Deformations and q-Convolutions. Old and New Results

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Abstract

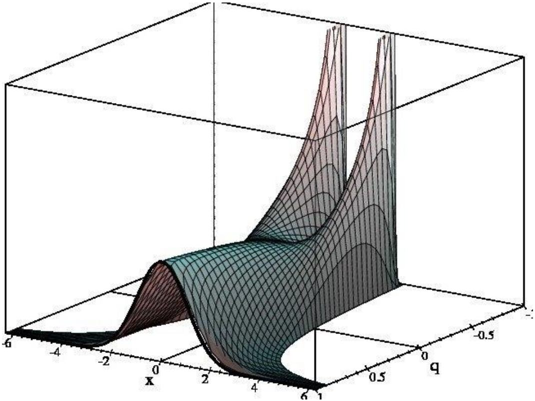


Figure: Density of q-Gaussian measure

Plan of the presentation

1. q-CCR(CAR) relations for $|q| > 1$, and q-continuous Hermite polynomials.
2. Combinatorial results on 2-partitions of $\{1, 2, \ldots, 2n\}$ – $P_2(2n)$.
3. q-discrete Hermite polynomials of type I, II.
4. q-analogue of classical convolutions of Carnovale and Koornwinder for $0 \leq q \leq 1$, ($q = 0$, Boolean convolution, $q = 1$ classical convolution).
5. Braided Hopf algebras of Kempf and Majid.
6. The construction of q-Discrete Fock space $\mathcal{F}_q^{disc}(\mathcal{H})$ and q-Discrete Brownian motions corresponding to q-Discrete Hermite polynomials of type I ($0 \leq q \leq 1$).
7. Matrix version of Khintchine inequalities.

q-CCR(CAR) relations for $|q| > 1$, and q-continuous Hermite polynomials

Continuous q-Hermite are defined as:

$$xH_n^q(x) = H_{n+1}^q(x) + \frac{q^n - 1}{q-1} H_{n-1}^q(x), H_0 = 1, H_1 = x$$

$$[n]_q!\delta_{n,m} = \int_{-\frac{!}{\sqrt{1-q}}}^{\frac{!}{\sqrt{1-q}}} H_n^{(q)}(x) H_m^{(q)}(x) d\mu_q^c(x),$$

where

$$d\mu_q^c(x) = \frac{1}{2\pi} q^{-\frac{1}{8}} \theta_1(\frac{\theta}{\pi}, \frac{1}{2\pi i} \log q) dx =$$

$$= \frac{1}{\pi}\sqrt{1-q}\sin(\theta) \prod_{n=1}^{\infty} (1-q^n)|1-q^n \exp(2\pi\theta)|^2 dx$$

for $0 \leq q < 1$, θ_1 – Jacobi theta one function.
$2\cos v = x\sqrt{1-q}, \quad \mathrm{supp}\, \mu_q^c = [-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]$

q-CCR(CAR) relations for $|q| > 1$, and q-continuous Hermite polynomials

Theorem (Bożejko+Yoshida)

If $-1 \leq q \leq 1$, $s > 0$, then there exist operators $A^{\pm}(f) = A^{\pm}_{q,s}(f)$, $g, f \in \mathbb{R}^N$, $N = \infty, 1, 2, \ldots$:

$$A(f)A^{+}(g) - (sq)A^{+}(g)A(f) = s^N < f, g > I.$$

$$A(f)\Omega = 0.$$

$\mathcal{F}_q(\mathcal{H}) = \oplus_{n=0}^{\infty}\mathcal{H}^n$ with scalar product

$$< f_1 \otimes \ldots \otimes f_n | g_1 \otimes \ldots \otimes g_n >_q = < P_q^{(n)}(f_1 \otimes \ldots \otimes f_n) | g_1 \otimes \ldots \otimes g_n > .$$

and $P_q^{(n)} = \sum_{\sigma \in S(n)} q^{inv(\sigma)}\sigma$.

q-CCR(CAR) relations for $|q| > 1$, and q-continuous Hermite polynomials

Construction

Take q-CCR operators: $a_q^{\pm}(f) = a(f)$.

$$a(f)a^+(g) - qa^+(g)a(f) =< f, g > I,$$

on $\mathcal{F}_q(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, *as in* [Bożejko+Speicher] *and put*

$$A_{q,s}(f) = s^{N-1} a_q(f), \ s > 0.$$

where N on $\mathcal{H}^{\otimes n}$ *is defined as:*

$$N(x_1 \otimes \ldots \otimes x_n) = n(x_1 \otimes \ldots \otimes x_n)$$

Combinatorial results on 2-partitions of $\{1, 2, \ldots, 2n\}$ – $P_2(2n)$.

Definition (q-conditions cummulants - Ph.Biane, M.Anshelevich)

If μ – probability measure on $\mathbb{R}$ with all moments, then the q-continuous cummulants are defined as follows:

$$\mu \to \left(R_\mu^{(q)}(n)\right)_{n=1}^{\infty}$$

in such a way that:

$$\int_{-\infty}^{\infty} x^n d\mu(x) = \sum_{\mathcal{V} \in P(n)} q^{cr(V)} R_\mu^{(q)}(\mathcal{V}), \qquad (1)$$

where $P(n)$ is the set of all set-partitions on $\{1, 2, \ldots, n\}$, and

Combinatorial results on 2-partitions

Definition

$$R_\mu^{(q)}(\mathcal{V}) = \prod_{B \in \mathcal{V}} R_\mu^{(q)}(|B|),$$

where $\mathcal{V}$ – partition of $\{1, 2, \ldots, n\}$, and $cr(\mathcal{V})$ is a number of of hyperbolic (restricted) crossings defined by Ph. Biane.

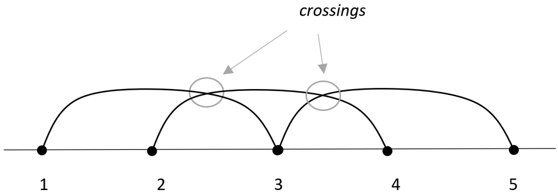


Figure: Crossings

Combinatorial results on 2-partitions

Remark

A.Nica defined left-reduced number of crossing $c_0(\mathcal{V})$ *as:*
$c_0(\mathcal{V}) = \#\{(m_1, m_2, m_3, m_4) : 1 \leq m_1 \leq m_2 \leq m_3 \leq m_4 \leq n :$
$(m_1, m_3) \in \mathcal{V}, (m_2, m_4) \in \mathcal{V}, (m_2, m_3) \notin$
$\mathcal{V},\ \text{each } m_1, m_2 \text{ minimal in the class of } \mathcal{V} \text{ containing it}\}$, *then Nica's q-cummulants* $\widetilde{R}_\mu^{(q)}(n)$ *come from (1), where* $cr(\mathcal{V})$ *is replaced by* $c_0(\mathcal{V})$*.*

If we define a „q-convolututon": $\mu = \mu_1 *_q \mu_2$: (Ph. Biane idea) is done as:

$$R_{\mu_1}^{(q)}(n) + R_{\mu_2}^{(q)}(n) = R_\mu^{(q)}(n), \quad n = 1, 2, 3, \ldots, \tag{2}$$

then we have the following open problem:

Combinatorial results on 2-partitions

Problem (open)

Is Ph. Biane „$*_q$-convolutions" positivity preserving?

Now, we are describing the new q-convolution corresponding to q-Discrete Hermite polynomials of the type I. We give also Wick formula for that case.

Combinatorial results on 2-partitions

Theorem (Bożejko–Yoshida (Wick formula))

If $G(f) = A(f) + A^{+}(f)$*, then*

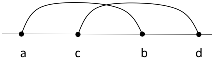
$$< G(f_1)\dots G(f_{2n})\Omega|\Omega >= \sum_{\mathcal{V}\in P_2(2n)} s^{\frac{1}{2}ip(\mathcal{V}} \cdot q^{cr(\mathcal{V})} \prod_{(i,j)\in\mathcal{V}} < f_i|f_j >$$

where $ip(\mathcal{V}) = \sum_{(i,j)\in\mathcal{V}} inpt(i,j)$*,* $inpt(i,j) = \#\{$*of* k *with* $i < k < j\}$

$$= \sum_{k=1}^{n}(j_k - i_k - 1), \;\; \text{if } \mathcal{V} = \{(i_1, j_1), \dots, (i_n, j_n)\} \in P_2(2n).$$

Combinatorial results on 2-partitions

We are recalling the crossing number definition for 2-partitions $\mathcal{V}$:

$$cr(\mathcal{V}) = \#\{(a,b),(c,d) \in \mathcal{V} : a < c < b < d,$$ 

$$\}.$$

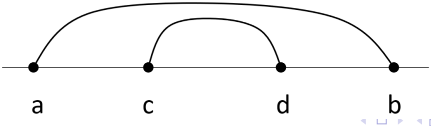
Theorem (Bozejko)

If $\mathcal{V} \in P_2(2n)$, then

$$cr(\mathcal{V}) + pbr(\mathcal{V}) = \frac{1}{2} ip(\mathcal{V})$$

where

$pbr(\mathcal{V}) = \#\{(a,b),(c,d) \in \mathcal{V} : a < c < d < b\} = nest(\mathcal{V}).$



Combinatorial results on 2-partitions

Theorem (Bozejko)

If $q \geq 1$, then there exist operators on a proper Fock space satisfying the (q-CCR):

$$B(f)B^+(g) - B^+(g)B(f) = q^N < f, g > I, \quad f, g, \in \mathcal{H} \text{ (Hilbert space)},$$

where $N(x_1 \otimes \ldots \otimes x_n) = n(x_1 \otimes \ldots \otimes x_n)$.
Moreover $\widetilde{G}(f) = B(f) + B^+(f)$:

$$< \widetilde{G}(f_1) \ldots \widetilde{G}(f_{2n})\Omega | \Omega >= \sum_{\mathcal{V} \in P_2(2n)} q^{pbr(\mathcal{V})} \prod_{(i,j) \in \mathcal{V} < f_i | f_j >} .$$

Proof's idea: Consider $A_{1/q,q}(f)$, $q \to 1/q$, $s = q$, where $B^{\pm}(f) = A^{\pm}_{1/q,q}(f)$, $f \in \mathcal{H}$ were constructed previously.

We recall the definition of q-Discrete Hermite polynomial of type I and type II for $0 \leq q \leq 1$ as:

I type: $h_0 = 1$, $h_1(x) = x$,
$xh_n(x) = h_{n+1}(x) + q^{n-1}[n]_q h_{n-1}(x)$,
$[n]_q = \frac{q^n-1}{q-1} = 1 + q + \ldots + q^{n-1}$.
Later we will denote $h_n(x; q) = h_n(x)$.

II type: $\widetilde{h}_n(x; q) = i^{-n} h_n(ix; q^{-1})$, where $i = \sqrt{-1}$.

q-discrete Hermite polynomials of type I, II $(0 < q < 1)$

Now we recall the definition of two exponential functions

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{(q:q)_k}, \quad E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{n}{2}} x^k}{(q:q)_k}$$

where $(a:q)_k = (1-a)(1-aq)\dots(1-aq^{k-1})$.

$$[k]_q! = \frac{(q:q)_k}{(1-q)^k}, \quad \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[j]_q![n-j]_q!} \text{ (Gauss symbol).}$$

Facts (see Andrews et al.

1. $E_q(z) = \prod_{n=0}^{\infty}(1 + q^n z)$, $z \in \mathbb{C}$,
2. $e_q(z)E_q(-z) = 1$, $z \in \mathbb{C}$,
3. (I type) $\int_{-1}^{1} h_m(x : q)h_n(x : q)E_{q^2}(-q^2x^2)d_qx = b_q \cdot q^{\binom{n}{2}}(q : q)_n\delta_{n,m}$,
4. (II type) $\int_{-\infty}^{\infty} \widetilde{h}_m(x : q)\widetilde{h}_n(x : q)e_{q^2}(-x^2)d_qx = c_q \cdot q^{-n^2}(q : q)_n\delta_{n,m}$, where

$$\int_0^x f(x)d_q(x) = (1 - q)\sum_{k=0}^{\infty} f(q^k x)q^k x$$

is well known Jackson integral for functions with support $supp(f) \subset \mathbb{R}^+$, and for arbitrary $f : \mathbb{R} \to \mathbb{C}$ we define

$$\int_{-\infty}^{\infty} f(x)d_q(x) = (1-q)\sum_{k=-\infty}^{\infty}\sum_{\varepsilon=\pm 1} q^k f(\varepsilon q^k); \quad supp(f) \subset \mathbb{R}.$$

Commutation relations in the Fock representation of type I discrete Hermite polynomials.

In Theoreom 1 put $s = q$, $q = q$, $0 \leq 1 \leq 1$, then operators

$$A_q^{\pm}(f) = A_{q,q}^{\pm}(f), \quad \widehat{G}(f) = A(f) + A^{+}(f).$$

appears in the following theorem:

Theorem

1. *If* $||f_i|| = 1$, $i = 1, 2, \ldots, 2n$, *then*

$$< \widehat{G}(f_1) \ldots \widehat{G}(f_{2n})\Omega|\Omega >= \sum_{\mathcal{V} \in P_2(2n)} q^{\frac{1}{2} ip(\mathcal{V}) + cr(\mathcal{V})} \prod_{(i,j) \in \mathcal{V}} < f_i | f_j >$$

Theorem

2

$$\int x^{2n} d\mu_q^I(x) = < \widehat{G}(f_1)^{2n}\Omega|\Omega > = [1]_q[3]_q \dots [2n-1]_q =$$

$$= \sum_{\mathcal{V} \in P_2(2n)} q^{e_0}(\mathcal{V}),$$

where $e_0(\mathcal{V})$ *was introduced by de Médicis+Viennot, where*

$$e_0(\mathcal{V}) = pbr(\mathcal{V}) + 2cr(\mathcal{V}) = \frac{1}{2} ip(\mathcal{V}) + cr(\mathcal{V}).$$

Theorem

3. *Moreover*

$$A_q(f)A_q^+(g) - q^2 A_q^+(g)A_q(f) = q^N < f, g > I.$$

for $f, g \in \mathcal{H}$.

Problems:

1. Prove positivity of q-Discrete (continuous) convolutions for $0 < q < 1$?
2. Describe q-Discrete Poisson measure (process)?
3. Calculate the operator norm of $||\widehat{G}(f_i)|| =?$, $i = 1, 2, \ldots$
4. If Ω is faithful state in the corresponding Fock space?

q-analogue of classical convolutions of Carnovale and Koornwinder for $0 \leq q \leq 1$, ($q = 0$, Boolean convolution, $q = 1$ classical convolution)

Let us define Jackson „q-moments" for „good" function $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$m_n^{disc}(f) = q^{\binom{n}{2}} \int_{-\infty}^{\infty} f(x) x^n d_q(x),$$

and „q-Discrete convolutions" of Carnowale and Koornwinder

$$(f \otimes_q g)(x) = \sum_{n=0}^{\infty} \frac{(-1)^n m_n^{disc}(f)}{[n]_q!} (\delta_q^n g)(x)$$

where

$$\delta_q f(x) = \begin{cases} \frac{f(x)-f(qx)}{x-qx}, & x \neq 0, \lim_{q \to 1} \delta_q f(x) = f'(x), \\ f'(0), & x = 0. \end{cases}$$

q-analogue of classical convolutions

Note that if $q = 1$, we have

$$\left(\int_{-\infty}^{\infty} dtf(t)\frac{(-1)^n t^n}{n!}\right) g^{(n)}(x) = \int_{-\infty}^{\infty} dtf(t)\left(\sum_{n=0}^{\infty}\frac{(-t)^n}{n!} g^{(n)}(x)\right) =$$

$$= \int_{-\infty}^{\infty} dtf(t)\, g(x-t) = (f * g)(x).$$

which is the classical convolution.

q-analogue of classical convolutions

Theorem (Carnovale+Koornwinder)

*For „good” functions $f, g : \mathbb{R} \to \mathbb{R}$ q-Discrete convolution is **associative** and **commutative**. Moreover*

$$m_n^{disc}(f \otimes_q g) = \sum_{n=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q m_k^{disc}(f)\; m_{n-k}^{disc}(g).$$

If $q = 0$, we get Boolean convolution.
If $q = 1$, we get classical convolution on $\mathbb{R}$.

Problem: Find characterization q-Discrete moments sequence $m_n^{disc}(f)$, i.e. for $f \geq 0$

$$m_n^{disc}(f) = \int_{-\infty}^{\infty} f(x) x^n d_q(x)?$$

Braided Hopf algebras of Kempf and Majid

Definition

Braided line is a braided algebra $\mathcal{A} = \mathbb{C}[[x]]$ formal power series in variable x which has braiding

$$\Phi(x^k \otimes x^l) = q^{kl} x^l \otimes x^k,$$

commultiplation: $\Delta(x^k) = \sum_{j=0}^{k} \left[{k \atop j} \right]_q x^{k-j} \otimes x^j$

co-unit $\varepsilon(x^k) = \delta_{k,0}$

braided antipode

$$S(x^k) = (-1)^k q^{\binom{k}{2}} x^k = (-1)^k q^{\frac{k(k-1)}{2}} x^k,$$

and then we get the q-analogue of of Taylor's formula:

$$\Delta(f(x)) = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!} \otimes \delta_q(f(x)).$$

Braided Hopf algebras of Kempf and Majid

Theorem (Kemp+Majid)

If $Qf(x) = f(qx)$, then we have

$$(f *_q g)(x) = (f \otimes id)(m \otimes id)[id \otimes Q \otimes id](id \otimes S \otimes id)(id \otimes \Delta)(f \otimes g)(x)$$

Moreover as observed by Koornwinder we have

$$\Delta(e_q(x)) = e_q(x) \otimes e_q(x),$$

$$S(e_q(x)) = E_q(-x),$$

$$\varepsilon(e_q(x)) = 1.$$

The construction of q-Discrete Fock space $\mathcal{F}_q^{disc}(\mathcal{H})$ and q-Discrete Brownian motions corresponding to q-Discrete Hermite polynomials of type I $(0 \leq q \leq 1)$

Now we present for $0 \leq q \leq 1$ the construction of q-Discrete Fock space $\mathcal{F}_q^{disc}(\mathcal{H})$ for q-Discrete Hermite of Type I, which is the completion of the full Fock space $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\oplus n} = \mathbb{C}\Omega \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \dots$ under the positive inner product on $\mathcal{H}^{\oplus n}$ done by:

$$< x_1 \otimes \dots \otimes x_n | y_1 \otimes \dots \otimes y_m >_q =$$

$$= \delta_{n,m} q^{\binom{n}{2}} \sum_{\pi \in S(n)} q^{inv(\pi)} < x_1 | y_{\pi(1)} > \cdot \dots \cdot < x_n | y_{\pi(n)} > .$$

We define creation operator $A_q^+(f)\xi_n = f \otimes \xi_n$, $f \in \mathcal{H}$, $\xi_n \in \mathcal{H}^{\otimes n}$ and the annihilation operator

$$A_q(f) x_1 \otimes \dots \otimes = q^{n-1} \sum_{k=1}^{n} q^{k-1} < x_k | f > x_1 \otimes \dots \otimes \check{x}_k \otimes \dots \otimes x_n$$

The construction of q-Discrete Fock space...

In the paper [B-Y] we have more general construction

$$A_{q,s}(f)x_1 \otimes \ldots \otimes = s^{2(n-1)} \sum_{k=1}^{n} q^{k-1} < x_k | f > x_1 \otimes \ldots \otimes \check{x}_k \otimes \ldots \otimes x_n$$

If we put $s^2 = q$ we get our q-Discrete Fock space $\mathcal{F}_q^{disc}(\mathcal{H})$.

Remark

For $f, g \in \mathcal{H}$ we have the following q-Discrete Commutation Relation:

$$A(f)A^+(g) - q^2 A^+(g)A(f) = q^N < f, g > I.$$

The construction of q-Discrete Fock space...

We recall q-Discrete Gaussian random variables $\widehat{G}_q(f) = A_q(f) + A_q^+(f)$. We get q-version of Wick formula

$$< \widehat{G}_q(f_1) \ldots \widehat{G}_q(f_{2n})\Omega|\Omega >= \sum_{\mathcal{V} \in P_2(2n)} q^{\frac{1}{2} ip(\mathcal{V})} \cdot q^{cr(\mathcal{V})} \prod_{(i,j) \in \mathcal{V}} < f_i | f_j > .$$

Our Gaussian $\widehat{G}_q(f)$ at the vacuum state Ω has the spectral measure μ_q^{disc} corresponding to q-Discrete Hermite polynomials of type I as it was defined as

$$xh_n(x) = h_{n+1}(x) + q^{n-1}[n]_q h_{n-1}(x), \;\; [n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \ldots + q^{n-1}$$

The construction of q-Discrete Fock space...

We recall q-Discrete Gaussian random variables
Now we define q-Discrete Brownian motion BM_t as follows.
Take $\mathcal{H} = L^2(\mathbb{R}^+, dx)$ and $f = \chi_{[0,t)}$,

$$f(x) = \begin{cases} 1 & \text{for } x \in [0, t), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$BM_t = \widehat{G}_q(\chi_{[0,t)})$$

is our q-Discrete Brownian motion.

Remark

Case $q = 1$ is the classical Brownian motion, and $q = 0$ is the Boolean Brownian motion.

Problem: Is the von Neumann algebra $BB_q =$ WO-closure of $\{BM_t : t \geq 0\}$ is factorial, that means it has no center?
This BB_q algebra corresponds to q-Discrete Hermite polynomials of type I, but the corresponding problem for continuous q-Hermite polynomials was solved by Bozejko, Kuemmerer and Speicher.

Matrix version of Khintchine inequalities

We are looking for matricial version Khintchine inequalities for random variables $X_1, X_2, \ldots, X_n$,

$$\left\| \sum_{j=1}^{n} a_j \otimes X_j \right\| \cong \max \left\{ \left\| \sum_{j=1}^{n} a_j a_j^* \right\|^{\frac{1}{2}}, \left\| \sum_{j=1}^{n} a_j^* a_j \right\|^{\frac{1}{2}} \right\} \qquad (3)$$

for $n = 1, 2, \ldots$ and a_j are complex matrices of arbitrary sizes and the norms are operator norms.

Theorem

Inequality (3) holds for q-continuous, q-discrete Gaussian, Kesten Gaussian and many others examples.

Matrix version of Khintchine inequalities

Corollary

If $VN(X_1, X_2, \ldots, X_n)$ has trace, then for some $q = q(N)$, $VN(X_1, X_2, \ldots, X_n)$ is NOT injective and also it is a FACTOR.