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§2 Example 2: *Momentum twistor variety*

For dual conformal invariant scattering consider momentum twistors $Z_1, \ldots, Z_n \in \mathbb{C}P^3$. The system is parametrized by

$$\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l),$$

for $1 \leq i < j < k < l \leq n$ satisfying determinantal identities (Plücker relations). Identifying $\langle ijkl \rangle \mapsto P_{ijkl}$ the same equations determine the Grassmannian $\mathrm{Gr}_{4,n}$ (up to scaling, i.e. $(\mathbb{C}^*)^n$-action).

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In the non-dual conformal invariant setting introduce $Z_{n+1}, Z_{n+2} \in \mathbb{C}P^3$ to break symmetry and parametrize by

$$\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l), \quad \langle ij \rangle := \det(Z_i Z_j Z_{n+1} Z_{n+2}),$$

for $1 \leq i < j < k < l \leq n$ resp. $1 \leq i < j \leq n$.

§2 Example 2: *Momentum twistor variety*

This motivates us to define the *momentum twistor variety* as the subvariety of $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{4}-1}$ determined by the vanishing of determinantal identidies amoing the $\langle ijkl \rangle$ and $\langle ij \rangle$.

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Consider the map

$$\langle ijkl \rangle \mapsto P_{ijkl} \quad \text{and} \quad \langle ij \rangle \mapsto P_{ij}$$

So the momentum twistor variety $\mathcal{MT}_n$ is the partial flag variety $\mathcal{F}_{2,4;n}$.

§3 Cluster algebras [S. Fomin, A. Zelevinsky math/0104151]

Commutative algebras with special generators called *cluster variables* grouped in *cluster*, and special automorphisms called *mutations* that map clusters to clusters and are determined by *quivers*.

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Example: The cluster algebras for $1 \to 2$ is recursively generated by $\{x_1, x_2\}$ and the mutation rule $x_{i+1}x_{i-1} = x_i + 1$ for generating clusters $\{x_i, x_{i+1}\}$.

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$$x_1, x_2, x_3 = \frac{x_2 + 1}{x_1}, x_4 = \frac{x_1 + x_2 + 1}{x_1 x_2}, x_5 = \frac{x_1 + 1}{x_2}, x_6 = x_1, x_7 = x_2, \ldots$$

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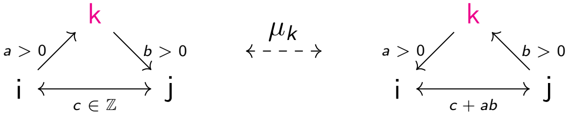
The cluster algebras is

$$A_{1\to 2} = \left\langle x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1 x_2}, \frac{x_1+1}{x_2} \right\rangle \subset \mathbb{C}(x_1, x_2)$$

III. Mutation [S. Fomin, A. Zelevinsky math/0104151]

The mutation rule is encoded locally in the quiver



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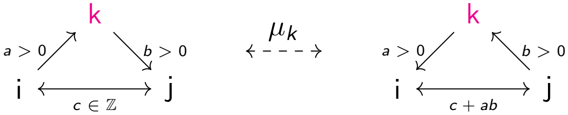

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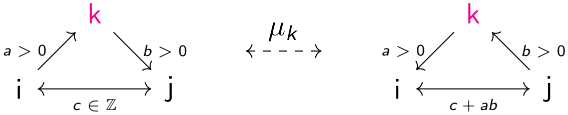

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where

$$x_k' = \frac{1}{x_k}\left(\prod_{i \to k} x_i + \prod_{k \to j} x_j\right)$$

A pair $s = (Q, \underline{x} = (x_1, \ldots, x_n))$ is called a *seed*.

III. Cluster algebra

Take away: to define a cluster algebra, I need

1. a quiver (directed graph, no loops, no 2-cycles), and
2. a tuple of algebraically independent elements (one for each vertex) in a field of rational functions, and
3. the mutation rule (encoded in the quiver).

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Recall the Plücker embedding

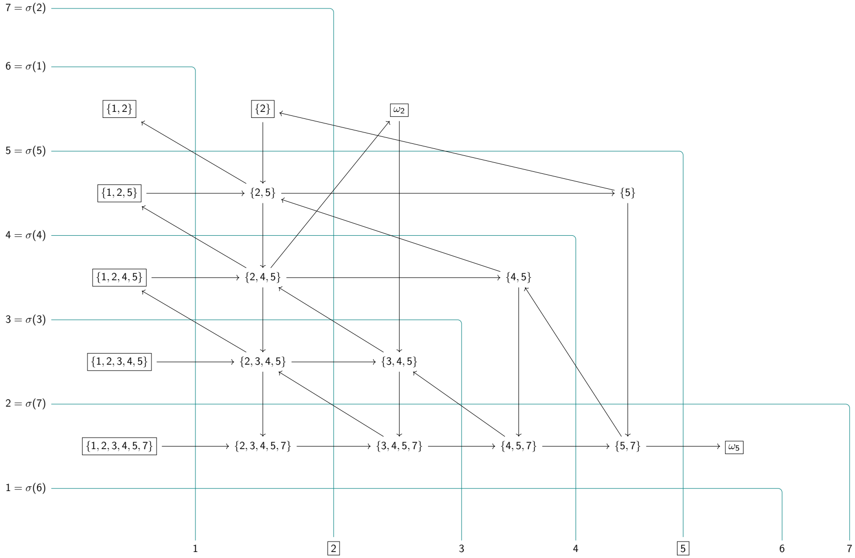
$$\mathcal{F}_{d_1,d_2;n} \hookrightarrow \mathbb{P}^{\binom{n}{d_1}-1} \times \mathbb{P}^{\binom{n}{d_2}-1}$$

with homogeneous coordinate ring $\mathbb{C}[\mathcal{F}_{d_1,d_2;n}]$

Theorem (C. Geiss, B. Leclerc, J. Schröer, Ann. Inst. Fourier 2008)

The homogeneous coordinate ring $\mathbb{C}[\mathcal{F}_{d_1,d_2;n}]$ is a cluster algebra.

III. Example $\mathcal{F}_{2,5;7}$



III. Example $\mathcal{F}_{2,5;7}$: cluster variables

Index sets are all of form $[i_j, d_j] \cup [i_{j+1}, d_{j+1}]$ and define right bound minors of a $n \times n$ unipotent upper triangular matrix.
Translating to Plücker coordinates yields the cluster variables:

$$\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]} \overset{Laplace}{=} \sum_{J \in \binom{[\ell,n]}{d_j - i_j + 1},\ J' = [\ell,n]\setminus J} (-1)^{\Sigma(i_j,d_j,J)} P_{[i_j-1]\cup J} P_{[i_{j+1}-1]\cup J'}$$

where $\ell = n - d_j - d_{j+1} + i_j + i_{j+1} - 1$.

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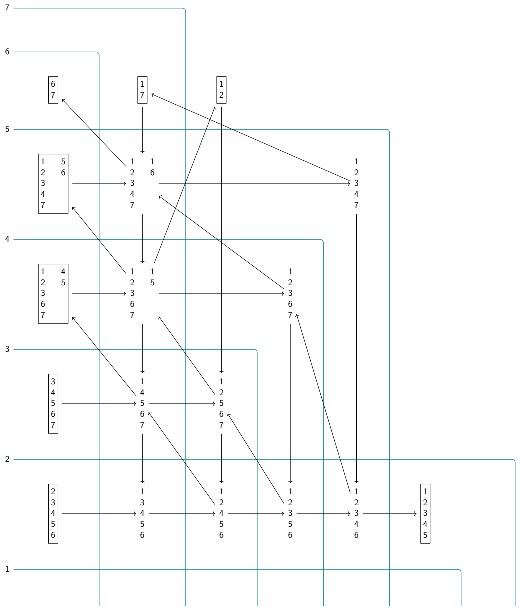
Translating to Plücker coordinates yields the cluster variables:

$$\Delta_{[i_j,d_j]\cup[i_{j+1},d_{j+1}]} \overset{Laplace}{=} \sum_{J\in\binom{[\ell,n]}{d_j-i_j+1},\ J'=[\ell,n]\setminus J} (-1)^{\Sigma(i_j,d_j,J)} P_{[i_j-1]\cup J} P_{[i_{j+1}-1]\cup J'}$$

where $\ell = n - d_j - d_{j+1} + i_j + i_{j+1} - 1$.Associate a two-column tableaux filled index sets of maximal term on RHS (tableaux of a cluster variable as an element of *Lusztig's dual canonical basis* [Lusztig '90, Chari–Pressley '91, Hernandez–Leclerc '10, Li '24])

Denote by $\mathcal{T}_{d_1,d_2;n}$ (resp. $\mathcal{T}_{d;n}$) the set of semistandard Young tableaux with columns of size d_1 or d_2 (resp. d), filled with $\{1,\ldots,n\}$.

III. Example $\mathcal{F}_{2,5;7}$



§3 Relation between the cluster structures

There's a combinatorial map on tableaux:

$$\varphi : \mathcal{T}_{d_1,d_2;n} \hookrightarrow \mathcal{T}_{d_2;n+d_2-d_1}$$

obtained by "filling up" d_1-columns to d_2 columns by adding $n+1,\ldots,n+d_2-d_1$.

Example: $\varphi : \mathcal{T}_{2,4;6} \hookrightarrow \mathcal{T}_{4;8}$, for $a<b$ and $i<j<k<l$ in $\{1,\ldots,6\}$

$$\begin{smallmatrix} a \\ b \end{smallmatrix} \mapsto \begin{smallmatrix} a \\ b \\ 7 \\ 8 \end{smallmatrix}, \qquad \begin{smallmatrix} i \\ j \\ k \\ l \end{smallmatrix} \mapsto \begin{smallmatrix} i \\ j \\ k \\ l \end{smallmatrix}$$

Theorem (L.B.-Jianrong Li, arxiv:2408.14956)

Let $s_0^F = (Q^F, \underline{x}^F)$ denote the initial seed for $\mathcal{F}_{2,n-2;n}$ (resp. $\mathcal{F}_{2,4;n}$). Then there exists a seed $s^{Gr} = (Q^{Gr}, \underline{x}^{Gr})$ for $Gr_{n-2;2n-4}$ (resp. $Gr_{4;n+2}$) so that

1. *Q^F is a full subquiver in Q^{Gr} obtained by freezing $n-5$ (resp. one) vertices and eliminating $(n-3)(n-5)$ (resp. 3) vertices*
2. *$\varphi(\underline{x}^F) \subset \underline{x}^{Gr}$.*

IV. Example: $n = 6$ s_0^{Gr} and s^{Gr}

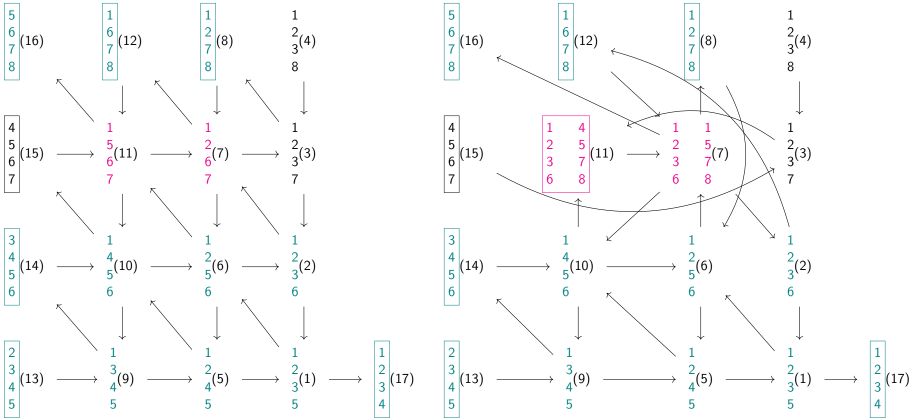


Figure: LHS: The initial seed for the Grassmannian $\mathrm{Gr}_{4;8}$. Mutation at (7), then (11) followed by freezing (11) yields the full subquiver depicted on the RHS on all vertices but (3),(4),(15) that coincides with the initial seed for $\mathcal{F}_{2,4;6}$.

IV. Example: $n = 8$ initial seed s_0^{Gr}

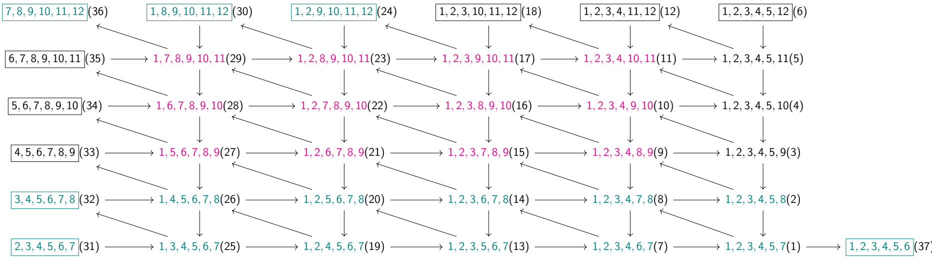


Figure: The initial seed for the Grassmannian $\mathrm{Gr}_{6,12}$. The central 3×4-grid from vertex (9) to (29) marked in magenta is the mutation grid. After the mutation sequence we freeze vertices (29), (16) and (10). Vertices marked in teal are those that already correspond to initial cluster variables for $\mathcal{F}_{2,6;8}$.

IV. Example: $n = 8$ mutated seed s^{Gr}

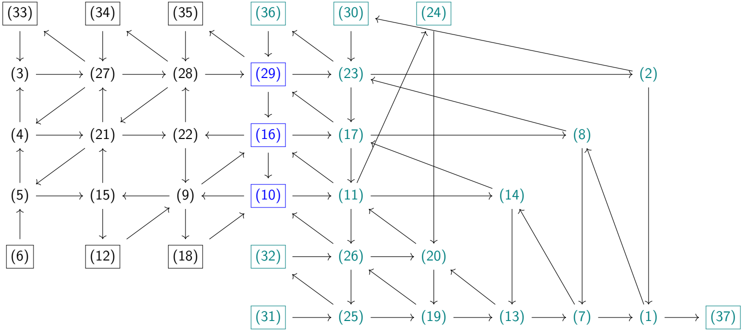
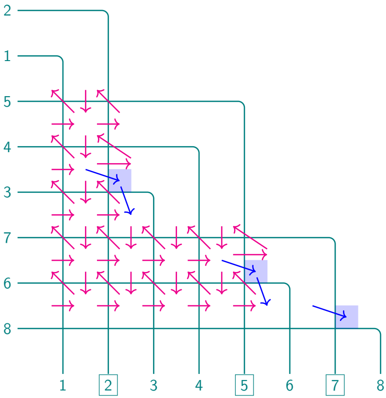


Figure: The quiver Q^{Gr} after freezing vertices 10,16 and 29. All colored vertices on the RHS yield the subquiver corresponding to the initial quiver Q^F for $\mathcal{F}_{2,6;8}$.

What's next?

- $\mathcal{SH}_n$ cluster structure relevant for the symbol alphabet for $n = 5$ [L.B.-Drummond-Glew], $n \geq 6$?
- Embeddings generalize for arbitrary partial flag varieties, conjecturally also compatible with cluster structure
- Application to cluster adjacency conjecture for the momentum amplituhedron
- Scattering with massive particles can be obtained from $\mathcal{MT}_n$ by imposing vanishing certain conditions (in terms of differential operators). What is the kinematic spaces? Does it have cluster structure?

Thank you!



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