Eilenberg-Schützenberger machines I: States, Σ-modules and applications.

Stars of the Plane.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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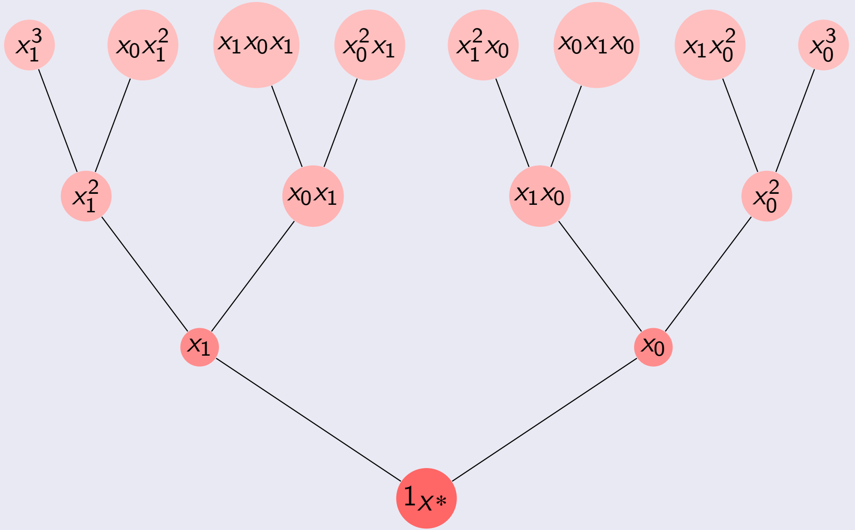
Introduction

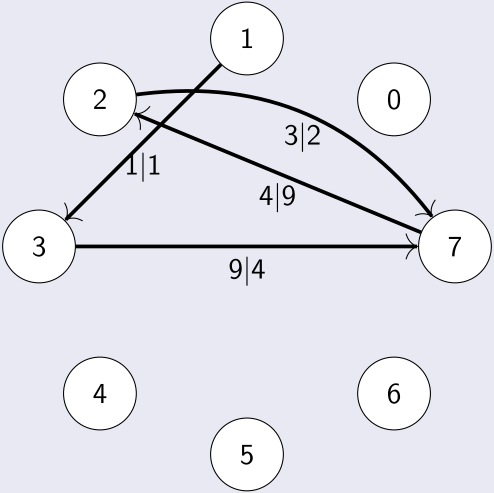
1. The story of automata theory (in the large, i.e. Eilenberg-Schützenberger machines) is all about states, ations (command letters), alphabets, transitions and multiplicities (outputs).
2. In this review, we will see several sets of states
    1. (Free) monoid on the alphabet $X = \{x_0, x_1\}$
    2. Numeral symbols on base b (i.e. $X = \mathbf{b} = \{0, \cdots, b-1\}$)
    3. (If times permits), the free group (on X)

The free monoid $\{x_0, x_1\}^*$.

▸▸ skip slide PNCDE



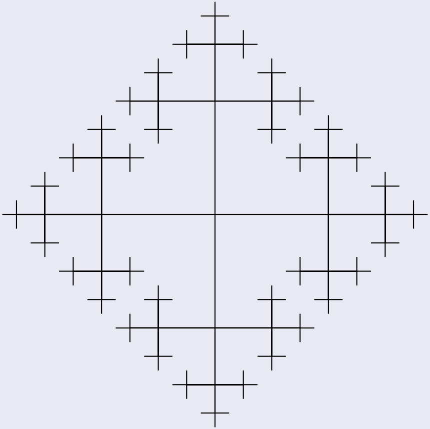
Numeral symbols $\mathbf{8} = \{0, \cdots, 7\}$



$$\begin{array}{l|l} \overline{1\ 1}\ 9\ 4\ 3 & 8 \\ \hline \quad 3\ 9 & 1\ 4\ 9\ 2 \\ \quad\quad 7\ 4 & \\ \quad\quad\quad 2\ 3 & \\ \quad\quad\quad\quad 7 & \end{array}$$

$$\begin{array}{r|l} 11 & 8 \\ \hline 3 & 1 \end{array} \quad \begin{array}{r|l} 39 & 8 \\ \hline 7 & 4 \end{array} \quad \begin{array}{r|l} 74 & 8 \\ \hline 2 & 9 \end{array} \quad \begin{array}{r|l} 23 & 8 \\ \hline 7 & 2 \end{array}$$

Free Group, here $\Gamma(a, b)$.



Factorizations

Last year (CAP10), H. Nakamura began his talk by some stringology i.e. the fact that any string (word) on the alphabet $\Sigma = \{X, Y\}$ could be written

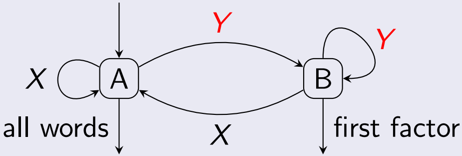
$$w = X^{h_1} Y X^{h_2} Y \cdots Y X^{h_d} Y \,|\, X^{h_\infty} \ . \qquad (1)$$

Doing this, save the last factor X^{h_∞}, we obtain a factorization into blocs of the form $X^h Y$. We will later write this set $X^* Y = Y + XY + X^2 Y + \ldots$, the (free) monoid they generate $(X^* Y)^* = 1 + (X^* Y)^+$. The set of all words, therefore, is

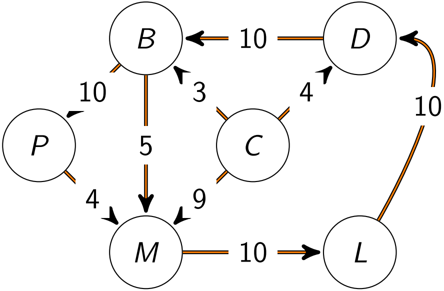
$$(X + Y)^* = (X^* Y)^* X^* = (X^* Y)^+ X^* + X^*, \qquad (2)$$

an instance of Lazard elimination theorem (discussed last year).

Factorization (1) can be computed by the following (boolean or $\mathbb{N}$-) automaton



A simple transition system: flow charts or flow diagrams



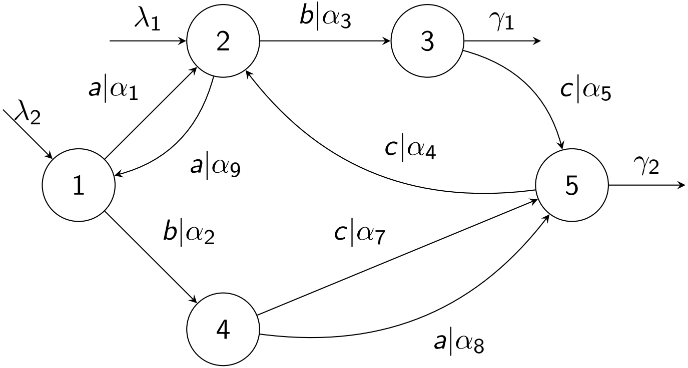
Directed graph weighted by numbers which can be lengths, time (durations), costs, fuel consumption, probabilities. This graph is equivalent to a square matrix. Coefficients are taken in different semirings (i.e. rings without the "minus" operation, as tropical or [min,+]) according to the type of computations to be done. Tropical semirings were so called by MPS school because they were founded by the Hungarian-born Brazilian mathematician and computer scientist Imre Simon. Evaluation is done by multiplications in series and addition in parallel.

Weighted (or multiplicity) automata: the forefathers

Samuel Eilenberg, *Automata, Languages, and Machines (Vol. A & B)* Acad. Press, New York, (1974)

Marcel-Paul Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*

Multiplicity Automaton (Eilenberg, Schützenberger)



Example: Evaluate 2.*bccabc*.

Multiplicity automaton (linear representation) & behaviour

Linear representation

Due to the left-to-right word reading, it is

$$\lambda = \begin{pmatrix}\lambda_2 & \lambda_1 & 0 & 0 & 0\end{pmatrix}, \qquad \gamma = \begin{pmatrix}0 & 0 & \gamma_1 & 0 & \gamma_2\end{pmatrix}^T$$

$$\mu(a) = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 0 \\ \alpha_9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mu(b) = \begin{pmatrix} 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

$$\mu(c) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_5 \\ 0 & 0 & 0 & 0 & \alpha_7 \\ 0 & \alpha_4 & 0 & 0 & 0 \end{pmatrix}$$

Multiplicities.

1. Multiplicities are taken within a semiring R. Each time you change R, you change your universe.
2. If $R = \mathbb{B}$, you get the theory of languages, if $R = \mathbb{N}$, you are able to count the paths for example.
3. If R is commutative, you have the theory of rational series and if R is a field, you get a way to compute within Sweedler's duals.
4. If the multiplicities are probabilities, you get stochastic automata.
5. But R does not need to be commutative
   1. If $R = \mathbf{k}\langle\Gamma\rangle$ for some alphabet Γ, you get transducers
   2. R can be a semiring of operators, this opens the door to application of rational identities to the plane of transition matrices.

Linear representation & Behaviour

Remark

For a right-to-left word reading, data have to be transposed.

Non commutative series

Series are functions $X^* \to R$ where R is a semiring (i.e. a ring without the "minus" operation as example the tropical semiring). We have different ways to consider a series, namely:

Math: Functions, elements of a dual (total, restricted, Sweedler's &c.)
Computer Sci.: Behaviour of a system (automaton, transducer, grammar &c.)
Physics: Non comm. diff. equations, evaluation of paths, normal orderings &c.

Behaviour of a "word machine", the series $\mathcal{B}(\mathcal{M})$.

$$\langle \mathcal{B}(\mathcal{M})|w\rangle = \lambda\,\mu(w)\,\gamma = \sum_{\substack{i,j\\ \text{states}}} \lambda(i) \underbrace{\left(\sum weight(p)\right)}_{\substack{\text{weight of all paths } (i) \to (j)\\ \text{with label } w}} \gamma(j) \tag{4}$$

Operations and definitions on series (R semiring).

Addition, Scaling: As for functions because $R\langle\langle X\rangle\rangle = R^{X^*}$ (viewed as R-R modules)
Concatenation: $f.g(w) = \sum_{w=uv} f(u)g(v)$
Polynomials: Series s.t. $supp(f) = \{w\}_{f(w)\neq 0}$ is finite.
The set of polynomials will be denoted $R\langle X\rangle$.
Pairing: $\langle S|P\rangle = \sum_{w\in X*} S(w)P(w)$ (S series, P polynomial)
Summation: $\sum_{i\in I} S_i$ summable iff f or all $w\in X^*$, $i\mapsto\langle S_i|w\rangle$ is finitely supported. In particular, we have

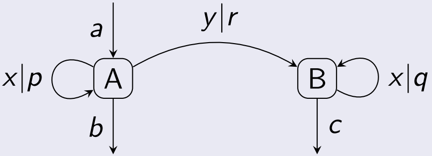
$$\sum_{i\in I} S_i := \sum_{w\in X*} \Big(\sum_{i\in I}\langle S_i|w\rangle\Big)\, w$$

Remark: This notion is exactly the one of limit of the net of partial sums $(\sum_{i\in F} S_i)_{F\subset_{finite} I}$ with respect to the sup-lattice of finite subsets of I, topology being the product of discrete topologies on R (see [12] "summable").

Operations and definitions on series (R semiring)/2

Star: For all series S s.t. $\langle S | 1_{X^*} \rangle = 0$, the family $(S^n)_{n \geqslant 0}$ is summable and we set $S^* := \sum_{n \geqslant 0} S^n = 1 + S + S^2 + \cdots$ (if R is a ring, we have $S^* = (1-S)^{-1}$) and the **plus-notation** $S^+ := \sum_{n \geqslant 0} S^n = S + S^2 + \cdots$ (again, if R is a ring we have $S^+ = S.(1-S)^{-1} = (1-S)^{-1}.S$).
Shifts: $\langle u^{-1} S | w \rangle = \langle S | uw \rangle$ and $\langle S u^{-1} | w \rangle = \langle S | wu \rangle$.

Let $\mathcal{M}$ be the automaton (p, q, r, a, b, c can be operators).



$$I = \begin{pmatrix} a & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} p.x & r.y \\ 0 & q.x \end{pmatrix} \quad F = \begin{pmatrix} b \\ c \end{pmatrix}$$

$$T^* = \begin{pmatrix} (p.x)^* & (p.x)^*.r.y.(q.x)^* \\ 0 & (q.x)^* \end{pmatrix}$$

$$\mathcal{B}(\mathcal{M}) = I.T^*.F = a.(p.x)^*.b + a.(p.x)^*.r.y.(q.x)^*.b$$

Rational series (Sweedler's duals & Schützenberger's shifts)

skip slide

Theorem A ($\mathbf{k}$ field, X finite).

Let $S \in \mathbf{k}\langle\langle X\rangle\rangle$ TFAE
i) The family $(Su^{-1})_{u\in X^*}$ is of finite rank.
ii) The family $(u^{-1}S)_{u\in X^*}$ is of finite rank.
iii) The family $(u^{-1}Sv^{-1})_{u,v\in X^*}$ is of finite rank.
iv) It exists $n \in \mathbb{N}$, $\lambda \in \mathbf{k}^{1\times n}$, $\mu : X^* \to \mathbf{k}^{n\times n}$ (a multiplicative morphism) and $\gamma \in \mathbf{k}^{n\times 1}$ such that, for all $w \in X^*$

$$(S, w) = \lambda\mu(w)\gamma \tag{5}$$

v) The series S is in the closure of $\mathbf{k}\langle X\rangle$ for $(+, conc, ^*)$ within $\mathbf{k}\langle\langle X\rangle\rangle$.

Definition

A series which fulfills one of the conditions of Theorem A will be called *rational*. The set of these series will be denoted by $k^{rat}\langle\langle X\rangle\rangle$. In the theory of Hopf algebras it is Sweedler's dual of $\mathbf{k}\langle X\rangle$.

Sweedler's duals & Kleene-Schützenberger's Theorem.

Remarks

1. (i $\leftrightarrow$ iii) needs $\mathbf{k}$ to be a field.
2. (iv) needs X to be finite, (iv $\leftrightarrow$ v) is known as the theorem of Kleene-Schützenberger (M.P. Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*, 245-270.)
3. For the sake of Combinatorial Physics (where the alphabets can be infinite), **(iv)** has been extended to infinite alphabets and replaced by **iv')** The series S is in the rational closure of $\mathbf{k}^X$ (linear series) within $\mathbf{k}\langle\langle X\rangle\rangle$.
4. When $\mathbf{k}$ is a ring, the rational closure of a subset $P \subset \mathbf{k}\langle\langle X\rangle\rangle$ is exactly the inverse-closed subalgebra of $\mathbf{k}\langle\langle X\rangle\rangle$ generated by P.
5. In the vein of (v) expressions like ab^* or identities like $(ab^*)^*a^* = (a+b)^*$ (Lazard's elimination) will be called rational.

Sweedler's duals & Kleene-Schüzenberger's Theorem./2

6. For the needs of CS, an analogue of Theorem A has been proved for **k** a commutative semiring (see [15, 11, 13]) where "is of finite rank" is replaced *mutatis mutandis* by "is contained in a shift-invariant submodule of finite type".
7. Contrariwise to the case when **k** is a field, the property of being a submodule of finite type is not hereditary (as soon as we only have a ring). It can then happen that the module generated by the shifts of a rational series be not of finite type. The case $\mathbf{k} = \mathbb{N},\ S = a^*a^* = \sum_{n\geqslant 0}(n+1)a^n$ is typical: when one computes the shifts *on the series* $S = a^*a^* = \sum_{n\geqslant 0} (n+1)a^n$ (considered as a function), we get a shift-invariant module of infinite type whereas, following Eilenberg [9], when we perform them on its rational expression a^*a^*, we get a FS automaton.
8. This theorem is linked to the following subjects: Representative functions on X^* (see Eiichi Abe [1], Chari & Pressley [3]), Sweedler's duals [7] &c.

From theory to practice: Schützenberger's calculus

From series to automata

Starting from a series S, one has a way to construct an automaton (finite-stated iff the series is rational) providing that we know how to compute on shifts and one-letter-shifts will be sufficient due to the formula $u^{-1}v^{-1}S = (vu)^{-1}S$.

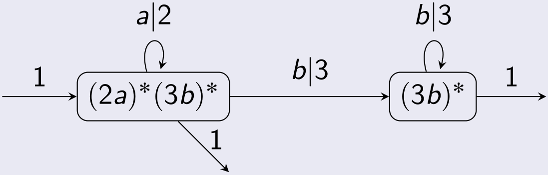
Calculus on rational expressions

In the following, x is a letter, E, F are rational expressions (i.e. expressions built from letters by scalings, concatenations and stars)

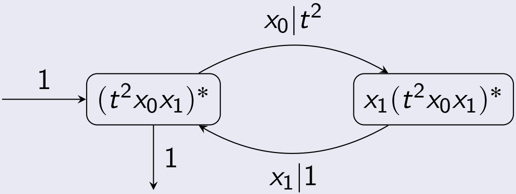
1. x^{-1} is (left and right) linear
2. $x^{-1}(E.F) = x^{-1}(E).F + \langle E|1_{X^*}\rangle x^{-1}(F)$
3. $x^{-1}(E^*) = x^{-1}(E).E^*$

Examples

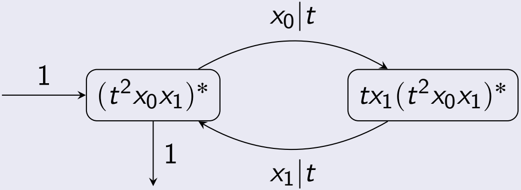
With $(2a)^*(3b)^*$; $X = \{a, b\}$



With $(t^2x_0x_1)^*$; $X = \{x_0, x_1\}$ (by shifts, unbalanced)



With $(t^2x_0x_1)^*$; $X = \{x_0, x_1\}$ (balanced)



From theory to practice: construction starting from S.

- **States** $\boxed{u^{-1}S}$ (constructed step by step)
- **Edges** We shift every state by letters (length) level by level (knowing that $x^{-1}(u^{-1}S) = (ux)^{-1}S$). Two cases:
  Returning state: The state is a linear combination of the already created ones i.e. $x^{-1}(u^{-1}S) = \sum_{v \in F} \alpha(ux, v) v^{-1}S$ (with F finite), then we set the edges

$$\boxed{u^{-1}S} \xrightarrow{x|\alpha_v} \boxed{v^{-1}S}$$

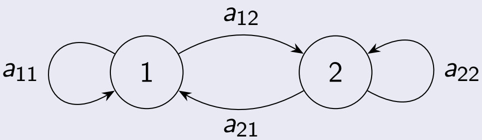
  The created state is new: Then

$$\boxed{u^{-1}S} \xrightarrow{x|1} \boxed{x^{-1}(u^{-1}S)}$$

- **Input** $\boxed{S}$ with the weight 1
- **Outputs** All states $\boxed{T}$ with weight $\langle T | 1_{X^*} \rangle$

Words and paths

Powers of a (generic) transfer matrix



$$T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$T^2 = \begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{21}a_{11} + a_{22}a_{21} & a_{22}^2 + a_{21}a_{12} \end{pmatrix}$$

$$T^n = \begin{pmatrix} \sum n\text{-paths } 1 \to 1 & \sum n\text{-paths } 1 \to 2 \\ \sum n\text{-paths } 2 \to 1 & \sum n\text{-paths } 2 \to 2 \end{pmatrix}$$

Star notation and Mc Naughton-Yamada formulae.

We set $T^+ := \sum_{n\geqslant 1} T^n$, $T^* := 1 + T^+ = 1 + T + T^2 + \cdots = \sum_{n\geqslant 0} T^n$. This matrix T^* is the (unique) solution $R \in \mathbf{k}\langle\langle a_{ij} \rangle\rangle$ of the self-reproducing equations

$$R = I + TR = I + RT$$

Mac Naughton-Yamada (with multiplicities) formulae.

Expressions

$$\text{With } T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ we have } T^* = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \text{ with} \tag{6}$$

$$\begin{aligned} A_{11} &= (a_{11} + a_{12}a_{22}^* a_{21})^* & A_{12} &= A_{11}\color{red}{a_{12}}a_{22}^* \text{ (or } = a_{11}^* a_{12} A_{22}) \\ A_{21} &= A_{22}\color{red}{a_{21}}a_{11}^* \text{ (or } = a_{22}^* a_{21} A_{11}) & A_{22} &= (a_{22} + a_{21}a_{11}^* a_{12})^* \end{aligned} \tag{7}$$

Applications of “word machines”.

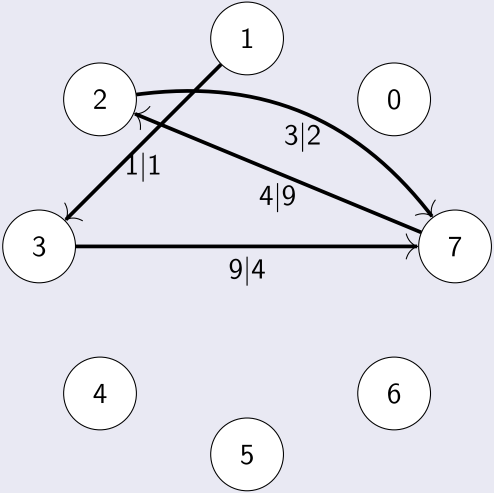
These expressions have many incarnations/applications. Among them

- Sweedler’s duals (and explicit/combinatorial computations within them)
- NCDE and, in particular, Hyper- (and Poly-) logarithms (today)
- Noncommutative geometry
- Geometrization of the Collatz conjecture (today)

Remarks

1. If the multiplicities of slide 13 are taken in some $\Sigma \times \mathbf{k}\langle\Gamma\rangle$ (resp. $\Sigma \times \Gamma$), we have a finite-state (resp. letter-to-letter) transducer.
2. Σ (resp. Γ) is called (and understood as) input (resp. output) alphabet.
3. If, in all loops, multiplicities belong to $\mathbf{k}_+\langle\langle\Gamma\rangle\rangle$ (i.e. series with no constant term), it is always possible to compute the star of the transfer matrix.
4. In a more general way, if multiplicities are taken in an augmented ring $(\mathcal{A}, \epsilon)$ which is complete (i.e. Hausdorff and complete with the topology defined by $\{(\mathcal{A}_+)^n\}_{n\geqslant 0}$) and $a_{ij} \in \mathcal{A}_+$ the generic matrix T possesses a star (computable by formulas Eq. 7). This is the case of many rings of formal series ($\mathbf{k}[[X]]$, $\mathbf{k}[[M]]$).
5. One obtains rational identities by factoring the sets of paths differently (see dual expressions of A_{12}, A_{21} in formulas formulas Eq. 7).

Application 1: Transducer



1 1 9 4 3	8
3 9	1 4 9 2
7 4	
2 3	
7	

With this simple transducer, we see that "states" can mean "cases". Here $\Sigma = \Gamma = \{0, \cdots 9\}$.

Application 2: Difference and differential equations

1. We have seen the shifts which give rise to a calculus on rational expressions, that we recall here
   1. x^{-1} is (left and right) linear
   2. $x^{-1}(E.F) = x^{-1}(E).F + \langle E|1_{X*}\rangle x^{-1}(F)$
   3. $x^{-1}(E^*) = x^{-1}(E).E^*$

   but not only, as transpose of right and left multiplication, they operate on series and can be used to set difference equations.
2. In the same way, we can consider differential equations of the type

$$\mathbf{d}(S) = MS \; ; \; \langle S|1_{X*}\rangle = 1_{\mathcal{A}} \tag{8}$$

   where $\mathbf{d}(S) = \sum_{w\in X*}(\langle S|w\rangle)'.w$ (term by term differentiation) and M, the multiplier, is a series without constant term. The case when $M = \sum_{x\in X} u_x\, x$ (homogeneous of degree one) is of particular interest and is used to better understand iterated integrals.

Construction of a solution: Picard iterations.

1. In the case when $(\mathcal{A}, d)$ admits a section (then $(\mathcal{A}, d, \int)$), one can construct a particular solution of

$$\begin{cases} \mathbf{d}(S) & = & M.S \text{ with } M \in \mathcal{A}_+\langle\langle X\rangle\rangle \\ \langle S|1_{X^*}\rangle & = & 1_{\mathcal{A}} \end{cases} \tag{9}$$

using Picard iterations.

$$S_0 = 1_{X^*} \ ; \ S_{n+1} = 1_{X^*} + \int M.S_n \tag{10}$$

Then, it is not difficult to see that S_n admits a limit S^{Pic} which satisfies (9).
The complete set of solutions of (9) is $S^{Pic}.\mathbb{C}\langle\langle X\rangle\rangle$.

Example of iterated integrals. ⏩ return ⏩ Lie group

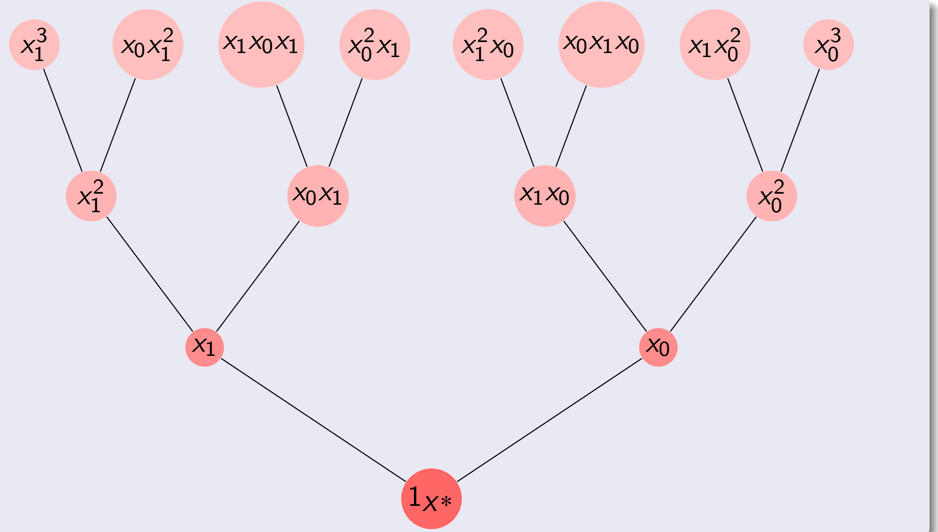
2. For example, let us consider a perturbated version of the polylogarithmic system (here $\Omega = \mathbb{C} \setminus (]-\infty, -1] \cup [1, +\infty[)$ and $S \in \mathcal{H}(\Omega)\langle\langle x_0, x_1 \rangle\rangle$)

$$\begin{cases} \mathbf{d}(S) = \left(\frac{x_0}{z} + \frac{x_1}{1-z} + h(z).[x_0, x_1]\right).S & (NCDE\text{-}Per1) \\ S(z_0) = 1_{X^*} & (Init.\ Cond.) \end{cases} \qquad (11)$$

$S_{z_0}^{Pic}(z)$ satisfies and can be computed by the following recursion

$$\langle S|w\rangle[z] = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_{z_0}^{z} \langle S|u\rangle[s] \frac{ds}{s} & \text{if} \quad w = x_0 u \\ \int_{z_0}^{z} \frac{ds}{1-s} = \log(\frac{1-z_0}{1-z}) & \text{if} \quad w = x_1 \\ \langle S|x_0 x_1 u\rangle[z] + \int_{z_0}^{z} \langle S|u\rangle[s].h(s)\, ds & \text{if} \quad w = x_1 x_0 u \\ \int_{z_0}^{z} \langle S|x_1 u\rangle[s] \frac{ds}{1-s} & \text{if} \quad w = x_1 x_1 u \end{cases}$$

Computation by levels and from left to right.



(Very) quick review of Polylogarithms.

3. Here we consider $\Omega = \mathbb{C} \setminus (]-\infty,-1] \cup [1,+\infty[)$
4. Classical polylogarithms are defined, for $k \geqslant 1, |z| < 1$, by

$$-\log(1-z) = \mathrm{Li}_1 = \sum_{n\geqslant 1} \frac{z^n}{n^1};\ \mathrm{Li}_2 = \sum_{n\geqslant 1} \frac{z^n}{n^2};\ \ldots;\ \mathrm{Li}_k(z) := \sum_{n\geqslant 1} \frac{z^n}{n^k}$$

5. Multiple polylogarithms extend classical ones twofold, they are indexed by words (i.e. lists) and satisfy the following system

$$\begin{cases} \mathbf{d}(S) = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right).S & (NCDE) \\ \lim_{\substack{z\to 0 \\ z\in\Omega}} S(z)e^{-x_0 log(z)} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle} & (Asympt.\ Init.\ Cond.) \end{cases} \qquad (12)$$

from the general theory (differential Galois group of NCDE + Lazard elimination), this system has a unique solution over Ω which is precisely Li (called G_1 in [5]).

Explicit construction of Li.

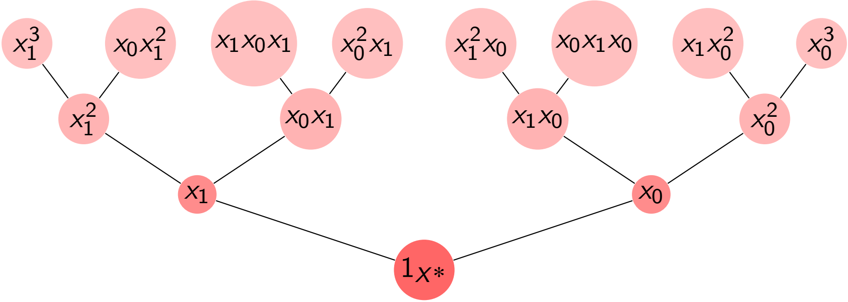
Given a word w, we note $|w|_{x_1}$ the number of occurrences of x_1 within w

$$\langle \mathrm{Li} \,|w\rangle[z] = \begin{cases} 1_\Omega & \text{if} \quad w = 1_{X^*} \\ \int_0^z \langle \mathrm{Li} \,|u\rangle[s] \frac{ds}{1-s} & \text{if} \quad w = x_1 u \\ \int_1^z \langle \mathrm{Li} \,|u\rangle[s] \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} = 0 \\ \int_0^z \langle \mathrm{Li} \,|u\rangle[s] \frac{ds}{s} & \text{if} \quad w = x_0 u \text{ and } |u|_{x_1} > 0 \end{cases}$$

The third line of this recursion implies

$$\alpha_0^z(x_0^n) = \frac{log(z)^n}{n!}$$

one can check that (a) all the integrals (improper for the fourth line) are well defined and (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)\, w$ is Li (G_1 in [1]).



Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $t_0 = 0$

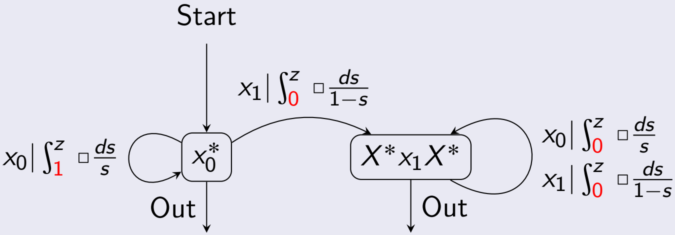
$$\langle S|x_1^n\rangle = \frac{(-log(1-z))^n}{n!} \quad ; \quad \langle S|x_0x_1\rangle = \underbrace{\mathrm{Li}_2(z)}_{cl.not.} = \mathrm{Li}_{x_0x_1}(z) = \sum_{n\geqslant 1}\frac{z^n}{n^2}$$

$$\langle S|x_0^2x_1\rangle = \underbrace{\mathrm{Li}_3(z)}_{cl.not.} = \mathrm{Li}_{x_0^2x_1}(z) = \sum_{n\geqslant 1}\frac{z^n}{n^3} \quad ; \quad \langle S|x_1x_0x_1\rangle = \mathrm{Li}_{x_1x_0x_1}(z) = \mathrm{Li}_{[1,2]}(z) = \sum_{n_1>n_2\geqslant 1}\frac{z^{n_1}}{n_1n_2^2}$$

$$\langle S|x_0x_1^2\rangle = \mathrm{Li}_{x_0x_1^2}(z) = \mathrm{Li}_{[2,1]}(z) = \sum_{n_1>n_2\geqslant 1}\frac{z^{n_1}}{n_1^2n_2} \quad ; \quad \langle S|x_0^n\rangle = \frac{\log^n(z)}{n!}$$

Computation of integrators by transducer

The two cases of the transducer are given by the languages x_0^* and $X^*x_1X^*$ and the generating series Li by the behaviour of the transducer



$$T = \begin{pmatrix} x_0 | \int_1^z \square \frac{ds}{s} & 0 \\ x_1 | \int_0^z \square \frac{ds}{1-s} & x_0 | \int_0^z \square \frac{ds}{s} + x_1 | \int_0^z \square \frac{ds}{1-s} \end{pmatrix}$$

$$\text{Alphabet} \ : \ \Sigma = \{x_0, x_1\} \times \mathrm{End}(W) \simeq \mathrm{End}(W).\{x_0, x_1\} \text{ with } W \subset \mathcal{H}(\Omega) \quad (13)$$

The space W.

1. We define $\mathcal{H}_0$ as the space of $f \in \mathcal{H}(\Omega)$ admitting an analytic continuation around zero. This space embeds naturally in $\mathcal{H}(\Omega)$. Then we define W as the algebra generated by $\mathcal{H}_0(\Omega)$ and $\log(z)$.
2. Due to the fact that $f \in W \setminus \{0\} \Longrightarrow f \sim_0 \alpha_k.z^k$ for some k and $\alpha_k \neq 0$, it is an easy exercise to see that W is a free $\mathcal{H}_0$-module with basis $\{\log^n(z)\}_{n\geqslant 0}$. We also remark that W is closed by all the integrators. More precisely, with splitting $\mathcal{H}_0 = \mathcal{H}_0^+ \oplus \mathbb{C}.1_\Omega$ w.r.t. the evaluation at zero (i.e. $\mathcal{H}_0^+ = \ker(\delta_0)$) we see that

$$W = W_+ \bigoplus \underbrace{\left(\oplus_{n\geqslant 0}\mathbb{C}.\log^n(z)\right)}_{W_r(=\text{rightmost branch})} = W_+ \bigoplus W_r \ . \tag{14}$$

   1. the integrator $\int_1^z, \square\frac{ds}{s}$ acts within W_r
   2. W_+ is made of sums $z^p\log^q(z)$ with $p \geqslant 1$ so that the other integrators (with lower bound 0) act in W_+
   3. $\int_0^z \square\frac{ds}{1-s}$ sends W_r to W_+.

Computation of the behaviour/1

Linear representation

Due to the fact that the action is on the left (i.e. right-left reading of the word), we have (with the alphabet $\mathrm{End}(W).\{x_0, x_1\}$)

$$\lambda = \begin{pmatrix} 1 & 1 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1_\Omega \\ 0 \end{pmatrix}$$

$$T = \begin{pmatrix} \int_{1}^{z} \square \frac{ds}{s}.x_0 & 0 \\ \int_{0}^{z} \square \frac{ds}{1-s}.x_1 & \int_{0}^{z} \square \frac{ds}{s}.x_0 + \int_{0}^{z} \square \frac{ds}{1-s}.x_1 \end{pmatrix}$$

Computation of the star/1

Applying formulas of Eq. (7), we get

$$T^* = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix}^* = \begin{pmatrix} a_{11}^* & 0 \\ a_{22}^* a_{21} a_{11}^* & a_{22}^* \end{pmatrix}$$

Computation of the star/2

This star can be factored, considering that

$$T = \begin{pmatrix} \int_1^z \square \frac{ds}{s}.x_0 & 0 \\ \int_0^z \square \frac{ds}{1-s}.x_1 & \int_0^z \square \frac{ds}{s}.x_0 + \int_0^z \square \frac{ds}{1-s}.x_1 \end{pmatrix} =$$

$$\begin{pmatrix} \int_1^z \square \frac{ds}{s} & 0 \\ 0 & \int_0^z \square \frac{ds}{s} \end{pmatrix}.x_0 + \begin{pmatrix} 0 & 0 \\ \int_0^z \square \frac{ds}{1-s} & \int_0^z \square \frac{ds}{1-s} \end{pmatrix}.x_1 =$$
$$T_0.x_0 + T_1.x_1$$

and using formula (2), we get

$$T^* = \Big((T_0.x_0)^* T_1.x_1\Big)^* (T_0.x_0)^* = \Big((T_0.x_0)^* T_1.x_1\Big)^+ (T_0.x_0)^* + (T_0.x_0)^* \tag{15}$$

About the asymptotic condition

3. We then have

$$\mathrm{Li} = \begin{pmatrix}1 & 1\end{pmatrix} T^* \begin{pmatrix}1_\Omega \\ 0\end{pmatrix} =$$

$$\begin{pmatrix}1 & 1\end{pmatrix} \left((T_0.x_0)^* T_1.x_1\right)^+ (T_0.x_0)^* \begin{pmatrix}1_\Omega \\ 0\end{pmatrix} + \begin{pmatrix}1 & 1\end{pmatrix} (T_0.x_0)^* \begin{pmatrix}1_\Omega \\ 0\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix}1 & 1\end{pmatrix} \left((T_0.x_0)^* T_1.x_1\right)^+ (T_0.x_0)^* \begin{pmatrix}1_\Omega \\ 0\end{pmatrix}}_{\mathrm{Li}^+ \text{ only words s.t. } |w|_{x_1}>0} + e^{x_0 \log(z)} \tag{16}$$

In this way $\mathrm{Li} = \mathrm{Li}^+ + e^{x_0 \log(z)}$ and we get

$$\lim_{z->0} e^{-x_0 \log(z)} \mathrm{Li} = \lim_{z->0} \mathrm{Li}\ e^{-x_0 \log(z)} = 1 \tag{17}$$

this allows to prove unicity by means of the differential Galois group of (12).

About the asymptotic condition/2

4. $\mathrm{Li} = G_1$ is a shuffle character (due to the fact that the multiplier and the asymptotic condition are grouplike i.e. characters).
5. For $a \notin]-\infty, 0]$, the integrator $\int_1^z \square \frac{ds}{s}$ can be replaced by $\int_a^z \square \frac{ds}{s}$, one then finds a series G_a which fullfils system (12) where the asymptotic initial condition is modified to $\lim_{\substack{z\to 0\\ z\in\Omega}} S(z)e^{-x_0(\log(z)-\log(a))} = 1_{\mathcal{H}(\Omega)\langle\langle X\rangle\rangle}$.
6. Due to the fact that, on the one hand the asymptotic counterterm $e^{-x_0(\log(z)-\log(a))}$ is grouplike (i.e. a shuffle character) and, on the other hand the multiplier is primitive (i.e. a shuffle infinitesimal character), one easily sees that all G_a are shuffle characters.
7. Computing $\langle G_a|x_0^*\rangle = \sum_{n\geqslant 0}\langle G_a|x_0^n\rangle = e^{(\log(z)-\log(a))} = z/a$, one sees that all shuffle characters G_a are different[a].

[a] More generally, the possibility of setting a series in the RHS place of a scalar product has been explored in [6].

Domain of Li (definition)

In order to extend Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S=\sum_{n\geqslant 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geqslant 0} Li_{S_n}(z)$ converges for the compact convergence in Ω (see [6]). One sets

$$Li_S(z) := \sum_{n\geqslant 0} Li_{S_n}(z) \tag{18}$$

The ladder (upper part)

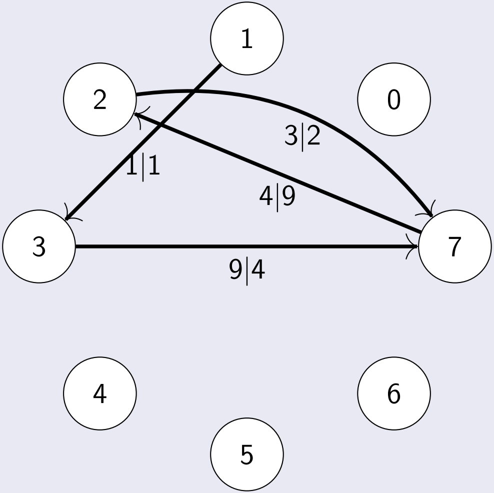
$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\ \mathrm{Li}_\bullet\ } & \mathcal{H}(\Omega) \\
\downarrow & & \downarrow \\
Dom(Li;\Omega) & \xrightarrow{\ \mathrm{Li}_\bullet^{(1)}\ } & \mathcal{H}(\Omega)
\end{array}$$

Examples

$$Li_{x_0^*}(z)=z,\ \ Li_{x_1^*}(z)=(1-z)^{-1}\ ;\ Li_{(\alpha x_0+\beta x_1)^*}(z)=z^{\alpha}(1-z)^{-\beta}$$

Last part: a two-state transducer for the Collatz function.

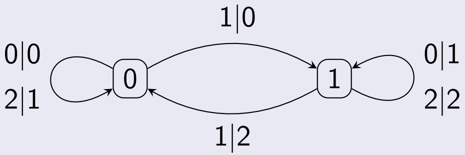
Recall: Division by 8 with transducer



1 1 9 4 3	8
3 9	1 4 9 2
7 4	
2 3	
7	

11	8	39	8	74	8	23	8
3	1	7	4	2	9	7	2

Division by 2 in base 3.



00	2	01	2	02	2
0	0	1	0	0	1
10	2	11	2	12	2
1	1	0	2	1	2

Collatz function, conjecture and plan.

Collatz function:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ odd} \end{cases}$$

Collatz conjecture:

For any $n > 0$, there exists $p \geqslant 0$ such that $f^p(n) = 1$

$$3 \xrightarrow{f} 10 \xrightarrow{f} 5 \xrightarrow{f} 16 \xrightarrow{f} 8 \xrightarrow{f} 4 \xrightarrow{f} 2 \xrightarrow{f} 1$$

Plan:

Explicit automaton realizing f^p (or equivalent) according to p.

Shortcut Collatz.

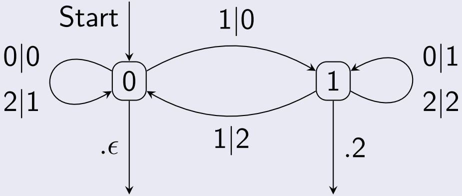
Shortcut Collatz function

$$g(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

Collatz conjecture with g:

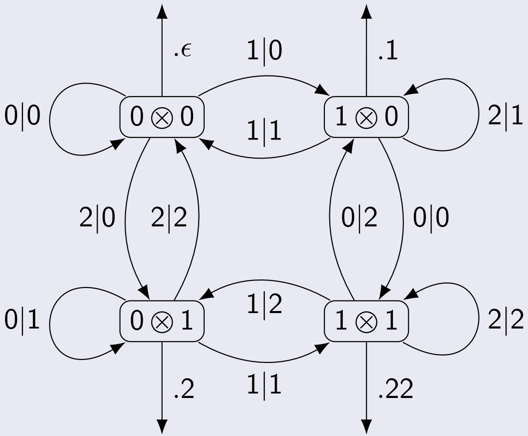
$$3 \xrightarrow{g} 5 \xrightarrow{g} 8 \xrightarrow{g} 4 \xrightarrow{g} 2 \xrightarrow{g} 1$$

and the corresponding transducer. We observe that the TS (transition structure) is that of the Division by 2 in base 3.



Shortcut Collatz/2

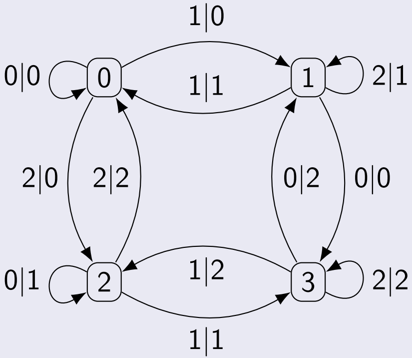
Transducer squared (for g^2)



(see MPS *On a theorem of R. Jungen*, 1962 [17]).

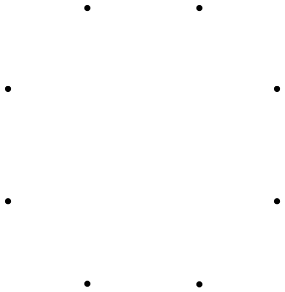
Shortcut Collatz/2

Transducer squared for g^2, transition structure.

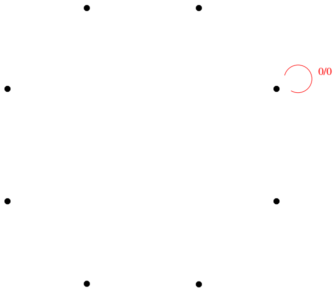


We get the division by 4 in base 3. Relabelling of the states done by reverse base 2 i.e. $a \otimes b \to 2b + a$.

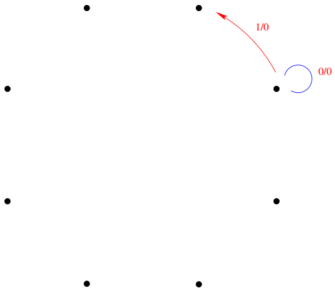
Division by 2^3 in base 3:



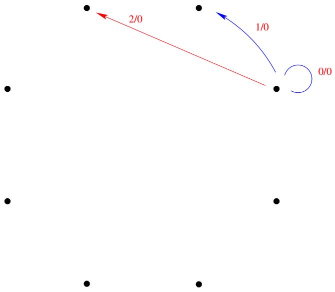
Division by 2^3 in base 3:



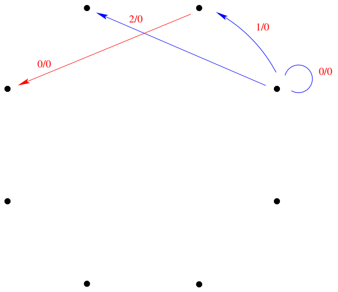
Division by 2^3 in base 3:



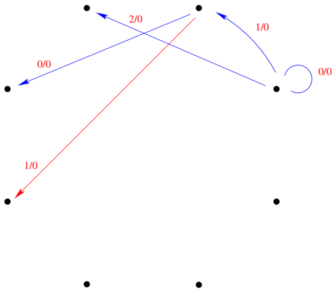
Division by 2^3 in base 3 :



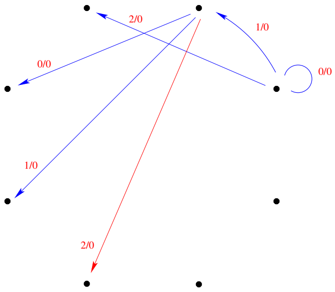
Division by 2^3 in base 3:



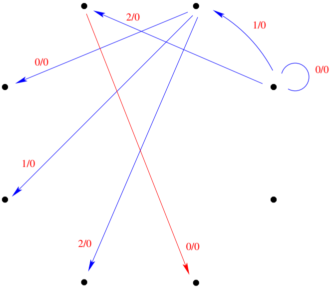
Division by 2^3 in base 3:



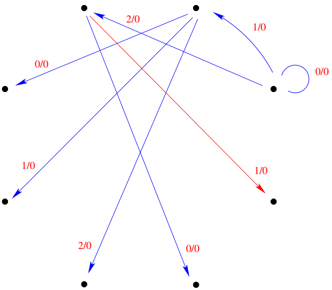
Division by 2^3 in base 3 :



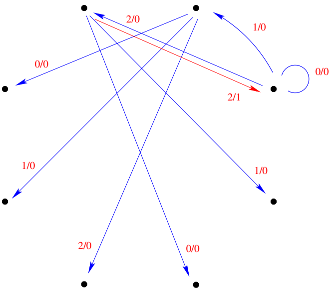
Division by 2^3 in base 3 :



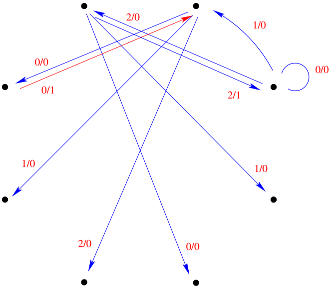
Division by 2^3 in base 3 :



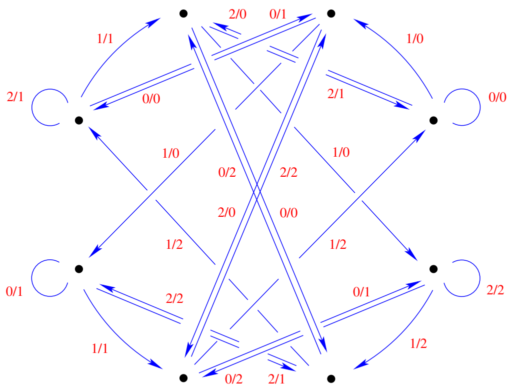
Division by 2^3 in base 3 :



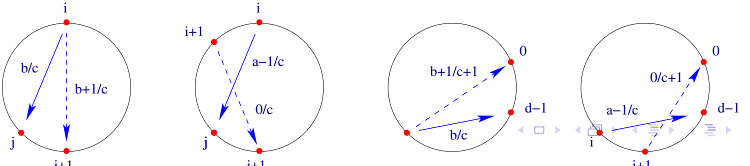
Division by 2^3 in base 3 :



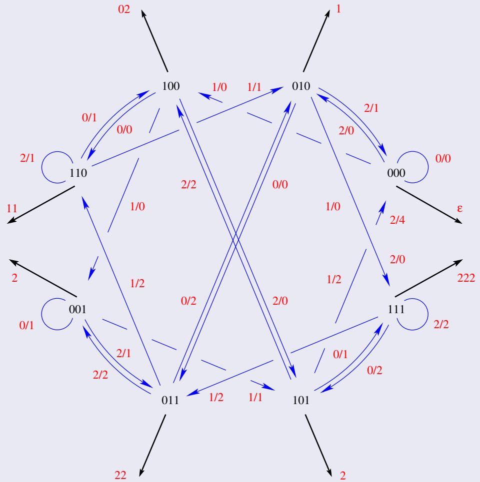
Division by 2^3 in base 3:



Construction:



Division by 2^3 in base 3 (TS of g^3) and terminal function.



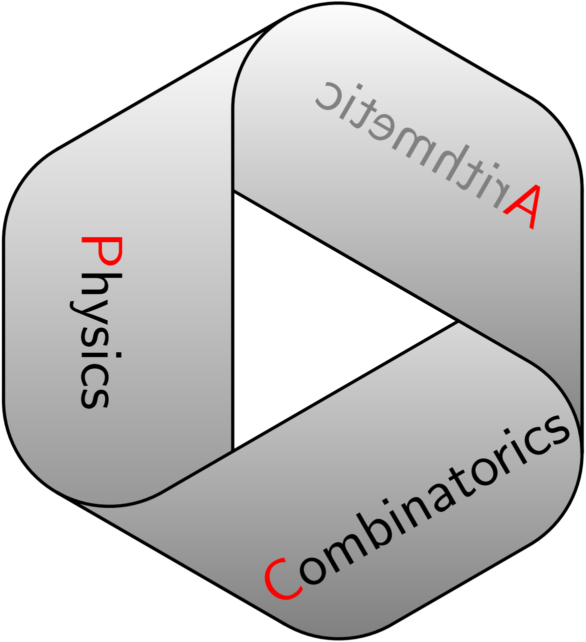
Concluding remarks

1. We have indicated the structure of automaton with multiplicities in a (non necessarily commutative) semiring R, following the original thought of Eilenberg and Schützenberger.
2. The computation of its behaviour, a generating series, entails that of the star of a matrix (in general with noncommutative coefficients).
3. When one specializes R to $R = \Sigma \times \mathbf{k}$ ($\mathbf{k}$ a ring of operators), one gets a powerful notion of Σ-action which is powerful enough to, for example, generate Hyperlogarithms and, through Lazard elimination, explain the asymptotic initial conditions.
4. When one specializes R to $R = \Sigma \times \mathbf{k}$ ($\mathbf{k}$ a commutative semiring), one gets the classical structure of automaton with multiplicities in $\mathbf{k}$, rational series, rational calculus.

Concluding remarks/2

5. If, moreover, **k** is a field, one can use the this rational calculus to compute within every Sweedler's dual of a **k** Hopf or bi-algebra.
6. The trick is the following. Let $\sigma: X \longrightarrow \mathcal{A}$ be an (indexed) generating family of $\mathcal{A}$, $\mu: \mathbf{k}\langle X\rangle \longrightarrow \mathcal{A}$ the corresponding (onto) morphism and $\mu^*: \mathcal{A}^* \hookrightarrow \mathbf{k}\langle\langle X\rangle\rangle$ its transpose. Then, due to the formula $\mu^*(f_{\mu(u)}) = \mu^*(f)_u$ we have $\mu^*(\mathcal{A}^\circ) = \mathbf{k}^{rat}\langle\langle X\rangle\rangle \cap Im(\mu^*)$ which allows the rational calculus within $\mathcal{A}^\circ$.
7. If the multiplicities are taken in some $R = \Sigma \times \Gamma$, one gets the classical "letter-to-letter" finite states transducer structure. In this way, states can encode cases.
8. We have described an unexpectedly simple two-state "letter-to-letter" transducer which produces the Collatz function.
9. This opens the door to a geometrization of the Collatz conjecture.
10. Such methods could shed a new light on Erdös conjecture [10].

THANK YOU FOR YOUR ATTENTION !


Arithmetic
Physics
Combinatorics

Bibliography. I

[1] Eiichi Abe, *Hopf Algebras*, Cambridge University Press, 3 juin 2004 - 300 pages.

[2] D. Caucal and C. Rispal, *On the Powers of the Collatz Function*, **Best Paper Award of MCU24**, to be published in LNCS series by Springer Verlag.

[3] V. Chari, A. Pressley, *A Guide to Quantum Groups*. Cambridge University Press (1994).

[4] M. Deneufchâtel, G. Duchamp, V. Hoang Ngoc Minh and A. I. Solomon, *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.
https://doi.org/10.48550/arXiv.1101.4497

Bibliography. II

[5] V. Drinfeld, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

[6] G. Duchamp, Quoc Huan Ngô and Vincel Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

[7] G. H. E. Duchamp and C. Tollu, *Sweedler's duals and Schützenberger's calculus*, In K. Ebrahimi-Fard, M. Marcolli and W. van Suijlekom (eds), Combinatorics and Physics, p. 67 - 78, Amer. Math. Soc. (Contemporary Mathematics, vol. 539), 2011. arXiv:0712.0125v3 [math.CO]

[8] G. Duchamp, C. Reutenauer, *Un critère de rationalité provenant de la géométrie non-commutative* (à la mémoire de Schützenberger), Inventiones Mathematicae, 128, 613-622, (1997).

Bibliography. III

[9] S. Eilenberg, *Automata, languages and machines, vol A*. Acad. Press, New-York, 1974.

[10] Lagarias, Jeffrey C. (2009), *Ternary expansions of powers of 2*, Journal of the London Mathematical Society, Second Series, 79 (3): 562–588,
arXiv:math/0512006,

[11] M. Fliess, *Matrices de Hankel*, J. Math. Pures Appl., t. 53, 1974, p. 197-222

[12] Darij Grinberg, Victor Reiner, *Hopf Algebras in Combinatorics*. `https://arxiv.org/abs/1409.8356`

[13] G. Jacob, *Représentations et substitutions matricielles dans la théorie matricielle des semigroupes*. Thèse, Univ. de Paris (1975).

Bibliography. IV

[14] H. N. Minh, M. Petitot, J. van der Hoeven, *Shuffle algebra and polylogarithms*, Discrete Maths, 225:217–230, 2000.

[15] Gearges Racinet, *Series generatrices non-commutatives de polyzetas et associateurs de Drinfeld*, Thèse (2000). `https://theses.hal.science/tel-00110891v1`

[16] M.P. Schützenberger, *On the definition of a family of automata, Inf. and Contr., 4 (1961)*, 245-270.

[17] M.P. Schützenberger, *On a theorem of R. Jungen*, Proc. Amer. Math. Soc. 13, 885-889, (1962).

Reserve/1

Proposition

Let $P \in \mathbb{C}\langle X\rangle$ and $f(z) = \langle Ł|P\rangle = \sum_{w\in X^*}\langle P|w\rangle \mathrm{Li}_w$.

1) The following conditions are equivalent

i) *f can be analytically extended around zero.*

ii) *$P \in \mathbb{C}\langle X\rangle x_1 \oplus \mathbb{C}.1_{X^*}$.*

2) In this case Ω itself[a] *can be extended to $\Omega_1 = \mathbb{C}\setminus(]-\infty,-1] \cup [1,+\infty[)$.*

[a] The domain, for z of Li_P.

A useful property

Proposition B

Let $\mathcal{B} = (k\langle X\rangle, conc, 1_{X^*}, \Delta, \epsilon)$ be a conc-bialgebra, then

1. The space $k^{rat}\langle X\rangle$ is closed by the convolution product $\diamond$ (here ${}^t\Delta$) given by
$$\langle S \diamond T|w\rangle = \langle S \otimes T|\Delta(w)\rangle \tag{19}$$
2. If k is a $\mathbb{Q}$-algebra and $\Delta_+ \ : \ k.X \to k.X \otimes k.X$ cocommutative, $\mathcal{B}$ is an enveloping algebra iff Δ_+ is nilpotent[a].
3. If, moreover k is without zero divisors, the characters $(x^*)_{x\in X}$ are algebraically independant over $(k\langle X\rangle, \diamond, 1_{X^*})$ within $(k\langle\langle X\rangle\rangle, \diamond, 1_{X^*})$.

[a] See CAP 2017

A useful property/2

Remark

Property (3) is no longer true if Δ is not moderate. For example with the Hadamard coproduct and $x \neq y$, one has $y \odot (x)^* = 0$.

Examples

Shuffle: $(\alpha x)^* \sqcup\!\sqcup (\beta y)^* = (\alpha x + \beta y)^*$

Stuffle: $(\alpha y_i)^* \uplus (\beta y_j)^* = (\alpha y_i + \beta y_j + \alpha\beta y_{i+j})^*$

q-**infiltration**: $(\alpha x)^* \uparrow_q (\beta y)^* = (\alpha x + \beta y + \alpha\beta\delta_{x,y} x)^*$

Hadamard: $(\alpha a)^* \odot (\beta b)^* = 1_{X^*}$ if $a \neq b$ and $(\alpha a)^* \odot (\beta a)^* = (\alpha\beta a)^*$

Starting the ladder

$$\begin{array}{ccc} (\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\ \downarrow & & \downarrow \\ (\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \end{array}$$

Domain of Li (definition)

In order to extend Li to series, we define $Dom(Li;\Omega)$ (or $Dom(Li)$) if the context is clear) as the set of series $S = \sum_{n\geqslant 0} S_n$ (decomposition by homogeneous components) such that $\sum_{n\geqslant 0} Li_{S_n}(z)$ converges for the compact convergence in Ω. One sets

$$Li_S(z) := \sum_{n\geqslant 0} Li_{S_n}(z) \tag{20}$$

Examples

$$Li_{x_0^*}(z) = z,\ Li_{x_1^*}(z) = (1-z)^{-1}\ ;\ Li_{(\alpha x_0+\beta x_1)^*}(z) = z^{\alpha}(1-z)^{-\beta}$$

Properties of the extended Li

Proposition

With this definition, we have

1. $Dom(Li)$ is a shuffle subalgebra of $\mathbb{C}\langle\langle X\rangle\rangle$ and then so is $Dom^{rat}(Li) := Dom(Li) \cap \mathbb{C}^{rat}\langle\langle X\rangle\rangle$
2. For $S, T \in Dom(Li)$, we have

$$\mathrm{Li}_{S \sqcup\!\sqcup T} = \mathrm{Li}_S \, . \, \mathrm{Li}_T$$

Examples and counterexamples

For $|t| < 1$, one has $(tx_0)^* x_1 \in Dom(Li, D)$ (D is the open unit slit disc), whereas $x_0^* x_1 \notin Dom(Li, D)$.
Indeed, we have to examine the convergence of $\sum_{n\geqslant 0} \mathrm{Li}_{x_0^n x_1}(z)$, but, for $z \in]0, 1[$, one has $0 < z < \mathrm{Li}_{x_0^n x_1}(z) \in \mathbb{R}$ and therefore, for these values $\sum_{n\geqslant 0} \mathrm{Li}_{x_0^n x_1}(z) = +\infty$.
In fact, in this case ($|t| < 1$)

Coefficients in the Ladder

$$\begin{array}{ccc}
(\mathbb{C}\langle X\rangle, ш, 1_{X^*}) & \xrightarrow{\mathrm{Li}_\bullet} & \mathbb{C}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*] & \xrightarrow{\mathrm{Li}_\bullet^{(1)}} & \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*} \\
\downarrow & & \downarrow \\
\mathbb{C}\langle X\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_0\rangle\rangle \, ш \, \mathbb{C}^{\mathrm{rat}}\langle\langle x_1\rangle\rangle & \xrightarrow{\mathrm{Li}_\bullet^{(2)}} & \mathcal{C}_{\mathbb{C}}\{\mathrm{Li}_w\}_{w\in X^*}
\end{array}$$

Were, for every additive subgroup $(H,+) \subset (\mathbb{C},+)$, $\mathcal{C}_H$ has been set to the following subring of $\mathbb{C}$

$$\mathcal{C}_H := \mathbb{C}\{z^\alpha(1-z)^{-\beta}\}_{\alpha,\beta\in H} \ . \tag{22}$$

Examples

$$Li_{x_0^*}(z) = z, \ Li_{x_1^*}(z) = (1-z)^{-1} \ ; \ Li_{\alpha x_0^* + \beta x_1^*}(z) = z^\alpha(1-z)^{-\beta}$$

The arrow $\mathrm{Li}_\bullet^{(1)}$

Proposition

i. The family $\{x_0^*, x_1^*\}$ is algebraically independent over $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})$ within $(\mathbb{C}\langle\langle X\rangle\rangle^{\mathrm{rat}}, ш, 1_{X^*})$.

ii. $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, x_1^*, (-x_0)^*]$ is a free module over $\mathbb{C}\langle X\rangle$, the family $\{(x_0^*)^{ш k} ш (x_1^*)^{ш l}\}_{(k,l)\in\mathbb{Z}\times\mathbb{N}}$ is a $\mathbb{C}\langle X\rangle$-basis of it.

iii. As a consequence, $\{w ш (x_0^*)^{ш k} ш (x_1^*)^{ш l}\}_{\substack{w\in X^* \\ (k,l)\in\mathbb{Z}\times\mathbb{N}}}$ is a $\mathbb{C}$-basis of it.

iv. $\mathrm{Li}_\bullet^{(1)}$ is the unique morphism from $(\mathbb{C}\langle X\rangle, ш, 1_{X^*})[x_0^*, (-x_0)^*, x_1^*]$ to $\mathcal{H}(\Omega)$ such that

$$x_0^* \to z,\ (-x_0)^* \to z^{-1} \text{ and } x_1^* \to (1-z)^{-1}$$

v. $\mathrm{Im}(\mathrm{Li}_\bullet^{(1)}) = \mathcal{C}_{\mathbb{Z}}\{\mathrm{Li}_w\}_{w\in X^*}$.

vi. $\ker(\mathrm{Li}_\bullet^{(1)})$ is the (shuffle) ideal generated by $x_0^* ш x_1^* - x_1^* + 1_{X^*}$.

Sketch of the proof for vi.

Let $\mathcal{J}$ be the ideal generated by $x_0^* ш x_1^* - x_1^* + 1_{X^*}$. It is easily checked, from the following formulas[a], for $k \geqslant 1$,

$$\begin{array}{rcl} w ш x_0^* ш (x_1^*)^{ш k} & \equiv & w ш (x_1^*)^{ш k} - w ш (x_1^*)^{ш k-1} \ [\mathcal{J}], \\ w ш (-x_0)^* ш (x_1^*)^{ш k} & \equiv & w ш (-x_0)^* ш (x_1^*)^{ш k-1} + w ш (x_1^*)^{ш k} [\mathcal{J}], \end{array}$$

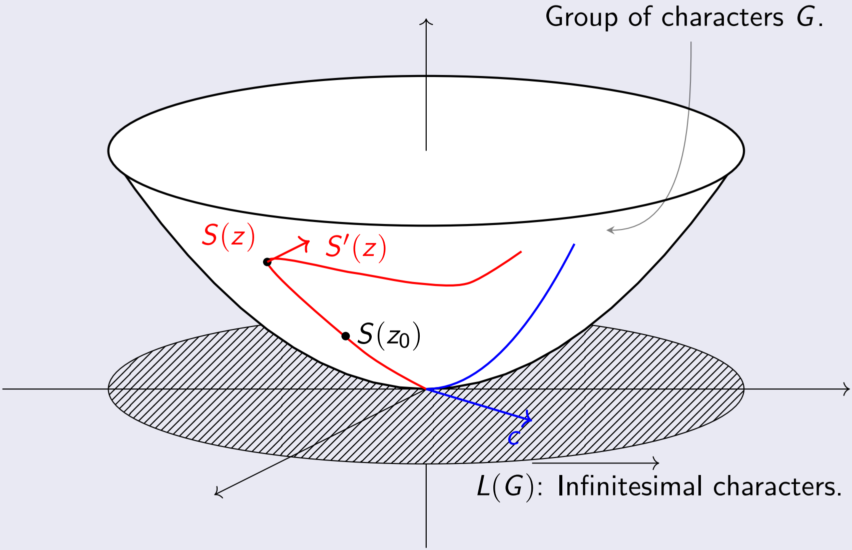
that one can rewrite $[\mathrm{mod}\, \mathcal{J}]$ any monomial $w ш (x_0^*)^{ш l} ш (x_1^*)^{ш k}$ as a linear combination of such monomials with $kl = 0$. Observing that the image, through $\mathrm{Li}_\bullet^{(1)}$, of the following family is free in $\mathcal{H}(\Omega)$

$$\{w ш (x_1^*)^{ш l} ш (x_0^*)^{ш k}\}_{(w,l,k) \in (X^* \times \mathbb{N} \times \{0\}) \sqcup (X^* \times \{0\} \times \mathbb{Z})} \qquad (23)$$

we get the result.

[a] In the Figure below, (w, l, k) codes the element $w ш (x_0^*)^{ш l} ш (x_1^*)^{ш k}$.

The Lie group of characters. ⏩ return



End of W

The easy exercise

We first prove that "it is a easy exercise to see that W is a free $\mathcal{H}_0$-module with basis $\{\log^n(z)\}_{n\geqslant 0}$"

1. We first prove that $(\sum_{i\in F} f_i.\log^{n_i} = 0) \Longrightarrow (\forall i \in F)(f_i = 0)$
2. Let $S = \{i \in F | f_i \neq 0\}$ be the support of $(f_i)_{i\in F}$. For all $f \in W \setminus \{0\}$, we have $f \sim_0 \alpha_k.z^k$ for a unique monomial $\alpha_k.z^k$ with $\alpha_k \neq 0$ (it is the valuation term of the Taylor series of f at zero). Then for $i \in S$, we have $f_i \log^{n_i}(z) \sim_0 \alpha_{k_i}.z^{k_i} \log^{n_i}(z)$. We then order S by $i \precsim j$ iff $k_i > k_j$ or, $k_i = k_j$, $n_i < n_j$ (it is the total order of the orders of infinity).

3. If $S = \emptyset$ we are done otherwise, let i_0 be the greatest element of S for $\precnsim$, we have

$$A(z) = \sum_{i \in F} f_i . \log^{n_i} = \sum_{i \in S} f_i . \log^{n_i} \sim_0 \alpha_{k_{i_0}} . z^{k_{i_0}} \log^{n_{i_0}}(z) \qquad (24)$$

whence $\lim_{z \to 0} \dfrac{A(z)}{z^{k_{i_0}} \log^{n_{i_0}}(z)} = \alpha_{k_{i_0}}$ but $\alpha_{k_{i_0}} \neq 0$ which proves that this case is impossible. □

Differential equations and BTT (Basic Triangle Theorem).

Theorem (DDMS [1])

Let $(\mathcal{A}, d)$ *be a k-commutative associative differential algebra with unit and* $\mathcal{C}$ *be a differential subfield of* $\mathcal{A}$ *(i.e.* $d(\mathcal{C}) \subset \mathcal{C}$*). We suppose that* $S \in \mathcal{A}\langle\langle X \rangle\rangle$ *is a solution of the differential equation*

$$\mathbf{d}(S) = MS \; ; \; \langle S | 1_{X^*} \rangle = 1_{\mathcal{A}} \qquad (25)$$

where the multiplier M *is a homogeneous series (a polynomial in the case of finite* X*) of degree* 1*, i.e.*

$$M = \sum_{x \in X} u_x \, x \in \mathcal{C}\langle\langle X \rangle\rangle \; . \qquad (26)$$

[1] *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, **M. Deneufchâtel, GHED, V. Hoang Ngoc Minh and A. I. Solomon**, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

Differential equations and BTT/2

Theorem (cont'd)

The following conditions are equivalent :

1. *The family* $(\langle S|w\rangle)_{w\in X^*}$ *of coefficients of* S *is free over* $\mathcal{C}$.
2. *The family of coefficients* $(\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ *is free over* $\mathcal{C}$.
3. *The family* $(u_x)_{x\in X}$ *is such that, for* $f\in\mathcal{C}$ *and* $\alpha_x\in k$
$$d(f)=\sum_{x\in X}\alpha_x u_x \Longrightarrow (\forall x\in X)(\alpha_x=0)\ . \tag{27}$$
4. *The family* $(u_x)_{x\in X}$ *is free over* k *and*
$$d(\mathcal{C})\cap span_k\Big((u_x)_{x\in X}\Big)=\{0\}\ . \tag{28}$$

The particular case of Hyperlogarithms.

3. Hyperlogarithms are analytic functions produced as coordinates of a solution of a Non Commutative Differential Equation (NCDE) of the type

$$S' = MS \tag{29}$$

where $S \in \mathcal{H}(\Omega)\langle\langle X \rangle\rangle$ and M, the multiplier, is of the form

$$M = \sum_{i=1}^{n} \frac{\lambda_i . x_i}{z - a_i},\ \ \lambda_i + 0 \tag{30}$$

(Polylogarithms are a subclass of Hyperlogarithms (see [4]).
In order to get a solution of this NCDE we have to pile up sections of the differential operators $\frac{1}{\lambda_i(z-a_i)}\frac{d}{dz}$ (i.e. integrators) of this differential equation labelled by letters. For this reason it is natural to use a transducer with values within endomorphism algebras (integrators). We illustrate it below with Polylogarithms.

Noncommutative Geometry

In 1992, I. M. Gelfand was in Paris with V. Retakh (IHES), we met him a lot of times during his visit, but we did not know that this would lead him to ask Alain Connes to better explain his conjecture about the rationality of some elements of the (reduced C^*-algebra) of the free group. We eventually solved it with Christophe, using tools and properties of automata theory and this resulted in publication [8]. Let me tell the story and detail the conjecture ...

Connes conjecture itself

Soit X be an alphabet (a set) and $\Gamma = F(X)$, the group freely generated by X. We define

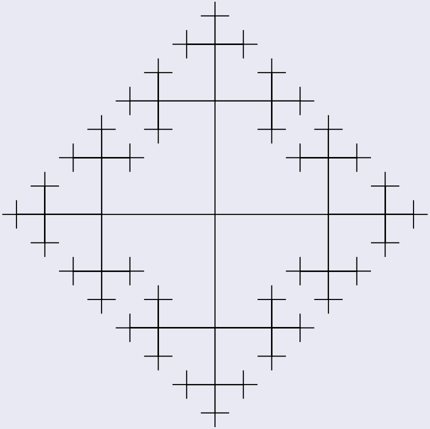
- for an arbitrary function $f : \Gamma \to \mathbb{C}$

$$||f||_2 = \sum_{g \in \Gamma} |f(g)|^2 \in [0, \infty] \tag{31}$$

- $l^2(\Gamma) = \{f : \Gamma \to \mathbb{C} \text{ t.q. } ||f||_2 < \infty\}$ (with the canonical basis $(\epsilon_g)_{g \in \Gamma}$)
- the regular representation by "shifts" $\Gamma \to \mathcal{L}(l^2(\Gamma))$ (bounded operators), extended faithfully to $\mathbb{C}[\Gamma]$ (the algebra of the free group) by linearity. Then $\mathbb{C}[\Gamma] \hookrightarrow \mathcal{L}(l^2(\Gamma))$
- $C_r^*(\Gamma)$ is the norm closure of $\mathbb{C}[\Gamma]$ within $\mathcal{L}(l^2(\Gamma))$ (called by specialists, the reduced C^* algebra of the free group).

Connes conjecture itself/3

Let's start with the free group (here F_2)



Connes conjecture itself/4

- shows that the free group can be realized by *reduced words* (the length of an element $g \in \Gamma$ is the length of the reduced word representing it)
  - take a barred (disjoint) copy of the alphabet $\tilde{X} = X \sqcup \bar{X}$ and form the quotient

$$F(X) = \tilde{X}^*/(x\bar{x} \equiv \bar{x}x \equiv 1) \tag{32}$$

    i.e. apply the rewrite rules $(x\bar{x} \mapsto 1,\ \bar{x}x \mapsto 1)$
  - see that the action on the nodes is (one step each)

$$a = \mathit{east},\ \ b = \mathit{north}, \bar{a} = \mathit{west},\ \ \bar{b} = \mathit{south}$$

- see the action by shifts on the (unoriented) edges.

Connes conjecture itself/5

The algebra $C_r^*(\Gamma)$ is the space where another sort of closure will take place.
Let $(C_r^*(\Gamma))_n$ be the smallest subalgebra $\mathbb{B} \subset C_r^*(\Gamma)$ such that $\Gamma \subset \mathbb{B}$ and

$$x \in M_n(\mathbb{B}) \cap [M_n(C_r^*(\Gamma))]^{-1} \Longrightarrow x \in [M_n(\mathbb{B})]^{-1} \tag{33}$$

You can imagine the making of this “rational completion of order n” by the following process from $\mathbb{B}_0 = \mathbb{C}[\Gamma]$. If $\mathbb{B}_k$ is constructed, then one obtains $\mathbb{B}_{k+1}$ as follows

1. form the matrices (with non commutative coefficients) $M_n(\mathbb{B}_k)$
2. take the ones which are regular (invertible), find the inverses of them, add their coefficients to $\mathbb{B}_k$ and make the closure by linear combinations and products

the wanted closure is $\cup_{n\geqslant 0} B_n$

One has

$$(C_r^*(\Gamma))_1 \subset (C_r^*(\Gamma))_2 \subset \cdots (C_r^*(\Gamma))_n \subset \cdots \subset C_r^*(\Gamma) \tag{34}$$

the rational completion of order n is $(C_r^*(\Gamma))^{\sim} = \cup_{n\geqslant 1}(C_r^*(\Gamma))_n$.
Note that these rational completions can be defined for the situations $\mathcal{A}_1 \subset \mathcal{A}_2$ where $\mathcal{A}_i$ are arbitrary k-AAU (we will use it later on), the usual rational closure is $(\ .)_1$.

Connes conjecture itself/7 : the operators P and F

Let $\Gamma^{(1)}$ be the set of unoriented edges of the Cayley graph of Γ. Such an edge is a pair $\{g, gx\}$ with $x \in \tilde{X}$ and $l(gx) = l(g) + 1$. We build a Γ-module with two sectors $\mathbb{H}^{\pm}$ by

$$\mathbb{H}^+ = l^2(\Gamma),\ \mathbb{H}^- = l^2(\Gamma^{(1)}) \oplus \mathbb{C} \tag{35}$$

and an involution F by

$$\begin{array}{c} \\ \mathbb{H}^+ \\ \mathbb{H}^- \end{array} \begin{array}{c} \begin{array}{cc} \mathbb{H}^+ & \mathbb{H}^- \end{array} \\ \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix} \end{array} = F \tag{36}$$

where $P : \mathbb{H}^+ \to \mathbb{H}^-$ is the isometry defined by

$$P\epsilon_1 = 1_{\mathbb{C}};\ P\epsilon_g = \{\epsilon_{\phi(g)}, \epsilon_g\} \text{ for } g \in \Gamma - \{1\} \tag{37}$$

Connes conjecture itself/8 : the finite rank condition

Note that $a \in \Gamma$ acts by isometries on $\mathbb{H}$ by

$$a\{g, gx\} = \{ag, agx\},\ \ a1_{\mathbb{C}} = 0,\ \ a\epsilon_g = \epsilon_{ag} \tag{38}$$

and then one can identify the elements of $C_r^*(\Gamma)$ with operators in Ł$(\mathbb{H})$.
We will denote $(C_r^*(\Gamma))_{fin}$ the algebra of elements $a \in C_r^*(\Gamma)$ such that $[F, a]$ is of finite rank.
The conjecture was

$$(C_r^*(\Gamma))_{fin} = (C_r^*(\Gamma))^{\tilde{}} \tag{39}$$

Connes conjecture itself/8 : steps of the proof

- first prove that

$$(C_r^*(\Gamma))_1 = \cdots = (C_r^*(\Gamma))_n = (C_r^*(\Gamma))^{\sim} \subset (C_r^*(\Gamma))_{fin} \qquad (40)$$

- remark that every $a \in C_r^*(\Gamma)$ can be represented by a series $a(1_\Gamma) = \sum_{g\in\Gamma} \alpha_g\, g$ (because $a(ug) = a(u)g$ and this series is in $l^2(\Gamma)$)
- denoting $m(g)$ the (unique) reduced word associated with $g \in \Gamma$, we establish that if the series $m(a) = \sum_{g\in\Gamma} \alpha_g\, m(g)$ is **rational** (i.e. that its orbit by shifts - i.e. letter cancellation - is of finite rank, in other words a rational series is a representative function on the free monoid) then $a \in (C_r^*(\Gamma))_1$
- if $a \in C_r^*(\Gamma)$ is such that $[F, a]$ is of finite rank, then $m(a)$ is rational

This ends the proof.

Rational series and duality

Let A be an alphabet and k a field, let us note A^*, the free monoid with base A, $k\langle A\rangle = k[A^*]$ its algebra (noncommutative polynomials), $k\langle\langle A\rangle\rangle = k^{A^*}$, the corresponding set of series and $(.,.) : k\langle\langle A\rangle\rangle \otimes k\langle A\rangle \to k$, the canonical pairing between series and polynomials. One defines canonical actions (see talk by Dominique Perrin) of A^* on series by

$$S.u = \sum_{w\in A^*} (S, uw)w, \;\; u.S = \sum_{w\in A^*} (S, wu)w \tag{41}$$

One has the following theorem.

Theorem

Let $S \in k\langle\langle A \rangle\rangle$ TFAE
i) The family $(S.u)_{u \in A^*}$ is of finite rank.
ii) The family $(u.S)_{u \in A^*}$ is of finite rank.
iii) It exists $n \in \mathbb{N}$, $\lambda \in k^{1\times n}$, $\mu : A^* \to k^{n\times n}$ (a multiplicative morphism) and $\gamma \in k^{n\times 1}$ such that, for all $w \in A^*$

$$(S, w) = \lambda\mu(w)\gamma \tag{42}$$

iv) (If A is finite, known as the theorem of Kleene-Schützenberger) The series S is in the rational closure of $k\langle A \rangle$ within $k\langle\langle A \rangle\rangle$.

Remarks 1) For the sake of Combinatorial Physics (where the alphabets are usually infinite, (iv) has been extended to infinite alphabets and replaced by
iv') The series S is in the rational closure of k^A (linear series) within $k\langle\langle A \rangle\rangle$.

Sweedler's duals

Remarks (cont'd) 2) This theorem is linked to the following (Representative functions on semigroups, from the book of Eichii Abe)

In fact, rational series are exactly representative functions on A^*. If one considers the multiplicative semigroup of a k-AAU and one restricts to the linar forms, one gets exactly the Sweedler's dual. Hence the rational series are also the Sweedler's dual $k\langle A\rangle^\circ$.

Charm of dualization

- States are linear forms
- Observation functions are linear forms
- Allows to swap between commutativity and co-commutativity (e.g. Connes Kreimer)
- Combinatorial interest (dual laws : shuffle, stuffle, infiltration, bases in duality)

So, one often wants to compute the dual of a Hopf algebra or a bialgebra $(\mathbb{H}, \mu, 1_{\mathbb{H}}, \Delta, \epsilon)$ and one should obtain some $(\mathbb{H}^{\circ}, \mu_{\Delta}, 1_{\epsilon}, \Delta_{\mu}, \epsilon_{1})$

Examples

- (Shuffle algebra) $(k\langle A\rangle, conc, 1_{A*}, \Delta, \epsilon)$ has (restricted) dual $(k\langle A\rangle, ш, 1_{A*}, \Delta_{conc}, \epsilon)$
- (Stuffle algebra) $(k\langle Y\rangle, conc, 1_{A*}, \Delta_{\uplus}, \epsilon)$ where $Y = \{y_i\}_{i\geqslant 1}$ and the $\Delta_{\uplus}$ is defined on the letters by

$$\Delta_{\uplus}(y_s) = y_s \otimes 1 + 1 \otimes y_s + \sum_{i+j=s} y_i \otimes y_j \tag{43}$$

- (deformations)

$$\Delta_{\uplus}^{(q)}(y_s) = y_s \otimes 1 + 1 \otimes y_s + q \sum_{i+j=s} y_i \otimes y_j \tag{44}$$

Note that some laws are better understood by their dual (shuffle, stuffle, infiltration).

One can always dualize a comultiplication by

$$\langle f *_\Delta g | w \rangle = \langle f \otimes g | \Delta(w) \rangle \tag{45}$$

but the same trick does not work for the products, and one has to find the domain of a possible comultiplication

$$\begin{array}{ccc}
\mathcal{A}^* & \xrightarrow{\ {}^t(\mu)\ } & (\mathcal{A} \otimes \mathcal{A})^* \\
\uparrow{\scriptstyle nat_1} & & \uparrow{\scriptstyle nat_2} \\
Dom({}^t(\mu)) & \xrightarrow{\ {}^t(\mu)\ } & \mathcal{A}^* \otimes \mathcal{A}^*
\end{array}$$

This domain is exactly the Sweedler's dual $\mathcal{A}^\circ$. Due to associativity, it has the very nice property that ${}^t(\mu)(\mathcal{A}^\circ) \subset \mathcal{A}^\circ \otimes \mathcal{A}^\circ$.

Perspectives

- The greatest dual of an algebra is the Sweedler's dual (mind that it can be (0), as for the Heisenberg-Weyl algebra)
- Many Hopf algebras of Combinatorics and Physics are free (commutative or noncommutative) and then Sweedler's dual separates the Hopf algebra
- Automata theory, by means of the rational expressions, provides a convenient language to harness this dual (which contains the other duals).