Monomial identities in the Weyl algebra [talk slides]

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joint work with Tom Roby, Stephan Wagner, Mei Yin
inspired by a question of Richard P. Stanley

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Abstract. The *Weyl algebra* (or *Heisenberg-Weyl algebra*) is the free algebra with two generators D and U and single relation $DU - UD = 1$. As a consequence of this relation, certain monomials are equal, such as $DUUD$ and $UDDU$. We characterize all such equalities over a field of characteristic 0, describing them in several ways: operational (by a combinatorial equivalence relation generated by certain moves), computational (through lattice path invariants) and in terms of rook theory. We also enumerate the equivalence classes and several variants thereof and discuss possible extensions to other algebras.

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Preprint:

- Darij Grinberg, Tom Roby, Stephan Wagner, Mei Yin, *Monomial identities in the Weyl algebra*, arXiv:2405.20492,

Slides of this talk:

- `https://www.cip.ifi.lmu.de/~grinberg/algebra/cap2024.pdf`

Items marked with $\circledast$ contain definitions/notations/terminology.

1. The Weyl algebra

- Let $\mathbf{k}$ be a field of characteristic 0.
- Let $\mathcal{M}$ be the free monoid with generators D and U. It consists of words with entries in $\{D, U\}$, such as $DUDDUUUD$.
- Let $\mathcal{W}$ be the Weyl algebra over $\mathbf{k}$ with generators D and U and relation $DU - UD = 1$.
- The Weyl algebra $\mathcal{W}$ acts on the polynomial $\mathbf{k}[x]$ by $D \mapsto \dfrac{d}{dx}$ and $U \mapsto x$. This action is faithful (since $\operatorname{char} \mathbf{k} = 0$).
- There is a canonical monoid morphism $\phi : \mathcal{M} \to \mathcal{W}$ sending D, U to D, U.
- Say that two words u and v in $\mathcal{M}$ are **ϕ-equivalent** if $\phi(u) = \phi(v)$ in $\mathcal{W}$.
- **Questions to be discussed** (posed by Richard P. Stanley)**:**
  1. When are two words ϕ-equivalent, in combinatorial terms?
  2. How efficiently can we test ϕ-equivalence?
  3. How many ϕ-equivalence classes are there, and how large are they?
  4. What is a minimal class of relations that generate ϕ-equivalence?
  5. What holds in other algebras or for other $\operatorname{char} \mathbf{k}$?

- We shall answer 1, 2, 3 fully and 4, 5 partly.

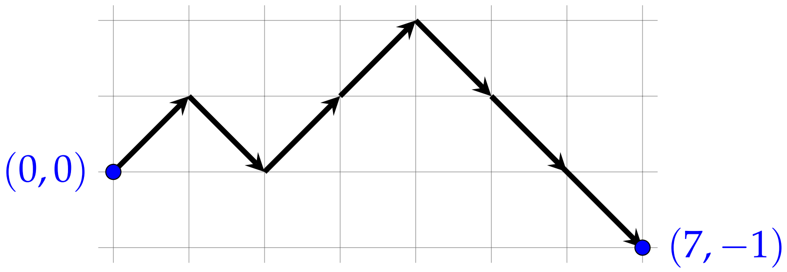
2. Words and paths

2.1. Definitions

- A **word** means a finite tuple of D's and U's. Thus, $\mathcal{M}$ is the set of all words.

- A word u is called a **prefix**, **suffix** or **factor** of a word w if and only if $w = uq$ or $w = pu$ or $w = puq$ (respectively) for some words p and q. These p and q can be empty.

- We encode words as **diagonal paths** (short: **paths**) on the plane, by reading each D as a step to the southeast ("**down**step") and each U as a step to the northeast ("**up**step"). For example, the word $UDUUDDD$ gives rise to the path

  This is a many-to-1 correspondence, as the starting point can be chosen freely. The **standard path** of a word w is the corresponding diagonal path starting at $(0,0)$.

  The word corresponding to a given path $\mathbf{p}$ is denoted $\mathrm{w}(\mathbf{p})$, and is called the **reading word** of $\mathbf{p}$.

- A word is **balanced** if it has equally many U's and D's.

- If $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ is a path, then

  - the **vertices** of $\mathbf{p}$ are $p_0, p_1, \ldots, p_k$;
  - the **NE-steps** of $\mathbf{p}$ are the vertices p_i of $\mathbf{p}$ with $p_i \nearrow p_{i+1}$;
  - the **SE-steps** of $\mathbf{p}$ are the vertices p_i of $\mathbf{p}$ with $p_i \searrow p_{i+1}$.

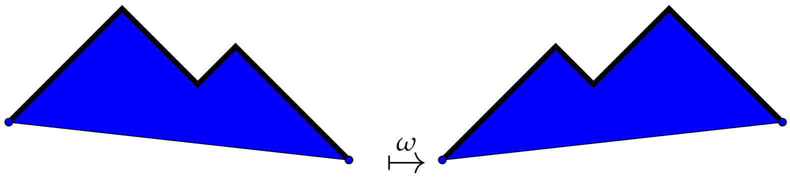
  For example, the above picture shows a path $\mathbf{p} = (p_0, p_1, \ldots, p_7)$ with $p_0 \nearrow p_1 \searrow p_2 \nearrow p_3 \nearrow p_4 \searrow p_5 \searrow p_6 \searrow p_7$.

2.2. The ω involutions

- Let $\omega : \mathcal{M} \to \mathcal{M}$ be the map that reads a word right-to-left and also changes each letter to the other possible letter ($D \mapsto U$ and $U \mapsto D$).

  Formally, ω is a monoid anti-automorphism.

- Pictorially, ω is reflection across a vertical axis:

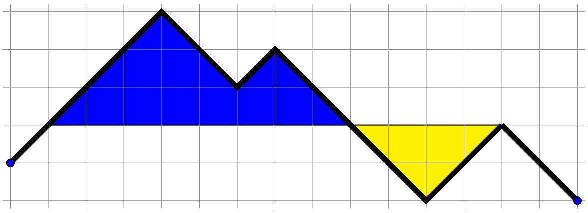
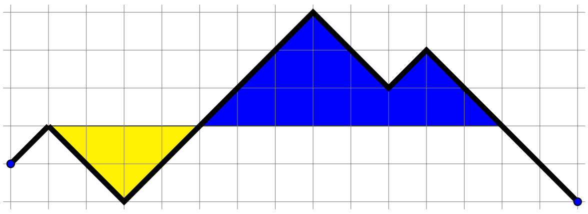
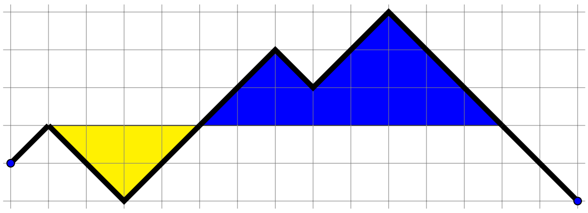


- Likewise, let $\omega : \mathcal{W} \to \mathcal{W}$ be the algebra anti-automorphism sending $D \mapsto U$ and $U \mapsto D$.

2.3. Balanced commutations and flips

- **Balanced commutations** and **balanced flips** are two ways to transform a word v into another word w.
- In a **balanced commutation**, we write our word v as $v = pxyq$ with two balanced factors x and y, and we set $w := pyxq$.
- In a **balanced flip**, we write our word v as $v = pxq$ with a balanced factor x, and we set $w := p\omega(x)q$.
- Both operations can be applicable to a word in many different ways.

- **Example:**

word		**standard path**
$U\underline{UUUDDUDD}\overline{DDUU}DD$		
balanced commutation	↓	balanced commutation
$UDDUU\underline{UUUDDUDD}DD$		
balanced flip	↓	balanced flip
$UDDUUUUUDDUDDDD$		

(NB: underlines and overlines mark the factors being transformed).

2.4. Heights

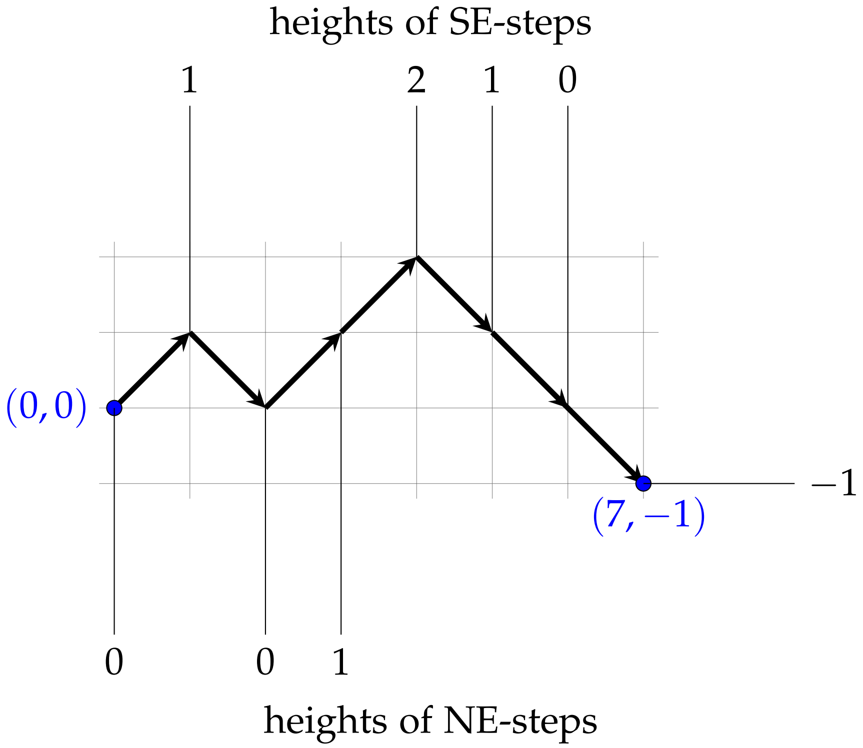
- The **height** $\operatorname{ht}(p)$ of a point p is its y-coordinate.
- The **initial height** and the **final height** of a path $\mathbf{p}$ are the heights of its starting and ending points.
- If $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ is any diagonal path, then we associate three Laurent polynomials (in the indeterminate z) to $\mathbf{p}$:

$$\underbrace{\sum_{i=0}^{k} z^{\operatorname{ht}(p_i)}}_{\text{the \textbf{height polynomial}}}, \qquad \underbrace{\sum_{p_i \text{ is an NE-step of } \mathbf{p}} z^{\operatorname{ht}(p_i)}}_{\text{the \textbf{NE-height polynomial}}}, \qquad \underbrace{\sum_{p_i \text{ is an SE-step of } \mathbf{p}} z^{\operatorname{ht}(p_i)}}_{\text{the \textbf{SE-height polynomial}}}.$$

These encode (respectively) the multiset of heights of all vertices, of all NE-steps, and of all SE-steps of $\mathbf{p}$.

* All concepts defined for paths can be applied to words (just replace a word by its standard path).

- **Example:** For the word $w = UDUUDDD$, the heights are shown in the following:



(The ending point is neither an SE-step nor an NE-step.) Thus, the word w has

$$\begin{aligned}
&\text{height polynomial } z^0 + z^1 + z^0 + z^1 + z^2 + z^1 + z^0 + z^{-1} \\
&\qquad = z^{-1} + 3 + 3z + z^2; \\
&\text{NE-height polynomial } z^0 + z^0 + z^1 = 2 + z; \\
&\text{SE-height polynomial } z^1 + z^2 + z^1 + z^0 = 1 + 2z + z^2.
\end{aligned}$$

- **Lemma:** Balanced commutations and balanced flips leave all three polynomials (NE-height, SE-height, height) as well as the final height unchanged.

3. The main theorems

* **Equivalence Theorem.**

  Let u and v be two words in $\mathcal{M}$. Then, the following seven statements are equivalent:

  $\mathcal{S}_\phi$: The words u and v are ϕ-equivalent (that is, $\phi(u) = \phi(v)$).

  $\mathcal{S}_{\text{pol}}$: The elements $\phi(u)$ and $\phi(v)$ act equally on the polynomial ring $\mathbf{k}[x]$. (Reminder: D acts as $\frac{d}{dx}$, and U as x.)

  $\mathcal{S}_{\text{NE}}$: The words u and v have the same final height and the same NE-height polynomial (i.e., the multiset of heights of the NE-steps of u is the same as for v).

  $\mathcal{S}_{\text{SE}}$: The words u and v have the same final height and the same SE-height polynomial (i.e., the multiset of heights of the SE-steps of u is the same as for v).

  $\mathcal{S}_{\text{ht}}$: The words u and v have the same final height and the same height polynomial (i.e., the multiset of heights of all vertices of u is the same as for v).

  $\mathcal{S}_{\text{comm}}$: The word u can be transformed into v by balanced commutations.

  $\mathcal{S}_{\text{flip}}$: The word u can be transformed into v by balanced flips.

* **Reflection Theorem.** Each balanced word u is ϕ-equivalent to $\omega(u)$.

- **Consequence (word problem):** The ϕ-equivalence of two words u, v of length $\leq n$ can be tested in $O(n)$ time and memory.

- **Remark (easy part of the Equivalence Theorem):** The lemma on the previous page shows that $\mathcal{S}_i \Longrightarrow \mathcal{S}_j$ for all $i \in \{\phi, \text{pol}\}$ and $j \in \{\text{NE}, \text{SE}, \text{ht}\}$.

- **Remark:** The implication $\mathcal{S}_{\text{comm}} \Longrightarrow \mathcal{S}_\phi$ is essentially known.

  Indeed, Dixmier (1968) observed that the Weyl algebra $\mathcal{W}$ is $\mathbb{Z}$-graded ($\deg U = 1$ and $\deg D = -1$) and that its 0-th graded component $\mathcal{W}_0$ is commutative (being spanned by the powers of DU). Hence, any two balanced words (upon application of ϕ) commute (as they lie in $\mathcal{W}_0$). Thus, balanced commutations don't change the ϕ-image.

- There are also other connections to existing results; see below.

4. Up-normal words

- To prove the main theorems, we need some more concepts.

⊛ A word w is called

  - **rising** if it has at least as many U's as D's;
  - **falling** if it has at most as many U's as D's.

  Note that balanced = rising $\wedge$ falling.

⊛ A **down-zig** means a word of the form UD^kU for some $k \geq 2$. Pictorially:

⊛ A word $w \in \mathcal{M}$ is said to be **up-normal** if it is rising and contains no down-zig as a factor.

- **Crucial lemmas.**

  1. Every up-normal word has the form

  $$D^a \, (UD)^{r_1} U \, (UD)^{r_2} U \, \cdots \, (UD)^{r_h} U \, D^b$$

     for some nonnegative integers a, b, h and $r_1, r_2, \ldots, r_h$.

  2. An up-normal word w is uniquely determined by its final height and its height polynomial.

  3. The height, NE-height and SE-height polynomials of a word w mutually determine each other, as long as the final height of w is known.

  4. If a rising word w is not up-normal, then we can write w as $w = upqv$, where u and v are two words, where p is a balanced word starting with a U, and where q is a balanced word starting with a D. (Thus, we can make w lexicographically smaller by applying a balanced commutation.)

- These easily yield:

- **Proposition.** Let $w \in \mathcal{M}$ be a rising word. Then, there exists a unique up-normal word $t \in \mathcal{M}$ that can be obtained from w by balanced commutations.

- Hence, $\mathcal{S}_{\text{ht}} \Longrightarrow \mathcal{S}_{\text{comm}}$ easily follows for rising w. If w is falling, then apply the same argument to $\omega(w)$.

- Also, $\mathcal{S}_{\text{comm}} \Longrightarrow \mathcal{S}_{\text{flip}}$ because any balanced commutation can be written as three balanced flips:

$$\begin{aligned} \ldots pq \ldots &\mapsto \ldots \omega(p) q \ldots \mapsto \ldots \omega(p) \omega(q) \ldots \\ &\mapsto \ldots \omega(\omega(p) \omega(q)) \ldots = \ldots qp \ldots \end{aligned}$$

- Altogether, $\mathcal{S}_{\text{NE}}, \ldots, \mathcal{S}_{\text{flip}}$ are thus equivalent.

5. Height polynomials and the Weyl action

- To link $\mathcal{S}_\phi$ and $\mathcal{S}_{\text{pol}}$ with $\mathcal{S}_3, \ldots, \mathcal{S}_{\text{flip}}$, we need the following easy fact:

- **Proposition.** Let w be a word and $\mathbf{p} = (p_0, p_1, \ldots, p_k)$ be a path corresponding to it. Let $h_i := \operatorname{ht}(p_i)$ for each $i \in \{0, 1, \ldots, k\}$. Then, for each $s \in \mathbb{Z}$, we have

$$(\phi(w))(x^s) = \left(\prod_{p_i \text{ is an SE-step of } \mathbf{p}} (s + h_k - h_{i+1}) \right) \cdot x^{s+h_k-h_0}.$$

  Here, we let $\mathcal{W}$ act on the Laurent polynomial ring $\mathbf{k}[x, x^{-1}]$ (extending the action on $\mathbf{k}[x]$ in the obvious way).

- An ω-reflected version also holds.

- Thus, $\mathcal{S}_{\text{pol}} \Longrightarrow \mathcal{S}_{\text{SE}}$ can be proved by uniqueness of roots of a polynomial.

  The rest of the proofs are easy.

6. Enumeration

6.1. All words

∗ For $0 \leq k \leq n$, let $a(n,k)$ be the number of ϕ-equivalence classes of words with k many D's and $n-k$ many U's.

- **Easy fact.** We have

$$a(n,0) = a(n,n) = 1 \qquad \text{and} \qquad a(n,k) = a(n,n-k).$$

- **Theorem.** For $n > 2k \geq 0$, we have

$$a(n,k) = a(n-1,k) + a(n-2,k-1).$$

- **Proposition.** For $k > 0$, we have

$$a(2k,k) = (k+3)2^{k-2} \quad \text{and} \quad a(2k+1,k) = (k+2)2^{k-1}.$$

- **Theorem.** We have

$$\sum_{n\geq 0} \sum_{0\leq k\leq n} a(n,k) t^k x^n = \frac{(1-3tx^2+t^2x^4)(1-tx^2)^2}{(1-tx-tx^2)(1-x-tx^2)(1-2tx^2)^2}$$

or equivalently

$$\sum_{n\geq 0} \sum_{0\leq k\leq n/2} a(n,k) t^k x^n = \frac{(1-tx^2)^3}{(1-x-tx^2)(1-2tx^2)^2}.$$

- **Theorem.** For all n and k with $0 \leq k \leq n/2$, we have

$$a(n,k) = \sum_{j=0}^{k} (k-j+1) \binom{n-k-1}{j}.$$

- **Corollary.** The total number of ϕ-equivalence classes of words of length $n > 0$ is

$$\sum_{k=0}^{n} a(n,k) = 2 \underbrace{F_{n+4}}_{\text{Fibonacci}} - \begin{cases} (3n+42)2^{n/2-3}, & \text{if } n \text{ even,} \\ (n+15)2^{(n-3)/2}, & \text{if } n \text{ odd.} \end{cases}$$

-

n	0	1	2	3	4	5	6	7	8	9	10
$\sum_k a(n,k)$	1	2	4	8	15	28	50	90	156	274	466

6.2. c-Dyck words

- ⊛ Fix a constant $c > 0$, and let $\mathcal{M}_c \subseteq \mathcal{M}$ be the set of words whose every prefix has at least c times as many U's as D's. This is a submonoid of $\mathcal{M}$.

  Let $a_c(n,k)$ be the number of ϕ-equivalence classes of words in $\mathcal{M}_c$ that consist of k D's and $n-k$ U's.

- **Lemma.** For every real constant $c \geq 1$ and any positive integers n and k with $n-1 \geq (c+1)k$, we have

$$a_c(n,k) = a_c(n-1,k) + a_c(n-2,k-1).$$

- **Theorem.** If c is a positive integer and n,k are positive integers with $n \geq (c+1)k$, then

$$a_c(n,k) = \binom{n-k-1}{k} - (c-2)\sum_{j=0}^{k-1}\binom{n-k-1}{j}.$$

- In particular,

$$a_1(n,k) = \sum_{j=0}^{k}\binom{n-k-1}{j} \qquad \text{and}$$
$$a_2(n,k) = \binom{n-k-1}{k},$$

  both of which can be proved bijectively.

6.3. The size of an equivalence class

- **Theorem.** Let $w \in \mathcal{M}$ be a word with NE-height polynomial $\sum_{i\in\mathbb{Z}} a_i z^i$ and SE-height polynomial $\sum_{i\in\mathbb{Z}} b_i z^i$. Then, the size of the ϕ-equivalence class containing w is

$$\prod_{i\geq 0}\binom{a_i+b_{i+2}-1}{b_{i+2}}\binom{b_{-i}+a_{-i-2}-1}{a_{-i-2}}$$
$$\cdot\begin{cases}\binom{a_0+b_0}{a_0}, & \text{if } w \text{ is balanced;}\\ \binom{a_0+b_0-1}{b_0}, & \text{if } w \text{ is rising and non-balanced;}\\ \binom{a_0+b_0-1}{a_0}, & \text{if } w \text{ is falling and non-balanced.}\end{cases}$$

7. Bond percolation

- Consider again the number of all ϕ-equivalence classes of words of length n:

n	0	1	2	3	4	5	6	7	8	9	10
$\sum_k a(n,k)$	1	2	4	8	15	28	50	90	156	274	466

This agrees with OEIS sequence A006727 "Bond percolation series for square lattice" up to $n = 11$ (but no further). Why?

- **Bond percolation on the directed square lattice** is one of the fundamental problems in statistical physics.

  In our language:

  - Fix a number $p \in [0,1]$.
  - Consider the integer lattice $\mathbb{Z}^2$.
  - A **site** is a lattice point (i,j) with $i \equiv j \bmod 2$.
  - The NE-arcs and the SE-arcs are called **bonds**. They connect sites.
  - Each bond is **open** with probability p and **closed** with probability $1-p$ (all bonds are independent).
  - Fluid is dropped at the origin $(0,0)$ and flows left-to-right across open bonds.
  - Which sites end up wet (i.e., eventually get some fluid)?

- We are interested in the infinite sum

$$S(p) := \sum_{\text{sites } (i,j)} \text{Prob}\left(\text{the site } (i,j) \text{ ends up wet}\right)$$

and its finite approximations

$$S_n(p) := \sum_{\substack{\text{sites } (i,j);\\ i \leq n}} \text{Prob}\left(\text{the site } (i,j) \text{ ends up wet}\right).$$

The latter is an explicit polynomial in p.

The former is a power series, and can be found as the coefficientwise limit $\lim_{n\to\infty} S_n(p)$.

- OEIS sequence A006727 "Bond percolation series for square lattice" is the sequence of coefficients of this series

$$S(p) = 1 + 2p + 4p^2 + 8p^3 + 15p^4 + 28p^5 + \cdots .$$

  Why do they agree with $\sum_k a(n,k)$ up to $n = 11$?

  There is some relation: $\sum_k a(n,k)$ is the total # of ϕ-equivalence classes of paths from $(0,0)$ to all sites (i,j) with $i = n$.

  Is there a clearer connection?

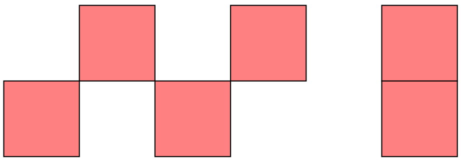
- Note that $S(p)$ has some negative coefficients, e.g. $-48816119038p^{50}$.

8. The rook theory connection

- The Weyl algebra $\mathcal{W}$ is known to be connected to **rook theory**.

⊛ Recall the standard definitions:

  - A **cell** is a pair (i, j) of positive integers, drawn as a 1×1-square.
  - A **board** is a finite set of cells, e.g.,



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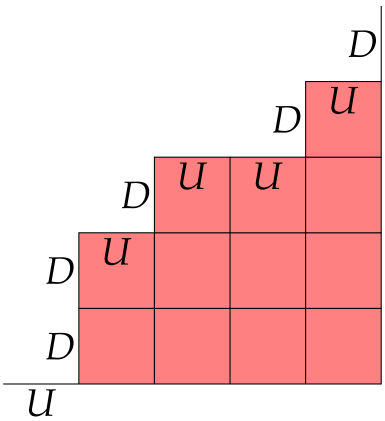
  - A **rook placement** of a board B is a subset S of B such that no two cells in S lie in the same row or column.
  - The **k-th rook number** $r_k(B)$ of a board B is the number of k-element rook placements of B.

    For example, the above example board has $r_0(B) = 1$ and $r_1(B) = |B| = 6$ and $r_2(B) = 8$ and $r_k(B) = 0$ for all $k > 2$.
  - Two boards B and C are said to be **rook-equivalent** if they share the same rook numbers (i.e., if $r_k(B) = r_k(C)$ for all $k \in \mathbb{N}$).
  - A well-known class of boards are the Ferrers boards:

    If w is a word in $\mathcal{M}$, then the **Ferrers board** B_w is a contiguous set of cells, whose bottom and right boundaries are straight lines, whereas the rest of its boundary is a jagged path that (when walked from southwest to northeast) takes a north-step for each D in w and an east-step for each U in w (reading the word w from left to right).

    For instance, if $w = UDDUDUUDUD$, then

$B_w =$



- **Theorem (Navon 1973).** Let $w \in \mathcal{M}$ be any word that contains n many D's and m many U's. Then, in $\mathcal{W}$, we have

$$\phi(w) = \sum_{k=0}^{\min\{m,n\}} r_k(B_w) U^{m-k} D^{n-k}.$$

- Consequently:

- **Equivalence Theorem +.** Let u and v be two words in $\mathcal{M}$. Assume that u and v have the same # of U's and the same # of D's. Then, the seven statements $\mathcal{S}_\phi, \mathcal{S}_{\text{pol}}, \ldots, \mathcal{S}_{\text{flip}}$ of the Equivalence Theorem are also equivalent to the following statement:

  - $\mathcal{R}_1$: The boards B_u and B_v are rook-equivalent.

- **Remark:** The Ferrers board B_w "does not see" any U's at the beginning of w and any D's at the end of w. Thus, ϕ-equivalence is stronger than rook-equivalence in general. Hence the need for the extra assumption in the Equivalence Theorem +.

- **Remark:** The implication $\mathcal{R}_1 \Longrightarrow \mathcal{S}_\phi$ is folklore among rook theorists. What else follows from rook theory?

  - A classical result of Foata and Schützenberger (1970) shows that each Ferrers board is rook-equivalent to a unique "increasing Ferrers board". These "increasing Ferrers boards" are somewhat similar to our up-normal words, but not quite in bijection.

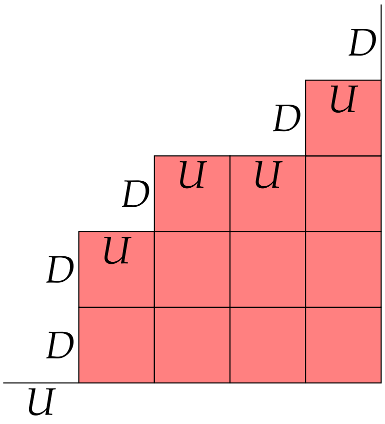
  - Foata and Schützenberger have their own kind of moves that they use to normalize a Ferrers board modulo rook equivalence: the "(k,k')-transforms". These appear to be close relatives of our balanced flips.

(*) Rook equivalence has a surprising interpretation (Haglund 1998 and Cotardo/Gruica/Ravagnani 2023):

For any finite field F, any nonnegative integers n and k, and any board $B \subseteq \{1,2,\ldots,n\}^2$, we define $P_k(B/F)$ to be the number of $n \times n$-matrices $A \in F^{n\times n}$ of rank k such that all entries of A in cells outside of B are zero.

- **Example:**

$B_w =$



$$\Longrightarrow \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & y & z & w \\ 0 & p & q & r & s \\ 0 & a & b & c & d \end{pmatrix}.$$

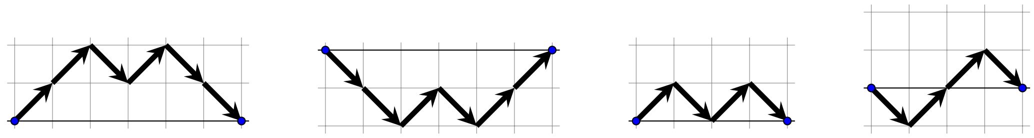
- **Equivalence Theorem ++.** Let u and v be two words in $\mathcal{M}$ that have the same # of U's and the same # of D's. Then, the seven statements $\mathcal{S}_\phi, \mathcal{S}_{\text{pol}}, \ldots, \mathcal{S}_{\text{flip}}$ of the Equivalence Theorem are also equivalent to the following statements:
  - $\mathcal{R}_1$: The boards B_u and B_v are rook-equivalent.
  - $\mathcal{R}_2$: For any finite field F and any $k \in \mathbb{N}$, we have $P_k(B_u/F) = P_k(B_v/F)$.
  - $\mathcal{R}_3$: For any finite field F, we have $P_1(B_u/F) = P_1(B_v/F)$.
- The proofs follow from Cotardo/Gruica/Ravagnani 2023, which also gives some further equivalent conditions (Corollary 3.2 loc. cit.).

9. Irreducible balanced words

- The Equivalence Theorem shows that ϕ-equivalence is "generated" by balanced commutations.
- But do we need **all** balanced commutations, or can we make do with a proper subset?
- The latter is the case. I don't know what the smallest sufficient set of balanced commutations is (I don't think it is even unique), but here is one that suffices:
- We define special classes of balanced words:
  - A **UIB word** means a balanced word that begins with a U and whose standard path never returns to the x-axis before its ending point.
  - A **DIB word** means a balanced word that begins with a D and whose standard path never returns to the x-axis before its ending point.

  Examples:

  UIB DIB neither neither



- An **irreducible balanced commutation** means a balanced commutation $pxyq \mapsto pyxq$ in which one of the words x and y is UIB and the other is DIB.
- **Theorem.** Any two ϕ-equivalent words can be transformed into each other by a sequence of irreducible balanced commutations.

10. Other algebras

- What if we replace the Weyl algebra $\mathcal{W} = \mathbf{k}\langle D, U \mid DU - UD = 1\rangle$ by other algebras?

10.1. Multivariate Weyl algebras

- For any $n \in \mathbb{N}$, there is an n-**Weyl algebra** $\mathcal{W}_n$, defined as the $\mathbf{k}$-algebra given by $2n$ generators $D_1, D_2, \ldots, D_n, U_1, U_2, \ldots, U_n$ and relations

$$\begin{aligned} D_iU_j &= U_jD_i && \text{for all } i \neq j; \\ D_iU_i &= U_iD_i + 1 && \text{for all } i; \\ D_iD_j &= D_jD_i && \text{for all } i, j; \\ U_iU_j &= U_jU_i && \text{for all } i, j. \end{aligned}$$

  This is $\cong$ the $\mathbf{k}$-algebra of differential operators on the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_n]$.

- It is also $\cong$ the n-fold tensor power $\mathcal{W}^{\otimes n}$ of the original Weyl algebra $\mathcal{W}$.

  Any word in the $D_1, D_2, \ldots, D_n, U_1, U_2, \ldots, U_n$ corresponds to a tensor product $w_1 \otimes w_2 \otimes \cdots \otimes w_n$ of n words $w_i \in \mathcal{M}$.

  The following theorem reduces the ϕ-equivalence problem to that in $\mathcal{M}$:

- **Theorem.** Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be $2n$ words in $\mathcal{M}$. Then,

$$\phi(u_1) \otimes \phi(u_2) \otimes \cdots \otimes \phi(u_n) = \phi(v_1) \otimes \phi(v_2) \otimes \cdots \otimes \phi(v_n) \text{ in } \mathcal{W}^{\otimes n}$$

  if and only if

$$\phi(u_i) = \phi(v_i) \text{ for all } i.$$

10.2. Characteristic p

- Hitherto we have assumed that the field $\mathbf{k}$ has characteristic 0.
- If $\mathbf{k}$ has characteristic $p \neq 0$ instead, some things go south:
  - the words $U^{p+1}D$ and UDU^p become p-equivalent despite not satisfying $\mathcal{S}_{\text{NE}}, \ldots, \mathcal{S}_{\text{flip}}$;

  - the action of $\mathcal{W}$ on $\mathbf{k}[x]$ is no longer faithful.

  The implications $\mathcal{S}_{\text{NE}}, \ldots, \mathcal{S}_{\text{flip}} \Longrightarrow \mathcal{S}_{\phi} \Longrightarrow \mathcal{S}_{\text{pol}}$ still hold, but both arrows are proper implications.

- Actually, U^p and D^p are central in $\mathcal{W}$ if $\operatorname{char} \mathbf{k} = p$.

  Thus, we get an additional kind of commutations where we swap a U^p or D^p factor with any neighboring factor.

  Question: Do these suffice? If not, what transformations are needed?

- The Weyl algebra $\mathcal{W}$ in characteristic p also has three quotients

$$\mathcal{W}^- := \mathcal{W} / (\mathcal{W}U^p\mathcal{W}), \qquad \mathcal{W}_- := \mathcal{W} / (\mathcal{W}D^p\mathcal{W}),$$
$$\mathcal{W}_-^- := \mathcal{W} / (\mathcal{W}U^p\mathcal{W} + \mathcal{W}D^p\mathcal{W}),$$

  and we can study a variant of ϕ-equivalence on each of them.

- **Question:** Find analogues of the main theorems for $\mathcal{W}$, $\mathcal{W}^-$, $\mathcal{W}_-$ and $\mathcal{W}_-^-$.

10.3. Down-up algebras

- We now return to the case when $\mathbf{k}$ is a field of characteristic 0.

- (*) The Weyl algebra $\mathcal{W}$ has several deformations and variations. One of the most general is the **down-up algebra** $\mathcal{A}(\alpha, \beta, \gamma)$ (due to Benkart/Roby 1998), defined for any three scalar parameters $\alpha, \beta, \gamma \in \mathbf{k}$. It is the $\mathbf{k}$-algebra with generators D and U and the two relations

$$D^2U = \alpha DUD + \beta UD^2 + \gamma D \qquad \text{and}$$
$$DU^2 = \alpha UDU + \beta U^2D + \gamma U.$$

- **Examples:**

  - $\mathcal{A}(2, -1, 0)$ is the **homogenized Weyl algebra** (with generators D and U and relations $[D, DU - UD] = 0$ and $[U, DU - UD] = 0$). This is also the universal enveloping algebra $U(\mathfrak{H})$ of the Heisenberg Lie algebra $\mathfrak{H}$.
  - $\mathcal{A}(2, -1, -2) = U(\mathfrak{sl}_2)$.

- If $\alpha + \beta = \gamma - \beta = 1$, then $\mathcal{W}$ is a quotient of $\mathcal{A}(\alpha, \beta, \gamma)$.

- Define a map $\phi : \mathcal{M} \to \mathcal{A}(\alpha, \beta, \gamma)$ in the same way as we defined $\phi : \mathcal{M} \to \mathcal{W}$. Surprisingly:
- **Proposition:** If two words u and v in $\mathcal{M}$ are ϕ-equivalent for $\mathcal{W}$, then they are also ϕ-equivalent for $\mathcal{A}(\alpha, \beta, \gamma)$.
- We can show that the converse holds whenever $\alpha + \beta = \gamma - \beta = 1$, but we suspect that it holds in more cases.

  But not always! e.g., not in:

  1. the case $(\alpha, \beta, \gamma) = (0, 1, 0)$ (here, we have $\phi(DU^2) = \phi(U^2D)$);
  2. more generally, the case $\alpha = \gamma = 0$ and arbitrary β (here we have $\phi(DU^4D) = \phi(U^2D^2U^2)$);
  3. the case $(\alpha, \beta) = (0, -1)$ and arbitrary γ (here we have $\phi(DU^4) = \phi(U^4D)$);
  4. the case $(\alpha, \beta) = (-1, -1)$ and arbitrary γ (here we have $\phi(DU^3) = \phi(U^3D)$);
  5. the case $(\alpha, \beta) = (1, -1)$ and arbitrary γ (here we have $\phi(DU^6) = \phi(U^6D)$).

10.4. Bonus question: Hecke algebras

- **Question:** Consider the Hecke algebra $\mathcal{H}_n(q)$ of the symmetric group S_n, where q is an indeterminate (i.e., generic). When do two words $i_1 i_2 \ldots i_k$ and $j_1 j_2 \ldots j_\ell$ over the alphabet $\{1, 2, \ldots, n-1\}$ satisfy $T_{i_1} T_{i_2} \cdots T_{i_k} = T_{j_1} T_{j_2} \cdots T_{j_\ell}$?

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