Reflection Equation algebras versus Quantum Groups

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Let us recall the notion of the second Sklyanin Poisson bracket. Let G be a classical Lie group and $Fun(G)$ be the space of functions on it. Then the bracket

$$\{f,g\}_{Sk} = \circ(\rho_l(r)(f\otimes g) - \rho_r(r)(f\otimes g))$$

is Poisson on G, if

$$r = \sum_{\alpha>0} X_\alpha \otimes X_{-\alpha} - X_{-\alpha}\otimes X_\alpha.$$

As usual we assume the generators X_α to be normalized so that

$$< X_\alpha, X_{-\alpha} >= 1.$$

If $G = SL(N)$, this bracket is well defined on the enveloping vector space (i. e. we cancel the condition $det T = 1$). We assume it to be defined on $Fun(gl(N)^*) = Sym(gl(N))$ and denote it $\{.,.\}_{Sk}$. This bracket possesses the following property

$$\{f,g\}_{Sk} = 0, \quad f,g \in Sym(gl(N))$$

for any two functions central w.r.t. to the linear gl bracket.
As such functions we can take TrT^k, where $T = \|t_i^j\|_{1\le i,j\le N}$ and

$$\{t_i^j, t_k^l\}_{gl(N)} = l_i^j\, \delta_k^l - l_k^l\, \delta_i^j.$$

It should be emphasized that the brackets $\{.,.\}_{Sk}$ and $\{.,.\}_{gl(N)}$ are not compatible. Instead, the bracket $\{.,.\}_{Sk}$ is compatible with its linearization, which is linear and is called the first Sklyanin bracket.

In the late 80's I studied another "r-matrix bracket"

$$\{f, g\}_r = \circ(r_{ad,ad}(f \otimes g)), \;\; f, g \in Sym(gl(N))$$

where $r \in \mathfrak{g}^{\otimes 2}$ is a bi-vector.
However, this expression is a Poisson bracket if r is skew-symmetric and satisfies the classical YB equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

As an example of such r we can take $H \wedge X \in sl(2)^{\wedge 2}$.

Drinfeld quantized such bi-vectors, i.e. for each such r he constructed a braiding R, i.e. an operator $R : V^{\otimes 2} \to V^{\otimes 2}$ such that it meets the braid relation

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23},$$

and is involutive $R^2 = I$.
So, this braiding cannot come from a QG.
However, by using it I introduced a "braided version of Lie algebra $gl(N)$" in the space $End(V)$ by putting

$$[X, Y] = \circ(X \otimes Y - \mathcal{R}(X \otimes Y)),$$

where $\mathcal{R} : End(V)^{\otimes 2} \to End(V)^{\otimes 2}$ is an extension of the involutive braiding R. (Note that $\mathcal{R}$ is also an involutive braiding.)

Fortunately, if $\mathfrak{g} = sl(N)$ and r is as above, the bracket $\{f, g\}_r$ can be slightly modified and thus we can get after a quantization an algebra related to the QG $U_q(sl(N))$.

If $\mathfrak{g} = sl(N)$ in the decomposition of the space $\mathfrak{g}^{\otimes 2}$ onto irreducible $\mathfrak{g}$-modules the component $\mathfrak{g}$ itself comes twice: once in $Sym^{(2)}(\mathfrak{g})$ and once in $\bigwedge^{(2)}(\mathfrak{g})$.

By considering a properly normalized $sl(N)$-covariant map

$$\alpha : Sym^{(2)}(\mathfrak{g}) \to \bigwedge^{(2)}(\mathfrak{g}),$$

and by adding it to the r-matrix above, we get a Poisson bracket onto the space $Sym(gl(N))$.

This Poisson bracket will be denoted $\{.,.\}_{RE}$ since its quantum counterpart is the Reflection Equation (RE) algebra.

It should be emphasized that the bracket $\{.,.\}_{RE}$ is compatible with that $\{.,.\}_{gl(N)}$ and their linear combination can be quantized via the so-called modified RE algebra, defined via the system

$$R\,\hat{L}_1\,R\,\hat{L}_1 - \hat{L}_1\,R\,\hat{L}_1\,R = R\,\hat{L}_1 - \hat{L}_1\,R,\;\; \hat{L} = \|\hat{l}_i^j\|.$$

This algebra is a two-parameter deformation of the algebra $Sym(gl(N))$.

For other classical Lie algebras such a map α does not exist: in the decomposition of $\mathfrak{g}^{\otimes 2}$ onto the irreducible components $\mathfrak{g}$ itself comes once.

This is the reason why a similar deformation of the algebra $U(\mathfrak{g})$ (and even $Sym(\mathfrak{g})$) for $\mathfrak{g}$ belonging to other series does not exist.

However, for the algebras $Fun(G)$ there exist two deformations: one of RTT type (arising from the Sklyanin bracket) and the other of RE type arising from the bracket $\{\,,\,\}_{RE}$, defined on the group.

What are quantum analogs of the Poisson brackets $\{.,.\}_{Sk}$ and $\{.,.\}_{RE}$?
For the former bracket it is the RTT algebra, generated by the entries of the matrix $T = \|t_i^j\|$ subject to

$$R\, T_1\, T_2 = T_1\, T_2\, R.$$

For the latter one it is the so-called Reflection Equation algebra, generated by the entries of the matrix $L = \|l_i^j\|$ subject to

$$R\, L_1\, R\, L_1 = L_1\, R\, L_1\, R.$$

The both algebras are unital.

The braiding R entering the both definitions is the so-called Hecke symmetry since it meets the Hecke condition

$$(R - q\,I)(R + q^{-1}\,I) = 0.$$

We assume q to be generic, i.e. such that $k_q = \frac{q^k - q^{-k}}{q - q^{-1}} \neq 0$. Nevertheless, there exists a lot of Hecke symmetries, which are not necessarily related to a Lie algebra. For such R the both algebras can be defined by the same formulas. We denote them correspondingly $\mathcal{T}(R)$ and $\mathcal{L}(R)$.
If R comes from the QG $U_q(gl(N))$, it and all corresponding objects are called standard.

In the both algebras it is possible to define quantum determinants $det_q\, T$ and $det_q\, L$. In all algebras $\mathcal{L}(R)$ the quantum determinants are central. In the algebras $\mathcal{T}(R)$ its centrality depends on R.
If R is standard, the quantum determinant in the RTT algebra is central.
As for the QG $U_q(sl(N))$ itself, it can be defined as the Hopf dual to the algebra $\mathcal{T}(R)/ < det_q\, T - 1 >$.
Observe that while the RE algebras $\mathcal{L}(R)$ for any R, deforming the usual flip P (for instance the Crammer-Gervais one), is a deformation of the enveloping algebra $U(gl(N))$, the QG $U_q(sl(N))$ deforms only the coalgebraic structure of $U(sl(N))$.

There exists a construction describing the center of the QG. In the paper [RTF] 1989 (Leningrad Math. J.) there are defined two matrices T^+ and T^- composed with entries belonging to the QG $U_q(gl(N))$ which are upper (resp., low) triangular and meet the RTT type relations

$$R\, T_1^{\pm}\, T_2^{\pm} = T_1^{\pm}\, T_2^{\pm}\, R, \;\; R\, T_1^{+}\, T_2^{-} = T_1^{-}\, T_2^{+}\, R.$$

Also, a condition on the diagonal components is imposed: $T^- = T^+$ on the diagonal.
Let us introduce the matrix $L = T^+(T^-)^{-1}$. Then according to RTF the elements $Tr_q\, L^k,\;\; k = 1, 2...$ belong to the center of the QG $U_q(gl(N))$.
Note that the matrix L generates the RE algebra. Thus, we get a map from the RE algebra to the QG. Note that this map is not unique.

Here Tr_q is the quantum trace, defined as follows

$$Tr_q A = Tr C_q\, A,$$

where C_q is a diagonal matrix, composed from powers of q. Thus, if $N = 2$, then it can be taken as $C_q = diag(q^{-1}, q)$.
However, for "exotic" Hecke symmetries, constructed by myself, the corresponding the matrix $C_q = C_R$ can be non-diagonal. The problem of classification of the matrices C_R is equivalent to classification of the Hecke symmetries.

Namely, the realization of the QG $U_q(sl(N))$ via the RTT relations from [RTF] was used in the papers by Jing, Liu, Molev "Eigenvalues of quantum Gelfand invariants" and "The q-immanants and higher Capelli identities".
By Gelfand invariants the authors mean the elements $Tr_q L^k$ (we call them power sums). They computed the values of the power sums in the irreps of the QG and thus they obtained a quantum analog of the Perelomov-Popov formula. The PP formula says that in the irreducible $U(gl(N))$-module V_λ labeled by the partition $\lambda = (\lambda_1 \geq \lambda_2 ... \geq \lambda_N)$, where λ_i are non-negative integers, the quantities $Tr\, L^k$ become scalar operators $\chi_\lambda(TrL^k)I$ and

$$\chi_\lambda(TrL^k) = \sum_i (\lambda_i + N - i)^k \prod_{p \neq i} \frac{\lambda_p - \lambda_i - p + i + 1}{\lambda_p - \lambda_i},$$

However, the problem of describing the center is more natural for the RE algebras than for the QG. Let R be any even (or $GL(N)$ type) Hecke symmetry (for instance the standard one). Then the generating matrix $L=\|l_i^j\|$ of the corresponding RE algebra is subject to the Cayley-Hamilton identity, i.e. a polynomial relation

$$\mathcal{P}(L)=0,\ \text{where } \mathcal{P}(t)=\sum_{k=0}^{K}(-1)^{N-k}a_k\,t^k$$

is a polynomial with central coefficients.
Then let us introduce "eigenvalues" μ_i of the matrix L in a natural way

$$\sum_i \mu_i=a_{N-1},\ \sum_{i<j}\mu_i\,\mu_j=a_{N-2}\ \dots\ \prod_i \mu_i=a_0.$$

They are assumed to be central in the extended algebra $\mathcal{L}(R)[\mu_1...\mu_K]$.

We assume the leading coefficient to be 1. (It is possible since R is even.)
If R is a deformation of the super-flip $P_{m|n}$, then $K = m + n$. In this case the leading coefficient is not a number. In this case we introduce two families of "eigenvalues" μ_i, $i = 1...m$ and ν_i, $i = 1...n$ which are assumed to be central in the extended algebra $\mathcal{L}(R)[\mu_i, \nu_i]$.
Anyway, a polynomial relation with central coefficients for a matrix with non-commutative entries is a very exceptional property. It is known for the generating matrices of the enveloping algebras of some Lie algebras and for the RE algebras. For a matrix composed of entries from the QG $U_q(gl(N))$ this property is possible since the corresponding RE algebra is embedded in the QG $U_q(gl(N))$. This construction is known only for the standard R.

The eigenvalues μ_i and ν_i are very useful tools for parametrization of the central elements from the RE algebra. Thus, the power sums can be parameterized via the eigenvalues μ_i and ν_i as follows

$$p_k(L) = \sum_{i=1}^{m} \mu_i^k\, d_i + \sum_{j=1}^{n} \nu_j^k\, f_j,$$

$$d_i = q^{-1} \prod_{p\neq i}^{m} \frac{\mu_i - q^{-2}\,\mu_p}{\mu_i - \mu_p} \prod_{j=1}^{n} \frac{\mu_i - q^2\,\nu_j}{\mu_i - \nu_j},$$

$$f_j = -q \prod_{i=1}^{m} \frac{\nu_j - q^{-2}\,\mu_i}{\nu_j - \mu_i} \prod_{p\neq j}^{n} \frac{\nu_j - q^2\,\nu_p}{\nu_j - \nu_p}.$$

Note that the eigenvalues μ_i, ν_i become scalar operators in irreps of $\mathcal{L}(R)$. Thus, we get three questions

1. How to construct irreps of $\mathcal{L}(R)$?
2. What are values of the quantities μ_i, ν_i in these modules?
3. What is Tr_R? More generally, what are unusual traces and in which algebras they have to be introduced?

Recently, Mikhail Zaitzev (Saponov's student) has computed the value of μ_i in the modules V_λ under condition that R is even

$$\chi_\lambda(\mu_i) = q^{-2(\lambda_i - i + K)}.$$

Here K is the degree of the CH polynomial for L.
This result enables us to compute the characters $\chi_\lambda(p_k)$ and the characters of the Schur polynomials.
In particular, the quantities $\chi_\lambda(p_k)$ are equal to these computed in [JLM] but in a much more general situation. Observe once more that we do not use any object of QG type. We are dealing only with the RE algebras, associated with any even Hecke symmetry.
Note that the PP result can be obtained by a "proper" passage to the limit $q = 1$.

Now, we compare the properties of the RTT algebra and these of the RE one.
First of all, in the algebra $\mathcal{L}(R)$ there is a center looking like that of $U(gl(N))$ if R is even or $U(gl(m|n))$ if R is not even.
In the algebra $\mathcal{T}(R)$ the center contains only $det_R T$ (for some R).
However, in this algebra there is a commutative subalgebra (we call it Bethe), generated by the following elements

$$Tr_{R(1...k)} T_1 ... T_k\, f(R_1, ..., R_{k-1}),$$

where f is any polynomial in $R_1, ..., R_{k-1}$.
Note that the center in the RE algebra $\mathcal{L}(R)$ is constructed by similar formulas

$$Tr_{R(1...k)} L_{\overline{1}} ... L_{\overline{k}}\, f(R_1, ..., R_{k-1}),$$

where

$$L_{\overline{1}} = L_1, \quad L_{\overline{k}} = R_k\, L_{\overline{k-1}}\, R_k^{-1}.$$

Also, note that in the definition of the Bethe subalgebra of the RTT algebra it is possible to use the usual trace instead of the R-trace. Let us explain this claim. We have

$$Tr_{R(1\dots k)} T_1 \dots T_k\, f = Tr_{(1\dots k)} C_1\, C_2 \dots C_k\, T_1 \dots T_k\, f$$

Let us denote $C_i\, T_i = \tilde{T}_i$. Thus we have

$$Tr_{(1\dots k)} C_1\, C_2 \dots C_k\, T_1 \dots T_k\, f = Tr_{(1\dots k)} \tilde{T}_1 \dots \tilde{T}_k\, f$$

. Also, observe that in virtue of the relation

$$R\, C_1\, C_2 = C_1\, C_2\, R,$$

we have that the matrices $\tilde{T}_i$ are subject to the RTT relation

$$R\, \tilde{T}_1\, \tilde{T}_2 = \tilde{T}_1\, \tilde{T}_2\, R.$$

However, in the RE algebras we have to use the R-trace.

Other discrepancies of these algebras are
1. The RE algebra (in the standard case) can be equipped with the adjoint action of the QG $U_q(sl(N))$ but the RTT algebra cannot.
2. The RTT algebra can be equipped with a bi-algebra structure whereas the RE algebra can be equipped with the braided bi-algebra structure.
3. On the RE algebra it is possible to introduce analogs of partial derivatives in the generators and thus to introduce an analog of the Weyl-Heisenberg algebra

These "quantum partial derivatives" are introduced via the following system

$$R\,L_1\,R\,L_1 = L_1\,R\,L_1\,R,$$
$$R^{-1}\,D_1\,R^{-1}\,D_1 = D_1\,R^{-1}\,D_1\,R^{-1},$$
$$D_1\,R\,M_1\,R = R\,M_1\,R^{-1}\,D_1 + R.$$

The first line defines a RE algebra $\mathcal{L}(R)$. The second line defines a RE algebra $\mathcal{D}(R^{-1})$. The third line is the so-called permutation relation between two algebras.
Namely, the entries of the matrix $D = \|\partial_i^j\|$, generating the algebra $\mathcal{D}(R^{-1})$, play the role of partial derivatives in l_i^j:

$$\partial_i^j = \frac{d}{d\,l_j^i}.$$

Note that in the classical limit (as $R = P$) the above system defines Weyl-Heisenberg algebra. Now, we describe its role in the Capelli identity and exhibit its quantum analogs.
First consider a classical low-dimensional example $n = 2$. Let

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; D = \begin{pmatrix} \partial_a & \partial_c \\ \partial_b & \partial_d \end{pmatrix}.$$

The matrix L has commutative entries, which generate $Sym(gl(2))$. The matrix D is composed from the partial derivatives. Then the matrix $\hat{L} = LD$ is subject to the relation

$$P\hat{L}P\hat{L} - \hat{L}P\hat{L}P = P\hat{L} - \hat{L}P.$$

This means that the matrix $\hat{L}$ is the generating matrix of the algebra $U(gl(2))$. Otherwise stated, the algebra $U(gl(2))$ is represented by differential operators. For $n > 2$ this construction is also valid.

In the current q-setting there is a similar statement.

Theorem.

Let $L = \|l_i^j\|_{1\le i,j\le N}$ be the generating matrix of an algebra $\mathcal{L}(R)$ and $D = \|\partial_i^j\|_{1\le i,j\le N}$ be the matrix composed of the partial derivatives, i.e. the matrices L and D are subject to the above system. Then the matrix

$$\hat{L} = L\,D$$

generates the corresponding modified RE algebra, i.e. it meets the relation

$$R\,\hat{L}_1\,R\,\hat{L}_1 - \hat{L}_1\,R\,\hat{L}_1\,R = R\,\hat{L}_1 - \hat{L}_1\,R.$$

This presentation of the matrix $\hat{L}$ is used for "quantum generalizations" of the Capelli identity. First, we remind the classical one.

Let $\hat{L} = L\,D$ in the classical setting. Then we have

$$rDet(\hat{L} + K) = detL\,detD,$$

where K is the diagonal matrix $diag(0, 1, ..., n-1)$ and $rDet$ is the so-called row-determinant.

Observe that the term $rDet(\hat{L} + K)$ in the l.h.s. can be written in the following form

$$cDet(\hat{L} + K) = detL\,detD,$$

where K is the diagonal matrix $diag(n-1, ...1, 0)$ and $cDet$ is the so-called column-determinant. Also, the matrix form

$$Tr_{1..N}A^{(N)}\hat{L}_1\,(\hat{L} + I)_2\,(\hat{L} + 2I)_3...(\hat{L} + (N-1)I)_N$$

is possible.

Proposition.

In the RE algebra the following holds

$$Tr_{R(1\dots m)}\, A^{(m)}\, \hat{L}_1\, (\hat{L}_{\overline{2}} + q\, I)(\hat{L}_{\overline{3}} + q^2\, 2_q\, I)\dots(\hat{L}_{\overline{m}} + q^{m-1}\, (m-1)_q\, I) =$$

$$q^{-m}\, det_R\, L\, det_{R^{-1}}\, D.$$

Here $(m|0)$ is the bi-rank of R. (Note that in the classical case $m = N$.)

Here, the determinants are the highest elementary polynomials, mentioned above (recall that R here is even). Now, we describe its far-going generalizations. In the classical setting they were obtained by A.Okounkov and called higher Capelli identities. They are based on the notion of immanants, also introduced by Okounkov.

Now, we are going to introduce a far-going generalization of the Capelli identity. Let us recall some basic notions.

Definition.

The *Hecke algebra* of A_{n-1} type $\mathbb{H}_n(q)$ is a unital associative algebra over $\mathbb{C}$, generated by the Artin generators g_i, $i = 1, ..., n-1$ that satisfy the following relations:

$$g_i g_j = g_j g_i, \quad |i-j| > 1,$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \quad 1 \le i \le n-2,$$

$$g_i^2 = 1_{\mathbb{H}_n(q)} + (q - q^{-1}) g_i, \quad 1 \le i \le n-2,$$

$q \in \mathbb{C} \setminus \{0, \pm 1\}$.

For a generic q the Hecke algebra $\mathbb{H}_n(q)$ is isomorphic to the group algebra of the symmetric group $\mathbb{C}[S_n]$. Consequently, $\mathbb{H}_n(q)$ is semisimple and is isomorphic to the product of matrix algebras over $\mathbb{C}$:

$$\mathbb{H}_n(q) \cong \prod_{\lambda \vdash n} Mat_{d_\lambda \times d_\lambda}(\mathbb{C}).$$

Here λ are partitions of the integer n. The symbol d_λ denotes the number of standard Young tableaux of the shape λ. This isomorphism takes diagonal matrix units $E_{ii}^{\lambda} \in Mat_{d_\lambda \times d_\lambda}(\mathbb{C})$ to primitive idempotents of the Hecke algebra, denoted by e_{ii}^{λ}.

A big role is played by the so-called *Jucys-Murphy elements* $\{j_k\}_{1\le k\le n}$; j_k defined by the following formula:

$$j_1 = 1_{\mathbb{H}_n(q)}, \qquad j_k = (g_{k-1}\dots g_1)(g_1\dots g_{k-1}), \quad 2\le k\le n.$$

Jucys-Murphy elements generate a maximal commutative subalgebra in $\mathbb{H}_n(q)$. Moreover, the Hecke algebra center $Z(\mathbb{H}_n(q))$ consists of all symmetric polynomials in the Jucys-Murphy elements.

Any primitive idempotent is an eigenvector for all Jucys-Murphy elements:

$$j_k e_{ii}^{\lambda} = q^{2c(k)} e_{ii}^{\lambda} = e_{ii}^{\lambda} j_k. \qquad (1)$$

Here $c(k) = j - l$ is the content of the box with the number k in the ith standard tableau of shape λ. Integers l and j are the row and column indexes of the kth box respectively.
The following elements are the central idempotents of the algebra $\mathbb{H}_n(q)$:

$$e^{\lambda} = \sum_{i=1}^{d_\lambda} e_{ii}^{\lambda}, \qquad 1_{\mathbb{H}_n(q)} = \sum_{\lambda \vdash n} e^{\lambda}.$$

The idempotent corresponding to the diagram (1^n) is called a *q-skew-symmetrizer* and is denoted by $A^{(n)}$.

The following is called "Quantum universal matrix Capelli identity".

Theorem.

In the Reflection Equation algebra, constructed from an arbitrary Hecke R-matrix, the matrix quantum Capelli's identity holds:

$$L_{\overline{1}}(L_{\overline{2}} + \frac{J_2^{-1} - 1}{q - q^{-1}}) \dots (L_{\overline{n}} + \frac{J_n^{-1} - 1}{q - q^{-1}}) =$$

$$M_{\overline{1}} \dots M_{\overline{n}} D_{\overline{n}} \dots D_{\overline{1}} J_1^{-1} \dots J_n^{-1}$$

Here, $J_k = \rho_R(j_k)$, and ρ_R is the R-matrix representation of the Hecke algebra.

Corollary. (JLM)

For any idempotend e_{ii}^{λ} the following equality holds:

$$\Big(L_{\overline{1}}(L_{\overline{2}} - q^{-c(2)}[c(2)]_q)\dots(L_{\overline{n}} - q^{-c(n)}[c(n)]_q)\rho_R(e_{ii}^{\lambda})\Big) =$$

$$\Big(q^{-2(c(1)+\dots+c(n))}M_{\overline{1}}\dots M_{\overline{n}}D_{\overline{n}}\dots D_{\overline{1}}\rho_R(e_{ii}^{\lambda})\Big)$$

By applying the R-traces $Tr_{R(1\dots n)}$ to the l.h.s. of the last formula, we get the so-called "quantum immanant".

Definition.

The element of the center of the RE algebra

$$Tr_{R(1\dots n)}\Big(L_{\overline{1}}(L_{\overline{2}}-q^{-c(2)}[c(2)]_q)\dots(L_{\overline{n}}-q^{-c(n)}[c(n)]_q)\rho_R(e_{ii}^{\lambda})\Big)$$

is called quantum immanant.

Note that this quantum immanant does not depend on the number i of the standard Young tableau.
Their classical counterparts were introduced by Okounkov.

Now, I want to discuss some new perspectives of the RE algebras in Integrable systems theory.
Recently D.Talalev has undertaken an attempt to generalize the approach by Deift, Li, Nanda, and Tomei in "The full symmetric Toda system" (1986).
The phase space of the system is the space of symmetric matrices L. The authors have shown that the ratios of coefficients in

$$\Delta_k(\lambda) = \det((L - \lambda I)_k)$$

Poisson commute among themselves with respect to the linear $gl(N)$ Poisson bracket.
Here, A_k means a sub-matrix of the matrix A in which the first k columns and the last k rows are removed.
Talalaev has generalized this result to the standard RE algebra in dimension 3 by using the Gelfand-Retakh quasi-determinants.

Let us recall the notion of a quasi-determinant.
For any $1 \le i, j \le m$ let r_i, c_j be the i-th row and the j-th column of X.
Let X^{ij} be the submatrix of X obtained by removing the i-th row and the j-th column from X. For a row vector r let $r^{(j)}$ be r without the j-th entry. For a column vector c let $c^{(i)}$ be c without the i-th entry.
Assume that X^{ij} is invertible. Then the quasi-determinant $|X|_{ij}$ is defined by the formula

$$|X|_{ij} = x_{ij} - r_i^{(j)} (X^{ij})^{-1} c_j^{(i)}.$$

If T is the generating matrix of the RTT algebra with the standard R, then the following holds

$$det_q T = |T|_{11} \, |T^{11}|_{22} \dots t_{NN}$$

and the factors in this product commute all together.
However, it is not so if R is not standard.
Besides, it is not so if L is the generating matrix if the RE algebra.
However, if R is standard, according to the Talalaev's construction we have the following.

Let us consider the RE algebra with R standard and the generating matrix

$$L = \begin{pmatrix} l_1 & a^+ & c^+ \\ a^- & l_2 & b^+ \\ c^- & b^- & l_3 \end{pmatrix} \qquad (2)$$

Let us set

$$J_1 = c^+, \;\; J_2 = l_2 - b^+(c^+)^{-1}a^+,$$

$$J_3 = c^- - (b^-, l_3)\begin{pmatrix} a^+ & c^+ \\ l_2 & b^+ \end{pmatrix}^{-1}\begin{pmatrix} l_1 \\ a^- \end{pmatrix}.$$

Then J_1, J_2 J_3 commute all together and their product equals $det_R L$.

However, for the higher dimensions the problem becomes much more difficult.

In conclusion, remark that the RE algebras enable us to quantize some bracket in a unusual way. Consider an example.
Let

$$\{h, x\} = 2x\, h,\ \{h, y\} = -2y\, h,\ \{x, y\} = h^2$$

be a Poisson bracket on $\mathbb{R}^3$.
Its quantum counterpart can be presented as follows
$\mathcal{L}(R)/ < Tr_R L >$, where R is standard.
The point is that this algebra is equipped with a unusual trace. (Recall that according to B.Shoikhet a quantization of a Poisson structure with the usual trace is possible if it is unimodular.)
However, it is not clear how is it possible to quantize the Poisson bracket

$$\{h, x\} = 2x\, h^2,\ \{h, y\} = -2y\, h^2,\ \{x, y\} = h^3$$

in a similar manner.

Many thanks